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Radial pulsations of distorted polytropes of non-uniform density

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A polytrope with index $n = 1.5$ is a good model for fully convective star cores like red giants, brown dwarfs, giant gaseous planets (like Jupiter), while a polytrope with index $n = 3.0$ is usually also used to model main-sequence stars like our Sun, at least in the radiation zone, corresponding to the Eddington standard model of stellar structure (for instance, see Eddington, A. S., 1926, The Internal Constitution of the Stars, Cambridge University Press, Cambridge). Eigen-frequencies (natural frequencies) of radial pulsations of differentially rotating and tidally distorted (DRTD) polytropic stellar models of polytropic indices 1.5 and 3.0 have been computed taking into account the effect of non-uniform densities inside the stellar interiors. The method utilizes an averaging technique of Kippenhahn and Thomas in conjunction with the concepts of Roche-equipotentials. The study compliments earlier studies of radial oscillations of DRTD stellar structures. The utility of the work comes from the necessity to include the effects of non-uniform densities that involve the Lane–Emden equation on eigen-frequencies of oscillation up to second order perturbations.

Keywords: Differential rotation, tidal distortion, polytropic models, oscillations, pulsations, equipotential surface

1 Introduction

Observations exhibit that certain stars undergo different types of periodic deformations due to radial and non-radial oscillations, rotation, glitches in their life time

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in order to reproduce radio, X-rays, gamma-rays and other electromagnetic perturbations (Glendenning, 2000; Shapiro & Teukolsky, 1983). Such observations have importance as per their astrophysical significance. The present work is concerned with the computation of pulsation periods of some polytropes having perturbations due to radial oscillations occurring at certain stages of stars lifetime. These may possibly happen due to several reasons such as tidal distortions on the primary star by the secondary star of a binary system, influence of interstellar medium, etc. (Glendenning, 2000; Shapiro & Teukolsky, 1983). The general pulsations analysis of stellar models was designed long ago by Chandrasekhar (1964). It was initially applied only to the polytropic equation of state, and some years later to more realistic models (Glass & Lindblom, 1983). Some radial oscillations frequencies were approximated for modern sets of equations of states of neutron stars by Kokkotas & Ruoff (2001). Radial pulsations for quark stars were generally analyzed using the bag model to build the equation of state for cold quark matter (Flores & Lugones, 2010). Their results suggest that the zero mode periods have been detected very low (Benvenuto, Vucetich & Horvath, 1991; Horvath, Benvenuto & Vucetich, 1991), which motivates the search for higher periods of oscillations (Flores & Lugones, 2014, 2018). In astrophysics, a polytrope refers to a solution of the Lane–Emden equation in which the pressure depends upon the density in the form $P = K\rho^{((n+1)/n)}$, where P is pressure, ρ is density and K is a constant of proportionality (Horedt, 2004). The constant n is known as the polytropic index. The polytropic index has an alternative definition with n as the exponent. According to this, P is a function of both ρ and T (the temperature), however, in the particular case described by the polytrope equation, there are other additional relations between these three quantities, which together determine the equation. Thus, this is simply a relation that expresses an assumption about the change of pressure with radius in terms of the change of density with radius, yielding a solution to the Lane–Emden equation. However, studies as above mentioned were carried out with an assumption that stars have uniform stellar structures, while for realistic stars, densities vary in the interior.

The pulsational properties of a star generally depend on the density distribution of the star and the ratio of specific heat of stellar material. The polytropic model for different polytropic indices n affords a convenient series of models for the study of pulsational properties. However, for most of the models, a better approximation to the density distribution of a star can be achieved by assuming a mass variation from the centre to the surface of the star. Therefore, it may be possible to obtain some conclusions regarding the effects of density distribution on the pulsation properties of DRTD polytropic models of different polytropic indices as well as composite polytropic models with interfaces of different radii. Singh (1969) has done some work on the radial oscillations of composite polytropes. However, this basic study was focused on non-rotating composite polytropes. Kumar et al. (2021) analysed oscillations of DRTD polytropes, but the study was mainly focused on the pressure and gravity modes of oscillations. This paper is concerned with the computation of stellar pulsations of DRTD polytropic stellar structures of stars of non-uniform densities. It is assumed that radial pulsations are excited for slowly rotating stars, and eigen-frequencies are influenced by the rotational effects.

With this object in view a number of DRTD polytropic stellar models have been taken for study from Lal, Mohan & Singh (2001). For study so that effects of non-uniform densities can be analyzed on their pulsation properties investigated. In order to analyse the effects of non-uniform densities on stellar radial pulsations as well as for drawing significant conclusions the results obtained in this study have been compared with the results earlier obtained by Lal, Mohan & Singh (2001) for the same stellar models but without incorporating the effect of mass variation.

2 Radial oscillations of DRTD polytropic stellar models of non-uniform densities

The mathematical problem of computing the eigen-frequencies of pulsations of a rotating star is quite complex. Mohan, Saxena & Agarwal (1991) determined radial and non-radial oscillations of distorted gaseous spheres adopting the approach used by Mohan & Agarwal (1987) to analyze the effect of rotation as well as tidal distortion on the configuration and periods of small adiabatic pulsations of composite models. The method utilizes the Kippenhahn & Thomas (1970) averaging approach to formulate the eigen-value problem of rotationally and tidally distorted stars in a manner that it can be used to determine the eigen-frequencies of various adiabatic modes of radial and non-radial oscillations. Authors such as Mohan, Lal & Singh (1998) and Saini, Kumar & Lal (2015) used this approach to formulate the eigen-value boundary valued problem for determining the periods of oscillations of radial modes of rotationally and tidally distorted stellar models.

Following Mohan, Lal & Singh (1998), adiabatic radial modes of oscillations of stellar models having solid body rotation, differentially rotating models rotating according to the law $\omega^2 = b_1 + b_2 s^2 + b_3 s^4$ (ω^2 is a nondimensional angular velocity of a fluid particle at distance s from the axis of rotation and b_i are suitably chosen constants, for $i = 1, 2, 3$) are obtained using their topological equivalent models developed considering the averaging concept of Kippenhahn & Thomas (1970). Following Mohan, Saxena & Agarwal (1991) and using the generalized approach of Kippenhahn & Thomas (1970), the Sturm–Liouville type differential equation determining the eigen-value problems of pulsations of rotating polytropic models of a star can be expressed as

$$\tau_1 \frac{d^2\eta}{dr_0^2} + \tau_2 \frac{d\eta}{dr_0} + (\tau_3 \nu *^2 - \tau_4)\eta = 0, \tag{1}$$

where

$$\begin{aligned}
\tau_1 =& \{1 - 8\mathcal{P}_1 r_0^3 - 12\mathcal{P}_2 r_0^5 - 2\left(7\mathcal{P}_3 - 24\mathcal{P}_1^2\right) r_0^6 - 16\mathcal{P}_4 r_0^7 - 18\left(\mathcal{P}_5 - 8\mathcal{P}_1\mathcal{P}_2\right) r_0^8 \\
&+ 8\left(21\mathcal{P}_1\mathcal{P}_3 - 32\mathcal{P}_1^3\right) r_0^9 - 2\left(11\mathcal{P}_6 - 96\mathcal{P}_1\mathcal{P}_4 - 54\mathcal{P}_2^2\right) r_0^{10} + \ldots\},
\end{aligned}$$

$$\begin{aligned}
\tau_2 =& \frac{1}{r_0}\{4 - 32\mathcal{P}_1 r_0^3 - 58\mathcal{P}_2 r_0^5 - 2\left(37\mathcal{P}_3 - 114\mathcal{P}_1^2\right) r_0^6 - 92\mathcal{P}_4 r_0^7 \\
&- 16\left(7\mathcal{P}_5 - 51\mathcal{P}_1\mathcal{P}_2\right) r_0^8 + 4\left(258\mathcal{P}_1\mathcal{P}_3 - 373\mathcal{P}_1^3\right) r_0^9 \\
&- 2\left(79\mathcal{P}_6 - 636\mathcal{P}_1\mathcal{P}_4 - 356\mathcal{P}_2^2\right) r_0^{10} + \ldots\} + \frac{N+1}{\theta_\psi}\frac{d\theta_\psi}{dr_0} r_0\{1 - 8\mathcal{P}_1 r_0^3 - 12\mathcal{P}_2 r_0^5 \\
&- 2\left(7\mathcal{P}_3 - 24\mathcal{P}_1^2\right) r_0^6 - 16\mathcal{P}_4 r_0^7 + 8\left(18\mathcal{P}_1\mathcal{P}_2 - 21\mathcal{P}_1\mathcal{P}_3 - 32\mathcal{P}_1^3\right) r_0^8 \\
&+ 8\left(21\mathcal{P}_1\mathcal{P}_3 - 32\mathcal{P}_1^3\right) r_0^9 - 2\left(11\mathcal{P}_6 - 96\mathcal{P}_1\mathcal{P}_4 - 54\mathcal{P}_2^2\right) r_0^{10} + \ldots\},
\end{aligned}$$

$$\tau_3 = \frac{N+1}{3\gamma r_{os}^3}\frac{\xi_u^2 k}{\theta_\psi}\left(\frac{\overline{\rho}}{\rho_c}\right),$$

$$\begin{aligned}\tau_4 = &-\left(3-\frac{4}{\gamma}\right)\frac{N+1}{\theta_\psi r_0}\frac{d\theta_\psi}{dr_0}\{1-5\mathcal{P}_1 r_0^3-7\mathcal{P}_2 r_0^5-8\left(\mathcal{P}_3-21\mathcal{P}_1^2\right)r_0^6-9\mathcal{P}_4 r_0^7\\ &-10\left(\mathcal{P}_5-6\mathcal{P}_1\mathcal{P}_2\right)r_0^8+\left(69\mathcal{P}_1\mathcal{P}_3-85\mathcal{P}_1^3\right)r_0^9\\ &-\left(12\mathcal{P}_6-78\mathcal{P}_1\mathcal{P}_4-43\mathcal{P}_2^2\right)r_0^{10}+...\},\end{aligned}$$

and

$$\nu *^2 = \frac{D^3 r_{os}^3 \sigma^2}{GM_0},$$

where

$$\begin{aligned}\mathcal{P}_1 =&\frac{1}{3t}b_1, \mathcal{P}_2=\frac{2}{15t}b_2, \mathcal{P}_3=\frac{4}{5t^2}q^2+\frac{4}{15t^2}b_1 q+\frac{19}{45t^2}b_1^2, \mathcal{P}_4=\frac{8}{105t}b_3,\\ \mathcal{P}_5 =&\frac{5}{7t^2}q^2+\frac{152}{315t^2}b_1 b_2+\frac{4}{21t^2}b_2 q,\\ \mathcal{P}_6 =&\frac{2}{3t^2}q^2+\frac{152}{35t^2}b_2 q+\frac{162}{105t^2}b_3 q+\frac{112}{315t^2}b_1 b_3+\frac{212}{1575t^2}b_2^2.\end{aligned}$$

Here $q = M_1/M_0$. M_0 and M_1 are the masses of primary and secondary components of the binary system, respectively, such that primary is supposed to be much larger than secondary ($M_0 >> M_1$). $t = M_0(r)/M_0$. The term $M_0(r)$ represents the mass in the interior of the sphere of radius r inside the primary component, D the distance separation between the centres of these stars, η the amplitude of pulsation that depends on variable r_0 (distance of an arbitrary point within the star from its centre). Also $\nu*^2$ is the dimensionless form of eigen-frequency σ, N the polytropic index and r_{os} is the value of r_0 at the free surface. In the above expressions values of the parameters ξ_u (dimensionless variable) are $3.65375, 6.89685$ for polytropes of indices $N = 1.5, 3.0$ respectively), ρ_c (central density), $\overline{\rho}$ (average density) and k are the parameters for the original undistorted polytropic stellar model, let γ be the ratio of specific heats, θ_ψ is the parameter depending upon the distance of the point under consideration from the centre of stellar model and G the universal gravitational constant. Differential equation (1) in non-dimensional form determines the eigen-frequencies of adiabatic radial oscillations of some differentially rotating polytropes, in which terms up to second-order of smallness in parameters b_1, b_2, b_3 are retained. For numerical computation of the eigen-frequencies, the second-order linear differential equation (1) is to be solved numerically, which require $\eta =$ finite, corresponding $r_0 = 0$ i.e. at the centre and $r_0 = r_{os}$ i.e. at the free surface.

3 Numerical evaluation of eigen-frequencies of pseudo radial modes of oscillations

Eigen-frequencies $\nu*^2$ of radial pulsations of differentially rotating polytropic gaseous spherical models can be determined by integrating equation (1), subjecting the condition requiring η to be finite at the centre as well as at the free surface. As the centre and the surfaces are both singularities, numerical computation was carried out using trial values of $\nu*^2$ to determine the eigen-frequencies for radial oscillations.

To start integrations from a point near the centre and another point near the surface, two series solutions have been developed at $x = 0.001$ and $x = 0.999$. Inward and outward integrations were performed using the Runge–Kutta method with a step length 0.0001. Few trails with different values of $\nu*^2$ were carried out till the absolute difference in the value of ratio $\eta/\left(d\eta/dx\right)$ at a pre-selected point in the interior of the model from the outward and inward integrations was found to be less than 0.0001. The series solution was developed to initiate integration from a point close to the centre outwards and from a point close to the surface inwards. Evaluations were made with the different trails until disparity between outward and inward values of η were less than 0.0001. Computations were performed to determine the fundamental and the first modes of pulsations of DRTD polytropic models of indices 1.5 and 3.0.

4 Analysis of results

The obtained eigen-frequencies of radial pulsations of DRTD stellar models of non-uniform densities are given in Table 1. The eigen-frequencies of radial modes of pulsations for polytropes of index 1.5 are found to be smaller in comparison to corresponding eigen-frequencies for polytropes of index 3.0. Effects of mass variation on the eigen-frequencies of tidally distorted polytropic models are shown with the help of Figure 1 to Figure 4. Excepting fundamental modes nonrotating model with polytropic index 1.5, eigen-frequencies of DRTD models of a star decrease, if it is considered as a spherical structure of non-uniform mass. The effect of mass variation is largest for model 9 (see Table 1) and is minimum for a nonrotating polytrope i.e. model 1. The effect of density variation causes the reduction in eigen-frequencies of the first mode of oscillations of DRTD model of polytropic star, comparing than fundamental mode. As the angular velocity increases from model 4 to model 6 with the same tidal effect, reductions in eigen-frequencies increase.

The results obtained are in good agreement with the results obtained by Lal, Mohan & Singh (2001). The eigen-frequencies have been compared in Figure 1 to Figure 4 to show the effect of density variation in stallers' interiors on the radial pulsations.

Table 1 Eigen-frequencies of radial modes of pulsations of DRTD polytropic models of stars [results in parenthesis are the eigen-frequencies of models of uniform density, cf. Lal, Mohan, & Singh (2001)].

Model	Values of rotational and tidal distortion parameters				$N = 1.5$		$N = 3.0$	
	b_1	b_2	b_3	q	ω_0^2	ω_1^2	ω_0^2	ω_1^2
1	0.0	0.0	0.0	0.1	2.692770 (2.69269)	12.40843 (12.51098)	9.205229 (9.26008)	16.82798 (16.95006)
2	0.1	0.0	0.0	0.1	2.585088 (2.66545)	11.70890 (12.32704)	8.852499 (9.17380)	15.94048 (16.72728)
3	0.0	0.1	0.0	0.1	2.684831 (2.69055)	12.31835 (12.48976)	9.181955 (9.25378)	16.72499 (16.92634)
4	0.0	0.0	0.1	0.1	2.691693 (2.69226)	12.39538 (12.50688)	9.202207 (9.25933)	16.81399 (16.94656)
5	0.1	0.1	0.0	0.1	2.577571 (2.66330)	11.62935 (12.30549)	8.829380 (9.16305)	15.84947 (16.69330)
6	0.1	0.1	0.1	0.1	2.576565 (2.66286)	11.61814 (12.30131)	8.826350 (9.16229)	15.83698 (16.69000)
7	0.1	0.2	0.1	0.01	2.582200 (2.66108)	11.68399 (12.28836)	8.844196 (9.15877)	15.90800 (16.67876)
8	0.1	0.2	0.1	0.1	2.569055 (2.66069)	11.53951 (12.27972)	8.802988 (9.15580)	15.74585 (16.66602)
9	0.1	0.2	0.1	0.15	2.554084 (2.65890)	11.39320 (12.26300)	8.754330 (9.15254)	15.57596 (16.65275)
10	0.1	-0.05	0.0	0.1	2.588858 (2.66604)	11.74920 (12.33478)	8.864102 (9.17458)	15.98599 (16.73334)
11	0.1	-0.1	0.05	0.1	2.592125 (2.66741)	11.78354 (12.34638)	8.873998 (9.17748)	16.02543 (16.74345)
12	0.1	-0.15	0.1	0.1	2.595405 (2.66776)	11.81825 (12.35202)	8.883998 (9.18066)	16.04942 (16.75430)
13	0.1	-0.02	0.4	0.01	2.597349 (2.66522)	11.81262 (12.32553)	8.892387 (9.17244)	16.06197 (16.72369)
14	0.1	-0.02	0.4	0.1	2.582580 (2.66434)	11.67953 (12.31578)	8.845115 (9.18003)	15.90749 (16.70877)
15	0.1	-0.02	0.4	0.15	2.566605 (2.66344)	11.53734 (12.30485)	8.793492 (9.16531)	15.74098 (16.69828)
16	0.04	-0.01	0.0625	0.1	2.649043 (2.68185)	12.11887 (12.43760)	9.063332 (9.22491)	16.46446 (16.85873)
17	0.1	0.02	-0.05	0.1	2.584087 (2.66505)	11.69860 (12.32363	8.849422 (9.17044)	15.92898 (16.71830)
18	0.1	-0.1	0.0	0.01	2.607839 (2.66830)	11.91594 (12.35671)	8.924364 (9.18046)	16.18049 (16.75478)
19	0.1	-0.1	0.0	0.1	2.592639 (2.66811)	11.78936 (12.35383)	8.875497 (9.17779)	16.03198 (16.74478)
20	0.1	-0.1	0.0	0.15	2.576308 (2.66585)	11.64856 (12.33254)	8.822988 (9.17289)	15.86696 (16.72891)
21	0.04	-0.16	0.16	0.01	2.674470 (2.68583)	12.35435 (12.47586)	9.141042 (9.23786)	16.73183 (16.90349)
22	0.04	-.016	0.16	0.1	2.659692 (2.68515)	12.23667 (12.46838)	9.094448 (9.23366)	16.59449 (16.89078)
23	0.04	-0.16	0.16	0.15	2.643074 (2.68339)	12.09355 (12.45283)	9.042480 (9.23079)	16.42942 (16.87934)

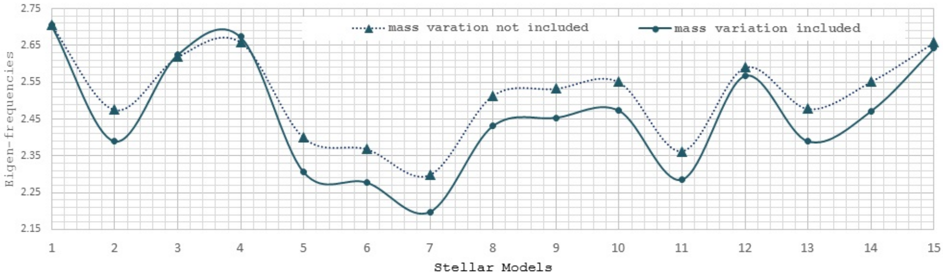

Figure 1 Eigen-frequencies of fundamental radial modes of oscillation for index 1.5.

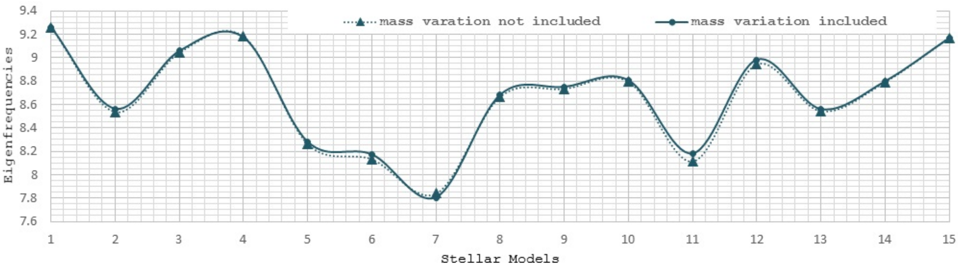

Figure 2 Eigen-frequencies of fundamental radial modes of oscillation for index 3.0.

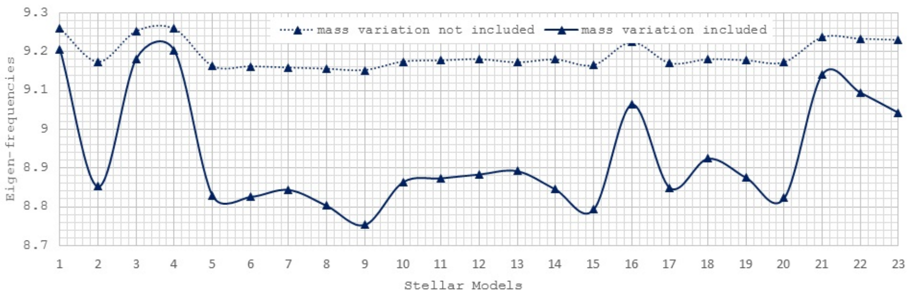

Figure 3 Eigen-frequencies of first radial modes of oscillations of stellar models of index 1.5.

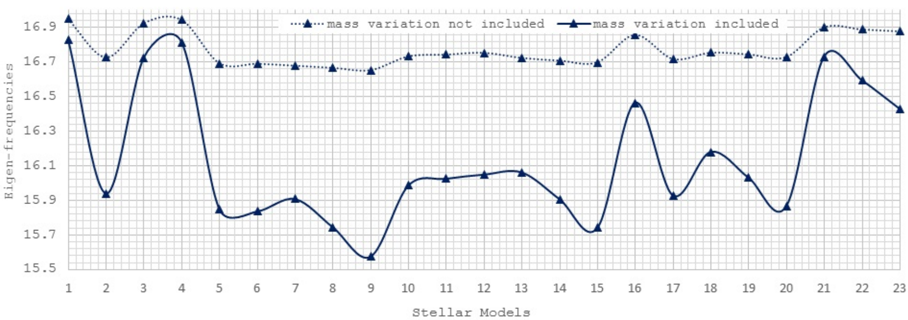

Figure 4 Eigen-frequencies of first radial modes of oscillation of stellar models of index 3.0.

5 Conclusion

In this study it is concluded that eigen-frequencies of the radial modes of pulsations of a star decrease with increase in angular velocity. Similar effects have also been observed on eigen-frequencies of a stellar model influenced by tidal effect and differential rotation as well. However, increment in tidal effect decreases the radial eigen-frequencies. Model 9 for which tidal effect is large and angular velocity rises rapidly towards surface, has lowest radial modes of oscillations for both indices. As the angular velocity of a star is increased, eigen-frequencies of radial modes of oscillations of stellar models are found to be decreased. On comparing of the results, it can be concluded that effect of non-uniform densities decrease the eigen-frequencies of pulsations for DRTD polytropes.

6 Appendix

Eigen-frequencies of radial modes of pulsations of DRTD stellar structures of non-uniform density

To determine the periods of radial modes of pulsations of rotationally and tidally distorted Roche model, Mohan, Lal & Singh (1998) formulated an eigen-value problem which was further used by Mohan, Saxena & Agarwal (1991) to formulate for some eigen-value problem to calculate the eigen-frequencies of radial and non-radial modes of oscillations of DRTD gaseous spheres and also by Saini, Kumar & Lal (2015), to make the eigen-value problem of determining radial modes of oscillations.

It is supposed that fluid elements oscillate in the unisons on an equipotential surface. Adopting the approach of Mohan, Lal & Singh (1998), equation (1) has been used to calculate eigen-frequencies of radial pulsations of DRTD stars related to eigen-value problem deciding eigen-frequencies of radial pulsations of equivalent model, and hence can be represented as:

$$\frac{d^2\eta}{dr_{o\psi}^2}+\frac{4-\mu}{r_{o\psi}}\frac{d\eta}{dr_{o\psi}}+\left\{\frac{\rho_{o\psi}\nu^2}{\gamma P_{o\psi}}-\left(3-\frac{4}{\gamma}\right)\frac{\mu}{r_{o\psi}^2}\right\}\eta=0,\ \text{where}\ \ \mu=-\frac{r_{o\psi}}{P_{o\psi}}\frac{dP_{o\psi}}{dr_{o\psi}}. \quad \text{(A-1)}$$

In the above mentioned equation, $r_{o\psi}$, $\rho_{o\psi}$ and $P_{o\psi}$ represents the values of radius r_ψ, density ρ_ψ and pressure P_ψ, respectively on surface $\psi = constant$. In the equanimity condition on surface $\psi = constant$, if, r_ψ, ρ_ψ and P_ψ, are used in lieu of $r_{o\psi}$, $\rho_{o\psi}$ and $P_{o\psi}$ and respectively, as well as to represent the equanimity of the surface $\psi = constant$ and using $r_0 = t/(\psi - q)$ in lieu of r and considering $\omega^2 = b_1 + b_2 s^2 + b_3 s^4$, equation (A-1) is reduced as:

$$\varkappa\frac{d^2\eta}{dr_o^2}+\left\{\frac{4-\mu}{r_o}\varrho-\varsigma\right\}\frac{d\eta}{dr_o}+\left\{\frac{R^2\nu^2\rho_\psi}{\gamma P_\psi}-\left(3-\frac{4}{\gamma}\right)\frac{\mu}{r_\psi^2}\vartheta\right\}\eta=0. \quad \text{(A-2)}$$

where

$$\begin{aligned}
\varkappa =&1 - 8\mathcal{P}_1 r_0^3 - 12\mathcal{P}_2 r_0^5 - 2\left(7\mathcal{P}_3 - 24\mathcal{P}_1^2\right) r_0^6 - 16\mathcal{P}_4 r_0^7 - 18\left(\mathcal{P}_5 - 8\mathcal{P}_1\mathcal{P}_2\right) r_0^8 \\
&+ 8\left(21\mathcal{P}_1\mathcal{P}_3 - 32\mathcal{P}_1^3\right) r_0^9 - \left(12\mathcal{P}_6 - 78\mathcal{P}_1\mathcal{P}_4 - 43\mathcal{P}_2^2\right) r_0^{10} + ..., \\
\varrho =&1 - 5\mathcal{P}_1 r_0^3 - 7\mathcal{P}_2 r_0^5 - \left(8\mathcal{P}_3 - 21\mathcal{P}_1^2\right) r_0^6 - 9\mathcal{P}_4 r_0^7 - 10\left(\mathcal{P}_5 - 6\mathcal{P}_1\mathcal{P}_2\right) r_0^8 \\
&+ \left(69\mathcal{P}_1\mathcal{P}_3 - 85\mathcal{P}_1^3\right) r_0^9 - \left(12\mathcal{P}_6 - 78\mathcal{P}_1\mathcal{P}_4 - 43\mathcal{P}_2^2\right) r_0^{10} + ..., \\
\varsigma =&\frac{1}{r_0}\left\{12\mathcal{P}_1 r_0^3 + 30\mathcal{P}_2 r_0^5 + 4\left(13\mathcal{P}_3 - 36\mathcal{P}_1^2\right) r_0^6 + 56\mathcal{P}_4 r_0^7 + 4\left(18\mathcal{P}_5 - 144\mathcal{P}_1\mathcal{P}_2\right) r_0^8\right. \\
&\left.- 36\left(21\mathcal{P}_1\mathcal{P}_3 - 32\mathcal{P}_1^3\right) r_0^9 + 10\left(11\mathcal{P}_6 - 96\mathcal{P}_1\mathcal{P}_4 - 54\mathcal{P}_2^2\right) r_0^{10} + ...\right\}, \\
\vartheta =&1 - 2\mathcal{P}_1 r_0^3 + 2\mathcal{P}_2 r_0^5 - \left(2\mathcal{P}_3 - 3\mathcal{P}_1^2\right) r_0^6 - 2\mathcal{P}_4 r_0^7 - 2\left(\mathcal{P}_5 - 3\mathcal{P}_1\mathcal{P}_2\right) r_0^8 \\
&- 2\left(3\mathcal{P}_1\mathcal{P}_3 - 2\mathcal{P}_1^3\right) r_0^9 - \left(2\mathcal{P}_6 - 6\mathcal{P}_1\mathcal{P}_4 - 3\mathcal{P}_2^2\right) r_0^{10} + ..., \\
\mu =&- \frac{r_\psi}{P_\psi}\frac{dP_\psi}{dr_\psi} = -\phi\frac{r_0}{P_\psi}\frac{dP_\psi}{dr_0}, \text{ such that} \\
\phi =&1 - 3\mathcal{P}_1 r_0^3 - 5\mathcal{P}_2 r_0^5 - 6\left(\mathcal{P}_3 - 2\mathcal{P}_1^2\right) r_0^6 - 7\mathcal{P}_4 r_0^7 - 2\left(4\mathcal{P}_5 - 19\mathcal{P}_1\mathcal{P}_2\right) r_0^8 \\
&- 3\left(15\mathcal{P}_1\mathcal{P}_3 - 16\mathcal{P}_1^3\right) r_0^9 - 2\left(5\mathcal{P}_6 - 26\mathcal{P}_1\mathcal{P}_4 - 15\mathcal{P}_2^2\right) r_0^{10} + ...
\end{aligned}$$

If there is no distortion i.e. ($b_1 = b_2 = b_3 = q = 0, D = R, P_\psi = P, \rho_\psi$ and $r_0 = x$), (A-2) will be reduced as:

$$\frac{d^2\eta}{dx^2} + \frac{4-\mu}{x}\frac{d\eta}{dx} + \left\{\frac{R^2\nu^2\rho}{\gamma P} - \left(3 - \frac{4}{\gamma}\right)\frac{\mu}{x^2}\right\}\eta = 0 \text{ where } \mu = -\frac{x}{P}\frac{dP}{dx}. \tag{A-3}$$

It determines eigen-frequencies of small adiabatic radial oscillations of gaseous sphere (for instance, see Roseland (1949)). P_ψ is the pressure on an equipotential surface, P is the pressure at an arbitrary point. Other symbols have their usual meanings as defined earlier.

Equation (A-2) have been used to analyse impacts of differential rotation with tidal distortion on periods of radial pulsations. The effect of differential rotation and tidal distortion on the stellar models have been incorporated through inclusion of the terms $\varkappa, \varrho, \varsigma, \vartheta, \phi$ and dependence of P_ψ and ρ_ψ on ψ. The eigen-value problem using equation (A-2), subject to initial conditions, $\eta = finite$ at centre and at free surface, has been solved. For computation, it is convenient to use

$$\eta = \frac{\zeta}{r_0} \text{ and } r_0 = x r_{os} \tag{A-4}$$

(r_{os} is value of r_0 at outer surface and x is taken as an independent variable, however, ζ is a dependent variable). The value of x will be zero at the centre while it will be one at outer surface. Initial condition $\eta = finite$ at the centre, and equation (A-2) will be reduced as:

$$\tau_1^*\frac{d^2\zeta}{dx^2} + \tau_2^*\frac{d\zeta}{dx} + \tau_3^*\zeta = 0, \tag{A-5}$$

where

$$
\begin{aligned}
\tau_1^*(b_1,b_2,b_3,q,x,t) &= \tau_1(b_1,b_2,b_3,q,x,r_{os},t),\\
\tau_2^*(b_1,b_2,b_3,q,x,t) =& \frac{4-\mu}{x}\varrho(b_1,b_2,b_3,q,x,r_{os},t) - \frac{2}{x}\tau_1(b_1,b_2,b_3,q,x,r_{os},t)\\
&- r_{os}\varsigma(b_1,b_2,b_3,q,x,r_{os},t),\\
\tau_3^*(b_1,b_2,b_3,q,x,t) =& \frac{R^2\nu^2\rho_\psi r_{os}^2}{\gamma P_\psi} - \left(3-\frac{4}{\gamma}\right)\frac{\mu}{x^2}\vartheta(b_1,b_2,b_3,q,x,r_{os},t)\\
&- \frac{1}{x}\tau_2^*(b_1,b_2,b_3,q,x,r_{os},t).
\end{aligned}
$$

Now the boundary conditions become:

$$
\begin{gathered}
\zeta = 0, \text{ at the centre } x=0\\
\text{and}\\
\zeta = finite, \text{ at the surface } x=1.
\end{gathered} \tag{A-6}
$$

To compute eigen-values, equation (A-5) is to be solved with the above boundary conditions. It is to be noted that centre and free surface both are the singularities for equation (A-5), therefore, it would be better to write two series solutions near these points to perform the integrations.

Series solution of ζ at the centre may be assumed as:

$$\zeta = \sum_{j=0}^{\infty} a_j x^{j+\lambda}. \tag{A-7}$$

Series solution of ζ near the centre may be assumed as:

$$\zeta = 1 + \sum_{j=0}^{\infty} c_j (1-x)^{j+\lambda}. \tag{A-8}$$

Integrating equation (A-5) numerically for some trial values of ν until a value of ν is obtained for which boundary conditions (A-6) to be satisfied and hence give eigenfrequency of radial modes of pulsations. It can be achieved by solving equation (A-5). Series solution (A-7) will provide the initial values near the centre. The optimization process will be continued with different values of ν until the value of $\zeta/(d\zeta/dx)$ achieved desired accuracy.

It should be noted that the eigen-value problem determines eigen-frequencies of pulsations of a DRTD polytropic gaseous sphere, rotating according to $\omega^2 = b_1 + b_2 s^2 + b_3 s^4$. On using $b_1 = 2n, b_2 = b_3 = 0, t = 1$, this mathematical model will be reduced to the model for radial modes of the frequencies of oscillations of solid body rotating stars. Although arbitrary precision on eigen-frequencies cannot be achieved, this would make no sense as resultant accuracy could not to be limited by machine precision, one therefore has to interpolate between the given points to construct staller background model. Different interpolation scheme can be applied to interpolate different modes of eigen-frequencies. Even though the parameters used to determine stellar structures are not very sensitive to actual interpolation scheme, the adiabatic index which is used into the oscillation, this quantity is sensitive to interpolation, especially if the equation of state changes quite abruptly. Using different interpolation schemes, such as spline interpolation or as linear logarithmic interpolation, it is noticed that the eigen-frequencies may vary up to about 0.5 per cent only.

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