

# Methods and Lessons Learned Determining the H-G Parameters of Asteroid Phase Curves

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## Abstract

The phase curve of an asteroid shows how its brightness changes as a function of solar phase angle. The project of determining an asteroid's phase curve is challenging because it combines four photometric objectives: determination of the asteroid's rotational lightcurve, monitoring the asteroid over a wide range of solar phase angles, high photometric accuracy, and FOV bridging or "all-sky" photometry to link the asteroid's brightness from night to night. All of these are important because the different shapes of phase curves (caused by different surface characteristics) have a subtle effect. The typical phase curve project requires photometric accuracy of about  $\pm 0.03$  mag or better; commits the observatory to a dozen nights, spread over about 2 months; and commits the astronomer to a set of data analyses that is significantly more extensive than is required for a typical "differential photometry rotational lightcurve" project. In this paper I will describe how I have gathered the necessary data, the procedures for data reduction, and some challenges in determining the slope parameter G.

## 1. Introduction

An asteroid's phase curve contains valuable information related to the surface characteristics of the asteroid. Amateur efforts to determine asteroid phase curves are a much-needed addition to our knowledge, because not many asteroids have well-determined phase curves, and few professional astronomers are doing such studies.

The purpose of this paper is to explore a few practical aspects of the asteroid phase curve project:

- What does a "good" phase curve look like?
- What range of solar phase angles must be covered, and how long is this likely to take?
- What photometric accuracy is required?
- Should magnitudes be transformed to the standard V-band, or left in instrumental v-band?
- Should the phase curve plot mean magnitude or peak magnitude of the lightcurve vs. solar phase angle?
- Does the lightcurve change as the solar phase angle changes?
- What procedure do the pro's use to determine phase curves from photometric data?
- For what level of accuracy should you strive when determining H and G?

I'll describe these topics in the context of two phase curve projects that I did in the past year. One of these, 1130 Skuld, was immediately successful (Buchheim, 2010). The other, 535 Montague, was

more troublesome, but was a useful learning experience. The result for it will be submitted to the *Minor Planet Bulletin* shortly.

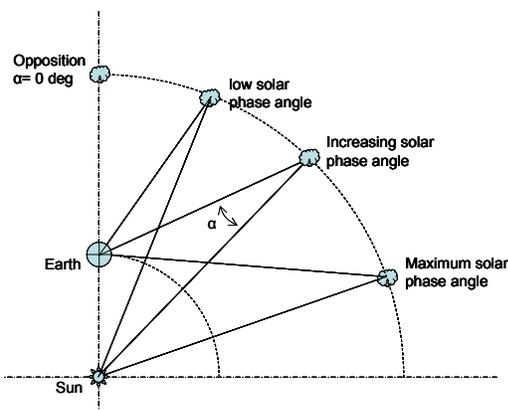
## 2. Phase Curve Background

The geometry of the observation of an asteroid is illustrated in Figure 1. The solar phase angle ( $\alpha$ ) is analogous to the moon's phase; when  $\alpha = 0^\circ$ , the asteroid is "full" (i.e. fully illuminated). When  $\alpha \approx 90^\circ$ , the asteroid is in quadrature and is illuminated in the same way that a first- or third-quarter Moon is in that half of its visible surface is in light and half is in darkness. Because of where they orbit, outside Earth's orbit, main belt asteroids don't reach solar phase angles much greater than about 20-30°. For example, think of Mars. It shows a "phase defect" but you never see a crescent Mars. The farther an object orbits from the Sun, the smaller the maximum observable solar phase angle. More distant objects (e.g. Jupiter Trojan asteroids) display a smaller range of solar phase angles and Kuiper-belt objects are so far away that Earth-bound observatories can observe them at solar phase angles of only  $\alpha \approx 0^\circ \pm 2^\circ$ .

On the other hand, near-Earth asteroids during their close approaches to Earth can be observed at quite large phase angles and, of course, spacecraft can arrange to observe their targets at large phase angles (Newburn et al., 2003).

The motion of Earth and the asteroid in their respective orbits around the Sun gradually alters the solar phase angle. For a typical main-belt asteroid,

the solar phase angle will go from about  $-20^\circ$  to nearly  $0^\circ$  (at opposition) over an interval of 2 to 3 months and then increase to about  $20^\circ$  over another 2 to 3 months.



**Figure 1: Geometry of orbits: Earth distance, Sun distance, and solar phase angle**

### 2.1 Characteristics of Bi-directional Reflectivity

Imagine a narrow beam of collimated light aimed at a flat surface. Perhaps the surface is covered with snow, or with dirt, or with rocks. The surface isn't smooth, so it won't reflect the light like a mirror. In most real surfaces, there is a "forward gloss" (a noticeable amount of the incident light comes off in the direction that a mirror would have sent it), a "backscatter" (a fair amount may be reflected back toward the light source), and a general diffuse reflection that goes off in all directions. The smoother the surface, the more light is likely to be directed into the forward gloss. In a perfectly smooth (glassy ice) surface, quite a bit of the reflected light is directed into the forward gloss, giving a nearly specular reflection from the mirror-like surface. If the incoming beam is directed exactly perpendicular to the surface, then the "forward gloss" is directed back toward the source. This is the geometric situation that you have at solar phase angle =  $0^\circ$ . The forward and reverse glosses are both directed toward the observer. There can be a pronounced increase in brightness near zero solar phase angle as a result of this phenomenon, which is the so-called "opposition effect".

There are at least three physical effects that give rise to the opposition effect: "shadow hiding", "multiple scattering", and "coherent backscatter" (see Lumme & Bowell, 1981).

### 2.2 Reduced Magnitude

Throughout an apparition, the solar phase angle isn't the only thing that changes. The Earth-asteroid and Sun-asteroid distance also change continuously. These changing distances naturally affect the asteroid's observed brightness and work in combination, not independently. This leads to the concept of "reduced magnitude" to account for the changing distances, which is defined by

$$V_R = V - 5\log(RD) \quad \text{Eq. 1}$$

Where

- V observed V magnitude
- R distance Sun to Asteroid (in AU)
- D distance Earth to Asteroid (in AU)

$V_R$  is the brightness that the asteroid *would have had* if it were placed at 1AU from the Sun, and observed from a distance of 1AU from the observer. By placing it at a standard distance,  $V_R$  "backs out" the effect of changing distance. Reduced magnitude is also sometimes written " $V_R(\alpha)$ " to show that it is a function of solar phase angle ( $\alpha$ ). Another standard nomenclature, used in the standard phase curve model, is that reduced magnitude is called " $H(\alpha)$ ", and the special value  $H(0)$  when solar phase angle is zero is called simply " $H$ " (the absolute magnitude of the asteroid).

For main belt asteroids, these distances change slowly, so that they can be treated as if they are invariant over a few nights. However, over the couple-month time duration of a phase curve project, they probably change noticeably. Plotting  $V_R$  vs.  $\alpha$  shows how the brightness is changing solely due to the phase effect. That is the essence of the phase-curve project.

Add to above the fact that the asteroid is also rotating, meaning that its brightness changes on a time scale of a few hours as it spins. If the asteroid's brightness is measured many times as the apparition progresses, its brightness changes due to all three effects.

### 2.3 Standard Phase Curve Model

As described in Bowell et al. (1989), the two-parameter " $H-G$ " model uses the following equation to describe the brightness (in reduced magnitude) of an asteroid as the solar phase angle changes:

$$H(\alpha) = H - 2.5\log[(1-G)\Phi_1(\alpha) + G\Phi_2(\alpha)] \quad \text{Eq. 2}$$

Where:

H The “reduced magnitude” at zero phase angle. It is sometimes written  $H_0$  to explicitly denote that it is  $\alpha = 0^\circ$ , or  $H(1,1,0)$  to denote that it is based on sun earth distances being 1 AU and  $\alpha = 0^\circ$ . These all mean the same thing.

$\phi_1, \phi_2$  These are functions that describe the single and multiple scattering of the asteroid’s surface. These functions are given in Appendix A, but if you are studying a main-belt asteroid and using *MPO Canopus*, you don’t need to deal with these equations because *MPO Canopus’* H/G Calculator handles them.

G The “slope parameter” that describes the shape of the phase curve. The fundamental goal of the phase curve project is to determine  $G$  by plotting the data points (H vs.  $\alpha$ ) and finding the value of  $G$  that is the best fit to the data.

### 2.4 What Does a “Good” Phase Curve Look Like?

I searched the NASA ADS abstract service for “asteroid phase curves” and spent a few days at the local university library skimming through *Icarus* and the *Astronomical Journal* to find several papers describing asteroid phase curves that were developed by professional astronomers. Figure 2 is an example from Harris et al. (1989) of a “good” phase curve. Note several features:

- The phase coverage is broad, from very low (near-zero) phase to phases greater than  $20^\circ$ .
- The phase coverage is dense, giving confidence that the data have captured the essential shape of the curve.
- The photometric accuracy is excellent. In this particular case, the error bars are barely larger than the plotted symbols.

That is a challenging quality level for which to strive, but one that seems to be important. There are strong indications that the phase curve, specifically the slope parameter ( $G$ ), is telling us something about the albedo and surface texture of the asteroid. However,  $G$  only changes by a few tenths, so the photometry and data analysis need to be quite accurate if the phase curve is to be reliable at this level. A few years ago, in my first attempt at a phase curve, the curve looked nice, but a combination of insufficient photometric accuracy (“only” about  $\pm 0.05$  mag) and

insufficient phase coverage ( $\alpha \approx 2\text{-}12^\circ$ ) left me unable to distinguish between two competing values ( $G = 0.15$  vs.  $G = 0.25$ ) that had been previously published.

It is also worthwhile noting that the data in Figure 21 deviates somewhat from the  $H$ - $G$  model curve, particularly in that the data show a larger and sharper opposition effect than does the  $H$ - $G$  model. This is not a unique example. Belskaya and Schevchenko (2000) show several phase curves where the data deviates from the  $H$ - $G$  model. Getting more and better data provides a better understanding of the opposition effect and of asteroid surface properties.

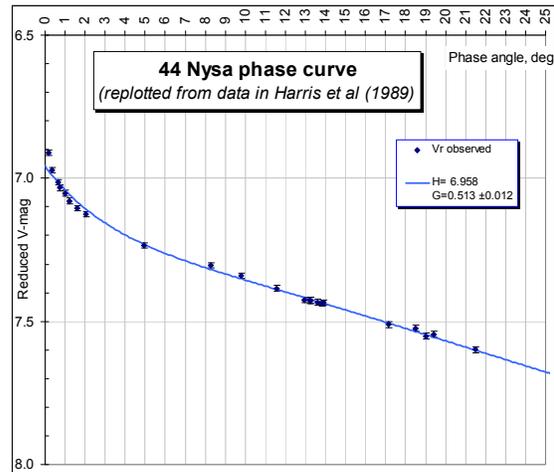


Figure 2: Example of a “professional” phase curve. Replotted from data in Harris et al. (1989).

### 2.5 What is the “Typical” Expected Range of G for Different Asteroid Types?

Both Harris (1989) and Lagerkvist and Magnusson (1990) determined slope parameters  $G$  for a goodly number of asteroids and correlated  $G$  to the asteroid taxonomic type. They found values ranging from  $G \approx 0.04 \pm 0.06$  for low albedo (e.g. C-type) asteroids to  $G \approx 0.45 \pm 0.04$  for high-albedo (e.g. E-type) asteroids. They showed a definite correlation that low-albedo objects had low  $G$  values and high-albedo objects had high  $G$  values.

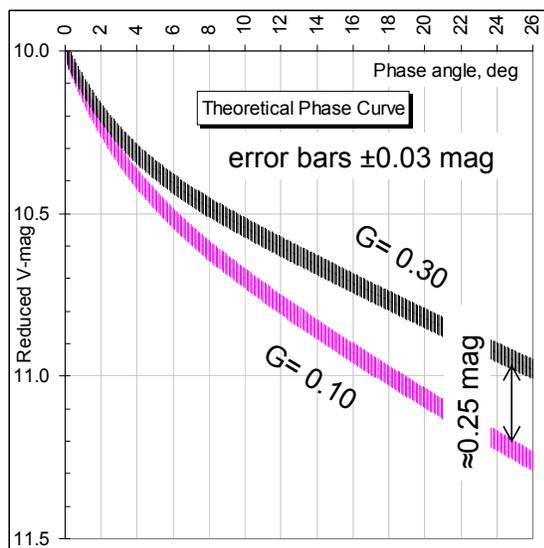
These results mean that we don’t expect to see a very wide range of  $G$  values. In order to provide a meaningful  $G$  value that can distinguish between different asteroid types or other asteroid properties, the accuracy of our determination must be pretty good, say within a formal error of  $\pm 0.05$  or better.

Before going on, it’s important to say that finding a value for  $G$  is not sufficient on its own to determine taxonomic class. Without other supporting evidence, the best being spectral data, the most one can say is that value of  $G$  that is found is consistent

with a particular taxonomic class or, more generally, objects of low or high albedo.

## 2.6 Can My Phase Curve Distinguish Between Different Values of G?

The nature of the need for quite good accuracy can be illustrated by Figure 3. Here, two phase curves are plotted, one with  $G = 0.1$  and the other with  $G = 0.3$ . How difficult is it to distinguish between these curves? If we have noise-free data ranging from  $\alpha \approx 0^\circ$  to  $\alpha \approx 30^\circ$ , it's easy to tell the two curves apart. For example, the brightness difference between the curves at  $\alpha \approx 25^\circ$  is about 0.25 mag. If we only had data going to, say,  $\alpha \approx 10^\circ$ , it would be harder to tell the two curves apart, since at  $\alpha \approx 10^\circ$  the two curves differ by only about 0.15 mag. So you can see that it's important to follow the asteroid out to fairly large solar phase angles.

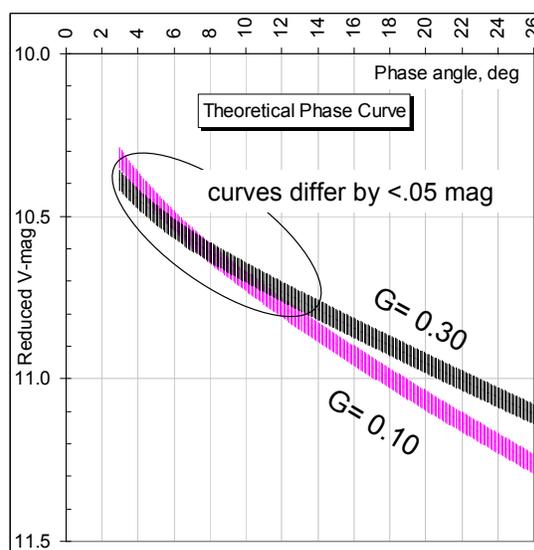


**Figure 3: Theoretical phase curves, illustrating shape effect of “G” value and importance of reaching large solar phase angle.**

Now suppose that we missed the nights near minimum solar phase angle and started observing the asteroid at  $\alpha \approx 3^\circ$ . We don't know what its brightness was at  $\alpha \approx 0^\circ$ , so can we tell the difference between the  $G = 0.1$  curve and the  $G = 0.3$  curve just by their shapes and slopes? That is the situation shown in Figure 4. The curves are virtually identical, differing by only 0.05 mag over the range  $\alpha = 3$ - $12^\circ$ . Without those critical “near-zero solar phase” data points, it is very difficult to distinguish between different values of  $G$ . The  $H/G$  calculator utility in MPO *Canopus* will give you an answer, but the uncertainty will be

large enough that the result can't be used to reliably distinguish between different taxonomic classes.

It is important to get those critical “minimum phase angle” data points, because the  $G$  calculated from observations only at  $\alpha > 5^\circ$  can be misleading. Hasegawa et al. (2009) noted this problem in their study of 4 Vesta. Using data from  $\alpha \approx 1.5^\circ$  to  $24^\circ$ , the inferred value was  $G = 0.32 \pm 0.04$ , but when the additional data points down to  $\alpha = 0.12^\circ$  were included, they determined the (presumably “correct”) significantly smaller value  $G = 0.23 \pm 0.02$ .



**Figure 4: Reaching very low solar phase angle ( $\alpha \approx 1^\circ$ ) is important, to distinguish G values and observe the “opposition effect”.**

## 2.7 Minimum, Maximum, or Average Magnitude?

Asteroids change brightness as they rotate, so the long-term change in brightness caused by the changing solar phase angle is superimposed on a (usually) much more rapid cyclic variation in brightness due to the rotational lightcurve.

Should the phase curve use the average, maximum, or minimum magnitude of the asteroid's rotational lightcurve? There seems to be no single answer to that question in the literature. The initial formal recommendation of the  $H-G$  system, reported in Marsden (1985), is silent on the subject. The seminal description of the  $H-G$  system (Bowell et al., 1989), explicitly relates it to the mean (average) V-band magnitude. Yet, some other examples of phase curves reported in the literature are based on maximum light, such as Harris (1989).

There are theoretical and mathematical reasons to expect that the slope parameter of the phase curve

may be slightly different depending on which points of the lightcurve are used. For example, Helfenstein and Veverka (1989) report calculations for the idealized cases of spheres and ellipsoids whose surfaces follow a standard reflectivity law. The phase curves for maximum, mean, and minimum magnitude have slightly different slope parameters. The difference isn't great, but it neither is it trivial. The difference between their "max light" vs. "min light" phase curves amounts to about 0.3 mag at a solar phase angle of 30° (after the rotational lightcurve effects are removed).

I also note that there is a subtle risk in the terminology regarding "mean" or "average" magnitude. First, the average magnitude is not the same as average brightness (or average light) – that logarithmic function in the definition of magnitude is important! Most published phase curves that use the mean magnitude state explicitly that it is 'mean magnitude', not 'mean light' that is being calculated. If the rotational lightcurve is complex, then the determination of the mean (average) magnitude may not be obvious. The formal definition of "mean magnitude" is that the rotational lightcurve encloses an equal area above and below the mean-magnitude line (Gehrels, 1956). That is, the mean magnitude is not necessarily the midpoint between the brightest peak and the faintest valley of the rotational lightcurve.

Considering that defining and identifying the "maximum" and "minimum" brightness of a real asteroid lightcurve is pretty unambiguous, and figuring that the possible difference between the phase functions based on "max" versus "min" brightness might be interesting, I've chosen to determine both the "max" and "min" brightness phase curves for my targets. As it worked out, in the case of 1130 Skuld there was almost no difference in  $G$  as determined by max vs. min brightness. For 535 Montague, the "max brightness" phase curve appears to have a significantly different slope than does the "min brightness" phase curve.

### 3. Phase Curve Error Analysis

If you are mathematically inclined, you can understand the importance of these features by reference to the linear error analysis given in *Asteroids II*.

Suppose that we have measurements of  $H(\alpha)$  at many different solar phase angles. Call the phase angles where we have measurements  $\alpha_i$ , with  $i = 1, 2, 3, \dots, N$ .

The phase curve is just the graph of  $H(\alpha)$  vs.  $\alpha$  and so the graph will have  $N$  data points. We will fit the data points to the curve of Eq.2 to get the best-fit value of  $G$ . The H/G Calculator utility in *MPO*

*Canopus* is a particularly convenient to do this curve fitting.

If we make a simplifying assumption that the photometric accuracy is the same for all data points, then the expected error in the estimated (best-fit) value of  $G$  is:

$$\Delta G \approx \sigma \sqrt{\frac{N}{N-2}} \cdot \frac{1}{\sqrt{\sum_i \alpha_i^2 - N\alpha_0^2}} \cdot \frac{1}{(0.0673 - 0.1132G + 0.0615G^2)} \quad \text{Eq. 3}$$

In this equation

$\sigma$  is the RMS photometric error in magnitudes (approximately  $1/\text{SNR}$ ). Obviously, smaller  $\sigma$  is better, achieved by higher SNR in the photometry.

$N$  is the number of solar phase angles at which we have data points. The square root term with "N-2" in the denominator demands that we make  $N$  (the number of data points) greater than 2. If  $N = 2$ , then we have no knowledge about the uncertainty in the slope parameter,  $G$ . This isn't so mysterious if you remember that the phase curve has 2 parameters ( $H$  and  $G$ ), hence with any two data points we can find a curve that is a perfect fit to the data, but there is no information about the probable error in the fit that might be caused by noise in the data. As long as  $N \geq 5$ , the square-root term is not much larger than 1.

$\alpha_i$  are the solar phase angles at which we have data (with  $i = 1, 2, 3, \dots, N$ ), and

$\alpha_0$  is the average phase angle,

$$\alpha_0 = (1/N) \sum_i \alpha_i$$

The term involving the sum of the squares of the phase angles at which data points are given ( $\sum \alpha_i^2$ ), and the average phase angle  $\alpha_0$  is a very important contributor to the error in  $G$ . A couple of observations about this term help to understand its significance. First, if we have data at only a single phase angle, then  $\alpha_i = \alpha_0$ , and, therefore, the denominator of this term equals 0. In this case, the uncertainty in  $G$  becomes indeterminate. That is, if we have only a single data point, then we know nothing about the shape of the curve and, hence, nothing about  $G$ . Second, the greater the range of solar phase angles covered by the data set, then the larger this denominator becomes and so the more accurately we'll know  $G$ .

For example, suppose that we have data at phase angles 0, 2, 4, and 6 degrees. The number of data points is  $N = 4$ , the sum-square is  $\sum \alpha_i^2 = 561$ , and the mean phase is  $\alpha_0 = 3$ . The resulting value of this term is  $1/[\sqrt{\dots}] = 0.22$ . Now, suppose we get two more data points, at  $\alpha = 8$  and 10 degrees phase angle. Then we have  $N = 6$  data points, the sum-square

phase is  $\sum \alpha_i^2 = 220$ , the average phase angle is now  $\alpha_0 = 5$ , and  $1/[\sqrt{\dots}] = 0.12$ . So, getting those two more data points at larger phase angles provided a nearly two-fold improvement in the accuracy of our determination of  $G$ .

There is a tricky point here: it isn't just the fact that we had additional data points, but also that they were spread over a wider range of phase angles. The dispersion is described by the term  $[\sum \alpha_i^2 - N\alpha_0^2]^{1/2}$ , and it is the dispersion that is important. Having data points spread over a wide range of solar phase angles makes this term larger and so improves our estimate of  $G$ . This is reasonable, since in order to determine  $G$  we are looking at the shape of the magnitude vs. phase graph, but at the curvature in the graph, and this requires both many data points (to get the benefit of averaging) and a wide spread in the phase angle (to better display the curvature).

Now, let's look at "typical" values of these terms.

$\sigma$  is the photometric accuracy. If we have SNR = 100 on both target and comp star, then we expect to be able to get  $\sigma \approx 0.014$  magnitude (assuming no systematic errors...)

The term  $\sqrt{\frac{N}{N-2}}$  ranges between about  $\approx 1.73$  to 1.12 (for  $N=3$  to 10 data points).

The term  $\frac{1}{\sqrt{\sum \alpha_i^2 - N\alpha_0^2}}$  ranged from 0.22 to 0.12 in the example above. If we had equally-spaced data points at 0, 2, 4 ... 20 degrees, then this term would be 0.048, and it would get as small as 0.030 if we could go all the way to phase angle 28 degrees. The term involving  $G$ ,

$$\frac{1}{(0.673 - .1132G + .0615G^2)}$$

varies between 14.9 (for  $G = 0$ ) to 42.3 (for  $G = 0.55$ ), over the reasonable range of expected values. Of course, you can't do anything to affect this factor.

Only  $\sigma$ ,  $N$ , and the range of  $\alpha$  values are under the observer's control, so by far the most significant things you can do to improve the accuracy of your estimate of  $G$  are:

- Get high signal-to-noise ratio, and properly calibrate your fields, to achieve good photometric accuracy. Shoot for  $\sigma \leq 0.02$  mag or better.
- Get data at as many phase angles as possible, over as wide a range as possible. Take advantage of opportunities when an asteroid is at very low phase angle ( $\alpha < 1^\circ$ ), and follow the asteroid as

long as practical to get up to high solar phase angle ( $\alpha > 15^\circ$ ).

If our goal is to get  $\Delta G \approx 0.05$  or better, then we need SNR > 100 and the dispersion term < 0.14. A little playing around with the numbers demonstrates that this implies a requirement for at least 5 data points, spread out between  $\alpha \approx 0^\circ$  to at least  $\alpha \approx 15^\circ$ , and photometric accuracy better than  $\pm 0.03$  mag.

#### 4. Determining the Phase Curve

The procedure I use for measuring the phase curve has the following steps.

##### Observations

- Planning and scheduling
- CCD photometry of the asteroid lightcurve
- Calibration of each night's comp stars at my observatory, or at a remote internet observatory

##### Data reduction

- CCD image reductions
- Differential photometry: Lightcurve reduction and Fourier curve model

##### Data Analysis

- Put differential lightcurves onto a single baseline and determine asteroid color index
- Download Asteroid dynamical parameters: solar phase angle, Earth distance, and Sun distance
- Translate lightcurves from V-mag to reduced magnitude  $V_R$
- Determining brightness at selected rotational phase points (max and min brightness) by using "actual" data points or "extrapolation" to min/max brightness using Fourier fit curve
- Plot the phase curve ( $V_R$  vs.  $\alpha$ )

This is a project for which you need to enjoy the time at your desk and computer as much as you enjoy your time in the observatory under the stars. Convince yourself that the data reduction and data analysis is really interesting! Constructing a phase curve will stretch your CCD photometry skill compared to differential photometry for rotational lightcurves. The project of determining an asteroid's phase curve is challenging, because it combines four photometric objectives: determination of the asteroid's rotational lightcurve, following the asteroid over a wide range of solar phase angles, doing either FOV bridging or "all-sky" photometry to link the asteroid's brightness from night to night, and achieving quite high photometric accuracy.

## 4.1 Observations

### 4.1.1. Planning and scheduling

There are two big differences between planning a “phase curve” project and planning a “lightcurve project”. The first difference is the number of nights involved. Whereas you can often make a fine lightcurve determination with a few nights’ data, the phase curve project requires that you follow the asteroid over a wide range of solar phase angles, which usually means devoting about a dozen nights over a couple of months. The second difference is the importance of getting lightcurve data on the nights of minimum phase angle. For a “lightcurve” project, it isn’t particularly important which night(s) you observe, but if you are going to create a good phase curve that captures the “opposition effect”, it is critical to get data on the nights near  $\alpha \approx 0^\circ \pm 2^\circ$ .

The most attractive targets are those few asteroids that reach a very low solar phase angle ( $\alpha \leq 1^\circ$ ) each year, since they offer the opportunity to measure the “opposition effect” that is a distinguishing feature of the phase curve. Each issue of the *Minor Planet Bulletin* contains a list of “low phase angle” candidates. You’ll need to sort through that list to identify targets that are appropriate for your location and equipment. I look for objects whose maximum brightness is at least 14 mag (for good SNR) and whose declination is higher than about 10 degrees (because from my 33 degree latitude and not-so-good southern horizon I don’t have many hours per night available for targets at low declinations). Because of the orientation of the ecliptic (where most of the main-belt asteroids are concentrated), this declination filter means that autumn and winter are the phase-curve season at my observatory.

I begin following my target a few nights before minimum phase angle. There is value in getting good data both “pre-opposition” and “post-opposition”, but there is a risk that the “pre-opposition” data may be orphaned if it’s cloudy on the few nights of minimum phase angle. From my backyard observatory on the coastal plain of southern California, it’s not unusual to lose a string of nights to clouds. I haven’t done a statistical analysis, but there does seem to be a surprising correlation between an asteroid reaching low phase angle and clouds settling over my neighborhood.

### 4.1.2. CCD photometry of the Asteroid Lightcurve

The project begins with fairly routine imaging for asteroid lightcurve observations. I usually use a

two-color (photometric filters) imaging sequence (R-R-V-V-... etc) near opposition, when the asteroid is brightest and I can get a good SNR with modest exposure duration. Far from opposition I turn to “clear” (unfiltered) images to maintain high SNR as the asteroid fades. If I’m imaging in the “clear” filter, I still scatter a few V- and R-band images into the sequence to help link to nights where the images are primarily V- or R-band.

Making several nights with two-color image sets enables me to determine the asteroid’s color (V-R). Color index is useful for two reasons. First, it is necessary to know the color index during data analysis in order to transform the C-band images to V-magnitudes. Second, it confirms that the asteroid’s color does not change as it rotates. (Yes, I know that the conventional wisdom, and all published data says that the full-disk color is essentially invariant at the  $\pm 0.05$  mag level, but who knows? There might be a surprise waiting to be discovered!)

I noted in the error analysis above that getting a good phase curve demands quite good photometric accuracy. Harris and Young (1989) is a strong example of this. They strove for photometric accuracy of “a few thousandths of a magnitude”. They also advised staying on the instrumental (b, v, r) system to maintain this level of differential photometric accuracy because once transformations are done (to get onto the standard B, V, R system), it is very difficult to achieve much better than 0.02 mag accuracy. Unfortunately, I wasn’t able to follow this advice fully (see the next section), and their guesstimate of 0.02-mag accuracy is about what I got overall.

### 4.1.3. Calibration of Each Night’s Comp Stars

Because the essence of the phase curve project is the determination of how the asteroid’s brightness changes, it is necessary to know the comp star brightness. There are two approaches that can be used: “linking” of comp stars from night to night (on the instrumental system), or “all sky photometry” to determine the B-V-R magnitudes of comp stars from all nights.

From a good observing site, the easiest way to do this is to link each nights comp stars to the preceding night. The idea is to take a short break near culmination, move the scope to the FOV of a “reference night”, take a few images (in all colors, if you’re doing two- or three-color series), and then return to the current night’s asteroid lightcurve series. Having images of “tonight’s comp stars” and the “reference night’s” comp stars, both at the same (low) air mass enables you to link the comp star brightness for all nights, relative to the reference night. It’s usually most convenient to make the first night of the project

the “reference night”, but it doesn’t really matter which night is chosen as the reference night.

For example, suppose that on night 1 we used star X as the comp star and on night 2 star Y was the comp star (and that the asteroid has moved fast enough that we can’t fit both into a single FOV). On night 2, near culmination, we took a few images of the FOV from night 1 (that contains star X). Make a table such as shown in Table 1.

Assume that we have a reasonable (catalog) value of the V-mag of star X = 12.65. This doesn’t have to be particularly precise value because we’re going to use it as an “anchor point” – all nights and all comp stars will refer back to it. If its assigned magnitude is off a bit, it will only affect the resulting calculation of *H*; it won’t affect the value of *G*, which is a function of the shape of the phase curve. Further, we’ll stay on the instrumental system and not try to bring each comp star into the standard BVR photometric system.

Now on night 2, we measured the instrumental magnitude (IM) of both star X and star Y at essentially the same air mass, and at nearly the same time, so that we can (hopefully) assume that the atmospheric conditions are the same on all images.

Since we are assuming that our sensor is linear, we can write:

$$[\text{mag of star Y}] - [\text{mag of star X}] = [Y_2 - X_2]$$

The V-mag of star Y is then calculated by:

$$[\text{mag of star Y}] = [\text{mag of star X}] + [Y_2 - X_2]$$

where

- Y2 the instrumental magnitude of star Y, as measured on night 2
- X2 the instrumental magnitude of star X, as measured on night 2
- MagX the “assigned V-mag” that we are using to anchor the calculations

By doing this for each night, and staying on our instrumental system, our determination of the phase curve won’t be infected by any problems doing transforms.

Any error in the assignment of a V-mag to the anchor star (star X) will result in a comparable error in the calculated asteroid absolute magnitude (*H*). In

this example, if the “true” magnitude of star X were 12.50 instead of 12.65, then our calculated value of *H* would be high (faint) by 0.15 mag. However, our determination of the phase curve parameter (*G*) would not be affected at all. Remember that *G* describes the shape of the curve and not its absolute brightness.

The good news about this is that you can stay on your instrumental system and that modest error in determining (or estimating) the V-mag of your reference night’s comp stars won’t upset the shape of the phase curve. The drawbacks are: (1) this method requires that each night that you do “linking” must be clear and stable so that changing sky conditions during the interval when you’re doing the linking don’t confuse the results, and (2) near opposition, the lowest-air-mass will occur at around midnight. The thing about that is that I have to be at work early the next morning. My preferred operating mode is to set the observatory to take a series of images of the target field all night – while I’m asleep – which means that doing the “linking” isn’t convenient.

Unfortunately, really clear and stable (“sort of photometric”) nights are infrequent at my backyard observatory, so I can’t be confident that this “linking” procedure will be satisfactory. It isn’t unusual to see atmospheric transparency change by a tenth of a magnitude or so over less than an hour on a “typical” night. So, I have used two different approaches to linking the comp stars by “all sky” photometry. When a nice clear and stable (“sort of photometric”) night arrives, I devote that night to calibrating all comp stars from all of the nights used for asteroid monitoring. This entails imaging of:

- a Landolt field near the horizon (air mass ≈ 2)
- a Landolt field near culmination
- each FOV used for asteroid photometry
- one or more additional Landolt fields

The first two Landolt fields enable me to determine the atmospheric extinction coefficient (using the “Hardie method”). Capturing one or two additional Landolt fields after the FOV imaging provides confirmation that the atmosphere was stable while the imaging sessions were being linked. The set of Landolt fields also enables determination of the system’s transforms. The imaging of the asteroid fields should be timed to put the FOV images as high in the sky as

Night	Star	IM (near culmination, at air mass = 1+ε)	Assigned V- mag	Calculation of “Linked” V-magnitude
1	X	-10.50	12.65	
2	Y	-10.650		$Y_2 = X_1 + [Y_2 - X_2]$
	X	-10.40		$Y_2 = 12.65 + [(-10.65) - (-10.40)]$

practical (i.e. at low air mass), to minimize the effect of any errors arising from the atmospheric extinction.

This can entail a fairly long night of imaging. For example, I usually do a R-V-V-R image sequence of each field of view, using 2 minute exposure in R and 3 minute exposure in V. Getting two Landolt fields to start, then 8 target FOVs, and wrapping up with two more Landolt fields adds up to 2 hours of “shutter open” time. Adding in the time required for centering on the target field, focus checks, image downloading, and occasional autoguiding errors that necessitate repeat of a field, this can add up to 3 to 4 hours of telescope time. (Those of you with fully-automated systems with high-quality mounts will have better efficiency; my system isn’t so fully automated).

This is a nice way to spend time under the stars, but I’ve found that even on nights that appear to be stable at my backyard observatory, there is a risk that the data ultimately shows that atmosphere conditions changed in the course of the observing session. That adds error and uncertainty to the resulting comp star calibrations, and sometimes it is painfully obvious that the data is wholly unreliable.

A modern alternative to this is use of a remote internet-accessible observatory. There are increasing numbers of these facilities located in high-altitude sites where very good conditions are routine. Most of them have telescopes that are outfitted with photometric filters, and hence are perfect solutions to the calibration of comp stars. I have used the Tzec Maun Foundation’s “Big Mak” telescope for this purpose, and it has been a delight! From high in the New Mexico mountains, the sky is frequently clear, dark, and stable (far better and more reliable than my home location). The field of the “Big Mak” (a 14-inch f/3.8 Maksutov-Newtonian with an ST-10 XME and photometric BVRI filters) is a good match to my home setup of 1 arc-sec pixels and 26 X 38 arc-min FOV. Its fast optics give good SNR with 1 to 2 minute exposures. The Tzec Maun observatory provides master flats, dark, and bias frames so the observer need not use telescope time to gather those.

This has turned out to be a fine solution to the challenge of calibrating comp stars.

The only drawback that I’ve found is that the overall efficiency (“shutter open” time vs. total clock time) is worse than at my home observatory. In a two hour session I typically get 1 hour of shutter-open time. Some of that efficiency loss is due to my own weak (but slowly-improving) skill at manipulating the remote telescope interface software.

## 4.2 Data Reduction

### 4.2.1. CCD image reductions

Regardless of where and how the images were gathered, they must be reduced in the normal way – bias, dark, and flat-field correction. Since we’re striving for the best possible photometry, don’t skimp on this step!

### 4.2.2. Differential Photometry for Lightcurve reduction and Fourier Curve model

The asteroid’s lightcurve is determined by differential photometry in the usual way using *MPO Canopus*. *Canopus*’ v.10’s “comp star selector” is a real aid because it helps record the catalog photometry of the comp stars (which is good, but not perfect) in the database

Merge all nights and determine a good lightcurve and rotation period for the asteroid in the usual way. Then, do two or three things:

- 1) Examine the lightcurve carefully to see if there are any systematic changes in the shape of the lightcurve as the solar phase curve changes. If there are, then divide the sessions into two groups (or more, if necessary) – one for “low phase angle” and another for “high phase angle” sessions. Asteroid 535 Montague is an example of an object whose lightcurve changes noticeably as the solar phase angle grows. Presumably this is a manifestation of shadowing effects from topography on the asteroid.
- 2) Run a Fourier fit of the lightcurve, at the determined lightcurve period, and record the Fourier coefficients (do this separately for the “low phase angle” and “high phase angle” groups if you have a case like 535 Montague, where the lightcurve changes noticeably).
- 3) Export the entire *MPO Canopus* database to a text file, from which it can be imported to Excel for further analysis.

### 4.2.3. Export MPO Canopus Observations File to Excel

With the lightcurve analysis done, I export all of the *MPO Canopus* observations to a text file, which can be opened in Excel for further analysis. The analysis includes:

- Replacing *MPO Canopus*’ estimates of comp-star magnitudes with “calibrated” magnitudes of the comp stars. The *MPO Canopus* star catalog is

pretty good, and the estimated comp star magnitudes are pretty good, but using “calibrated” magnitudes as described above improves the overall accuracy of V and R (and C) to about  $\pm 0.02$  mag (full range)

- Determination of the asteroid’s observed V-magnitude (at each data point for which a v-band image was taken), and R-magnitude (for each point at which an r-band image was taken). In the case of C-band images, they are converted to V-band using the procedure described above.
- Determination of the asteroid’s color index. For the projects reported here I used (V-R), although the same methods will work with any other color index.
- Conversion of C and R magnitudes to “V”, to create a master table of JD vs.  $V_{\text{observed}}$ .

The net result of this is a table of JD,  $V_{\text{observed}}$  that includes all data observed points.

The information on the asteroid’s orbital parameters from JPL Horizons is interpolated into this table, so that I can calculate the Sun distance (R), Earth distance (D), solar phase angle ( $\alpha$ ) at each observation time.

## 5. Getting the Asteroid’s Orbital Parameters

The data reduction demands that we translate the lightcurves into  $V_R$  using Eq.1. In order to do that we need to know the asteroid’s Sun distance (R) and Earth distance (D). Most planetarium programs will give you this information, but it is often more accurate and more convenient to download the data from the “Horizons” system of the NASA Jet Propulsion Laboratory (available on the internet at <http://ssd.jpl.nasa.gov/?horizons>). There, you can set your location, your target, a time interval, and receive a table of all the parameters you request for all the time ticks that you request. This table is easily imported into Excel to support your lightcurve analysis.

I downloaded the orbital information at 4 hour increments, selecting a time near the beginning of the night as a “reference time”. I then used a linear interpolation ( $d\alpha/dt$  and  $d[5\log(RD)]/dt$ ) to determine the parameters at the time of each observation.

## 6. Example: 1130 Skuld

This asteroid turned out to be a nice project. Its lightcurve is shown in Figure 5. Skuld’s lightcurve didn’t change noticeably during the time that I observed it (from  $\alpha = 0.3^\circ$  to  $\alpha = 17.6^\circ$ ), and its color

was also quite stable over the entire observed apparition. The period ( $P = 4.807$  hr) is short, so that I could get at least one maximum and one minimum each night, and for many all-night runs my lightcurve captured both the primary and secondary maxima and primary and secondary minima. The primary and secondary minima are virtually identical magnitude; the primary and secondary minima differ by only a small amount. This meant that I could get at least one, and sometimes two, “maxima” data points, and one or two “minima” data points each night that I observed the object.

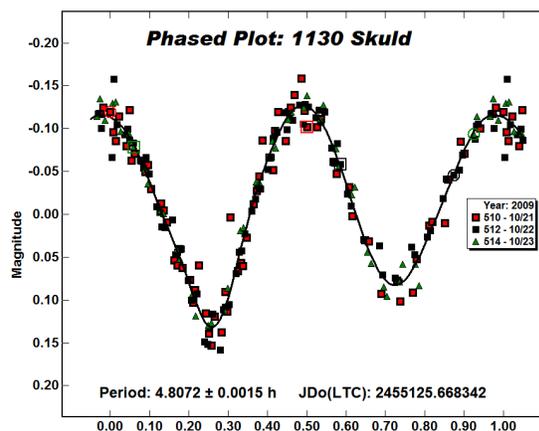


Figure 5: Lightcurve of 1130 Skuld.

The data analysis procedure was relatively straightforward. Each night’s lightcurve data was analyzed in *MPO Canopus* using several comp stars in the usual way. Then I used one particularly clear and stable night to do “all sky” photometry to determine the standard V and R band magnitudes of the comp stars for each lightcurve night. That enabled me to put each lightcurve on a common V-mag baseline. Overall, the photometric accuracy was about 0.04 mag, considering the inherent SNR of asteroid and star images, and the consistency across comp stars and nights.

For each night’s lightcurve, I could select the lightcurve “max brightness” data point, note the time, determined the V-mag, translate that into reduced magnitude  $V_R$ , and look up the solar phase angle at the time of “max brightness”. An example of all this is shown in Figure 6.

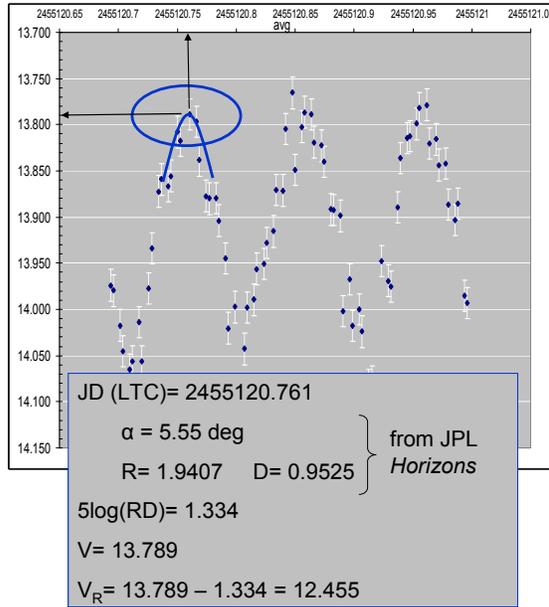


Figure 6: Translating observed V-mag to reduced magnitude.

I wanted to try to “average out” the inevitable photometric noise (which was about  $\pm 0.02$  mag), so I made a little quadratic fit to the half-dozen or so data points nearest the max/min of the lightcurve in order to get an “averaged” estimate of the lightcurve max and min. This worked nicely. In almost all cases the “averaged” estimate didn’t differ from the single brightest (or faintest) data point by more than a few hundredths of a magnitude, so the “averaging” procedure may not have been necessary at all since with or without it gave virtually identical phase curves.

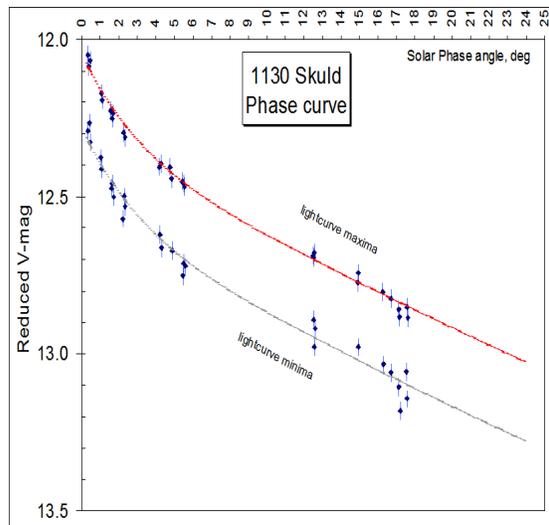


Figure 7: Phase curve of 1130 Skuld, showing “max” and “min” brightness curves.

By repeating that process for each “max” and “min” on each night of lightcurve data, I plotted the phase curve, shown in Figure 7 (from Buchheim, 2010).

Note that the slope parameter  $G$  derived from the “max” light is essentially the same as that derived from the “min” light. This  $G = 0.25$  is reasonable for Skuld’s reported classification as an S-class asteroid (NASA, 2008).

## 7. Example: 535 Montague

This project proved to be a bit trickier, but it forced me to learn about some considerations that weren’t needed in the case of 1130 Skuld. The period was long, so any single night might not have captured a lightcurve max or min, yet I still wanted to take advantage of the data gathered on every night. The shape of the lightcurve changed as the solar phase angle increased, so I had to account for that. The southern California weather didn’t provide any good (clear and stable) nights for linking the comp stars from the lightcurve nights, so I relied on all-sky photometry done at a remote observatory (the Tzec Maun Foundation) to calibrate the comp stars.

### 7.1 Using the Fourier fit to Interpolate and Extrapolate the Rotational Lightcurve

535 Montague has a rotational lightcurve period of  $P \approx 10.248$  h, and its primary and secondary maxima are significantly different in brightness, as are its primary and secondary minima. This means that on many nights, an all-night lightcurve might be missing either the maximum brightness or the minimum brightness, or both. How can we take good advantage of a night that gives only a “partial” lightcurve so that it contributes useful data points to the phase curve?

Harris et al. (1989) dealt with this problem, and I followed their procedure. The idea is to use the Fourier fit model of the lightcurve as a way of extrapolating to data points that weren’t actually measured on a given night. The concept goes like this: using several nights’ differential photometry data, construct a complete rotational lightcurve, with full coverage of the rotation. This is the standard “lightcurve” project and *MPO Canopus* makes it relatively easy. Once the full lightcurve and period are determined, create a Fourier fit to the lightcurve. *MPO Canopus* does this also, providing the Fourier coefficients for the model lightcurve. This Fourier model should use sufficient “orders” to capture all of the essential features of the lightcurve. For the fairly complicated shape of the lightcurve of 535 Montague, I found that 6 orders

were barely sufficient while using 8 orders gave a nice fit to the measured lightcurve.

I exported the Fourier coefficients into an Excel file (“fourier.xls”). The Fourier model is a nice “smoothed and averaged” mathematical representation of the lightcurve, given by:

$$V_R(t) = H(\alpha) + \sum_{n=1}^N a_n \sin\left(\frac{2\pi nt}{P}\right) + \sum_{n=1}^N b_n \cos\left(\frac{2\pi nt}{P}\right) \quad \text{Eq. 4}$$

Where

$V_{R(t)}$  V-mag reduced magnitude of the asteroid at time  $t$  and phase angle  $\alpha$ .

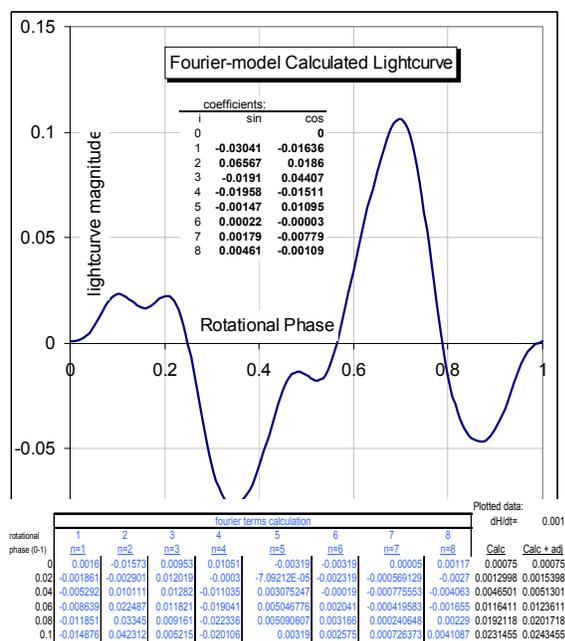
$t$  Time (hours or days)

$P$  Period (same units as  $t$ )

$a_n, b_n$  Fourier coefficients from Canopus

$H(\alpha)$  Average V-mag of the asteroid at phase angle  $\alpha$ .

This may look a bit complicated but it isn’t too hard to program an Excel spreadsheet that will calculate  $V_{R(t)}$  and plot a graph (see Figure 8).



**Figure 8: Excel spreadsheet can calculate Fourier-model curves.**

How do you find  $H(\alpha)$ , which is, after all, the value being sought? In the same Excel spreadsheet I make two columns containing the table of measurements for the night: JD and measured data (in V reduced magnitudes). The spreadsheet calculates the “Fourier fit” at each data point. This allows me to do two things. I plot the data (D) on the same graph as the Fourier curve [ $V_{R(t)}$ ], and manually iterate the

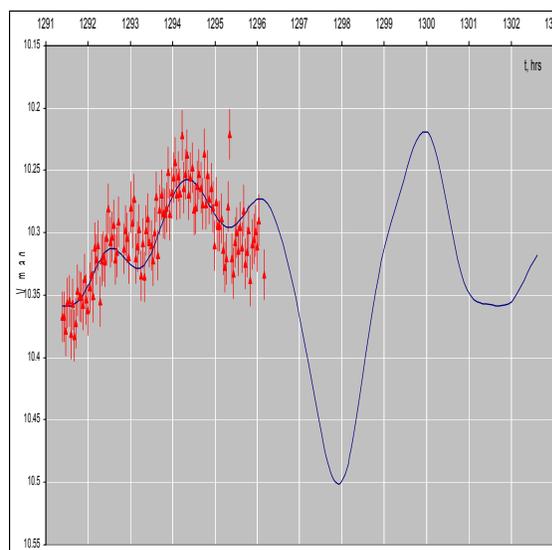
value of  $H(\alpha)$  until I find the value of  $H(\alpha)$  that gives the best fit. “Best fit” is judged by minimizing the squared error between the Fourier fit and the actual data, summed over all of the data points on a single night. The equation is:

$$\epsilon^2 = \sum_j [D(t_j) - V(t_j)]^2$$

is the squared error between data (D) and model (V). The summation extends over all data points (where the  $j^{\text{th}}$  data point was taken at time  $t_j$ ).

This Fourier fit has two uses. First, in a case like 1130 Skuld (where each night offers a max and min brightness), the Fourier fit is a convenient way to “average” the lightcurve shape, to smooth out photometric errors.

Second, the Fourier fit is used to deal with nights (such as 535 Montague) where neither the max nor the min brightness is available. The Fourier fit, matched to the night’s data, can be *extrapolated* to the next convenient lightcurve max or min. The concept is illustrated in Figure 9.



**Figure 9: Using the Fourier model to extrapolate to unobservable lightcurve maximum or minimum.**

The lightcurve extrapolated to the nearest maximum/minimum provides extremum magnitudes to use as data points on the phase curve. That way, each night for which you have data makes a contribution to the phase curve. Even if the asteroid would have set, or the Sun would have risen by the time of lightcurve maximum, the Fourier extrapolation tells you what the asteroid’s magnitude *would have been* if you could have observed its maximum. Look up the

solar phase angle at this time of maximum, and add the data point to your phase curve plot.

There is one important caveat to the extrapolation: it is based on the assumption that the average magnitude,  $H(\alpha)$ , doesn't change over the course of the night up to the "extrapolated" data point.

## 7.2 What if $H(\alpha)$ isn't Constant During a Night's Session?

For main-belt asteroids (and more distant objects), it is usually safe to *assume* that  $H(\alpha) \approx$  constant over any night's observations because the solar phase angle changes only very little over a single night. For example, in the case of 535 Montague, solar phase angle never changed by more than 0.02 deg/hr, which is less than 0.2 degrees over a night's observation period. If you assume a typical  $G = 0.15$ , the fastest that you expect  $V_R$  to change in this case is about  $|dV_R/dt| \leq 0.002$  mag/hr. That is, the changing phase during a single night won't alter the  $V_R$  by more than a few of hundredths of a magnitude, or roughly equal to my photometric accuracy. So, it is reasonable (barely) to assume that  $H(\alpha)$  is constant over any night, and the procedure described in Section 7.1 will work fine.

However, we know that in actuality  $\alpha$  is not constant; if it were, then there wouldn't be any phase curve to measure! Since  $\alpha$  changes from night to night, then it must also be a little different at dawn than it was at dusk. For the case of 535 Montague the effect wasn't clearly detectible compared with photometric noise. However, if you're chasing a near-Earth asteroid (NEA), whose solar phase angle might change very rapidly even over the course of a single night, then that fact needs to be accommodated in your analysis.

This is done by making an iterative calculation that Harris et al. (1989) described, and which is a little tricky. The equation given for the Fourier fit of the lightcurve (Eq. 4) can accommodate a non-constant  $H(\alpha)_i$  with no trouble. We just need to know how to calculate  $H(\alpha)$ , but we only know that after we've determined the phase curve parameter  $G$ !

So what Harris et al. (1989) did was to select a "provisional" value of the slope parameter (say,  $G_{prov} = 0.15$ ), and use that provisional value to calculate  $H(\alpha)$  over the range of  $\alpha$ 's on each night (individually), for use in the Fourier fit of Eq. 4. This was a simple calculation to add to my Excel spreadsheet. The Fourier fit is matched to the night's lightcurve data points in the same way as described in Section 7.1, with the only difference being that the Fourier fit now includes the  $H(\alpha)$  that is appropriate to the phase angle of each data point.

The plot of the overall phase curve is then used to determine a "revised/improved" value of  $G$ . That revised/improved estimate is plugged in place of the provisional value and the whole set of calculations are run again. The procedure is iterated until things converge on a stable value for  $G$ , which usually doesn't require more than a couple of iterations.

What I found – not surprisingly – was that this iteration wasn't really needed for my main-belt objects. The effect of  $\alpha$  varying in the course of a night and the resulting non-constant  $H(\alpha)$  amounted to less than a hundredth of a magnitude.

However, in the case of a near-Earth asteroid whose solar phase angle and distances may change significantly in the course of each night, this iterative adjustment is likely to be necessary to achieve the maximum possible accuracy.

## 7.3 Does the Rotational Lightcurve Change as Solar Phase Angle Increases?

Most of us select nights that are near an asteroid's opposition when measuring the rotational lightcurve to determine its synodic period. This makes sense because that is when the asteroid is brightest and you get the maximum rotational coverage (because the asteroid is visible most of the night).

However, does the lightcurve change its shape as the solar phase angle changes? One might expect that shadow effects from topography (hills or craters) on the asteroid would alter the lightcurve since their shadows become longer and cover greater areas on the surface of the asteroid at increasing phase angles. After all, you don't see shadows on the full Moon, when  $\alpha \approx 0^\circ$ , but at other phases the shadows of lunar mountains are distinct. Indeed this effect is reported in the professional literature. For example Gehrels (1956) noted that the rotational lightcurve of 20 Masalia at  $\alpha \approx 3^\circ$  was measurably different from that at  $\alpha \approx 0.5^\circ$ , and that it had changed character again by  $\alpha \approx 20^\circ$ . The differences manifested themselves as changes in the peak and valley brightness of about 0.03 mag and also changed the "sharpness" of the peak and valley – not huge differences but definitely measurable. Similarly, Groeneveld and Kuiper (1954) found a similar effect for 7 Iris and 39 Lutetia.

The photometric accuracy for which we're striving in order to determine the phase curve is quite sufficient to identify changes in lightcurve shape as the solar phase angle increases; therefore, the data analysis routine for determining the phase curve should be able to accommodate these changes.

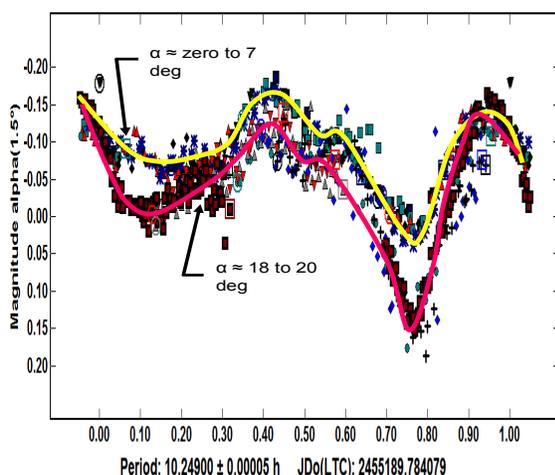
I have followed the approach suggested by Harris and Young (1979). I determine a Fourier fit to the lightcurve using a few nights of data near opposition

( $\alpha \approx 0^\circ$ ) and use this to represent the lightcurve at low solar phase angle. Then I look for obvious visual changes in the lightcurve from nights at increasing solar phase angle. If a definite change is visually apparent, I create a second Fourier fit to use at high solar phase angles.

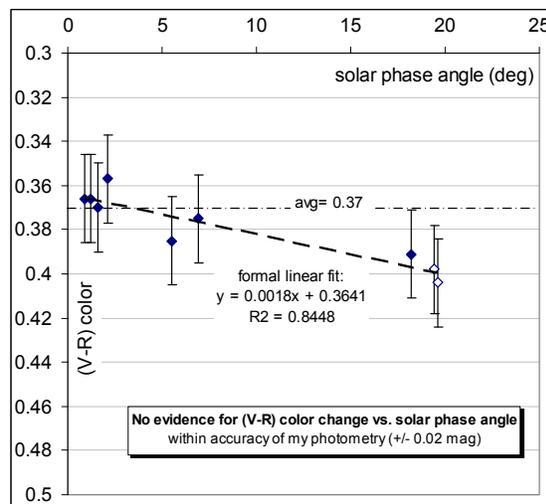
As it worked out, this was needed in the case of 535 Montague, where the shape and peak-to-peak amplitude of the lightcurve changed noticeably at  $\alpha > 9^\circ$  (Figure 10).

### 7.4 Does the Asteroid’s Color Change as Solar Phase Angle Increases?

There have been occasional hints in the literature that the color of some asteroids might change a bit as the solar phase angle changes. All of these purported color changes are very subtle and uncertain. For example, Belskaya et al. (2010) report that the (B-V) and (V-R) color of 21 Lutetia may increase very slightly ( $\approx 0.001$  mag/deg) with solar phase angle, and Tupieva (2003) reports that the (U-B) color of 44 Nysa decreases ( $\approx -0.011$  mag/deg) with increasing solar phase angle. I plotted my inferred color of 535 Montague versus solar phase angle (Figure 11). Fitting a simple linear trend line to the data does show a slight increase in (V-R) color with  $\alpha$ , but all the data points are consistent with  $(V-R) = 0.37 \pm 0.02$ , which is the indicated accuracy of my photometry. Hence, I conclude that I haven’t seen evidence of a change in color during the course of this apparition.



**Figure 10: 535 Montague’s Lightcurve shape changes as solar phase angle increases.**



**Figure 11: A hint, but no compelling evidence, for (V-R) color change with solar phase angle.**

## 8. Appendix A: Equations for $\Phi_1$ and $\Phi_2$

For the record, I give here the equations to calculate Eq 2, taken from Bowell et al (1989). All angles are to be interpreted as radians in these equations:

$$W = \exp[-90.56 \tan^2(\alpha/2)]$$

$$\Phi_1(\alpha) = W\phi_{1S} + (1-W)\phi_{1L}$$

where

$$\phi_{1S} = 1 - \frac{C_1 \sin \alpha}{0.119 + 1.341 \sin \alpha - 0.754 \sin^2 \alpha}$$

$$\phi_{1L} = \exp\left[-A_1 \left(\tan \frac{\alpha}{2}\right)^{B_1}\right]$$

$$A_1 = 3.332$$

$$B_1 = 0.631$$

$$C_1 = 0.986$$

and

$$\Phi_2(\alpha) = W\phi_{2S} + (1-W)\phi_{2L}$$

where

$$\phi_{2S} = 1 - \frac{C_2 \sin \alpha}{0.119 + 1.341 \sin \alpha - 0.754 \sin^2 \alpha}$$

$$\phi_{2L} = \exp \left[ -A_2 \left( \tan \frac{\alpha}{2} \right)^{B_2} \right]$$

$$\begin{aligned} A_2 &= 1.862 \\ B_2 &= 1.218 \\ C_2 &= 0.238 \end{aligned}$$

## 9. Acknowledgements

This research made use of the JPL Horizons System, for determination of orbital parameters of solar system objects; the NASA Planetary Data System, and the NASA Astrophysics Data System. Some of the observations reported here were made possible the internet-accessed telescopes of the Tzec Maun Foundation's New Mexico Observatory, to which I am very grateful. I am pleased to acknowledge the Science Library of the University of California at Irvine, for their helpful and enlightened support of researchers who are not associated with the University.

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