NEW FORMULAE FOR OPTIMUM MAGNIFICATION AND TELESCOPIC LIMITING MAGNITUDE

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ABSTRACT. New approximate formulae are presented for the optimum magnification and limiting magnitude of a telescope. The formulae are based on a new mathematical representation of the threshold illuminance of the eye as a function of source size and background brightness. The new results are compared with formulae derived from an alternative representation by Schaefer.

RÉSUMÉ. De nouvelles formules approximatives pour le calcul du grossissement optimal et de la magnitude limite d'un télescope sont présentées. Ces formules sont basées sur une nouvelle représentation du seuil d'illumination de l'oeil, en fonction de la dimension de la source et de la brillance du ciel. Les nouvelles formules sont comparés à celles dérivées de la représentation alternative par Schaefer.

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1. INTRODUCTION

The faintest magnitude star that can be seen through a telescope depends on the response of the eyes to incident radiation and on the properties of the telescope. Many authors have studied the response of the eyes. If i is the illuminance (light received per unit area) at the observer produced by a source and b is the surface brightness (light reflected, emitted, or transmitted per unit area per unit solid angle) of the background against which the source is observed, then there is a relation of the form $i = f(b)$ for the minimum illuminance that can be perceived by the observer. Some early workers used dependences of the type $f(b) = kb^{1/2}$ and $f(b) = i_0(1 + kb)^{1/2}$. An important advance was made by Hecht (1934), who suggested the relation:

$$i = f(b) = i_0(1+kb^{1/2})^2 , \qquad (1)$$

which has proven to be an excellent approximation in spite of having been proposed on the basis of a now discredited chemical theory of vision.

Hecht (1947) also proposed the use of two formulae of the same form: one for faint illuminances when the rods of the retina are dominant (the scotopic region), and one for bright illuminances when the cones of the retina are dominant (the photopic region). He based his values of i_0 and k on laboratory observations by Knoll *et al.* (1946) of sources one arcminute in diameter in fields whose brightness ranged from complete darkness to daylight. The eye has a resolving power of about one arcminute, so the sources were effectively point sources. Hecht's formulae were used by Weaver (1947) and Garstang (1986), and, with great success, by Schaefer (1990) in his exhaustive treatment of telescopic limiting visual magnitudes.

In visual photometry light emission is measured in candelas (abbreviated cd), where 1.0 cd is 1/60th the luminous intensity of 1.0 projected cm^2 of a black body at the temperature of melting platinum (2044 K). Luminous flux is measured in lumens, where 1.0 lumen is the flux from 1.0 cd into unit solid angle. In metric units, illuminances are measured in lux, where 1.0 lux is equivalent to 1.0 lumen falling on 1.0 square metre; lighting engineers and photographers often use foot-candles, with 1.0 foot-candle = 1.0 lumen falling on an area of 1.0 square foot. The units are related by the relation: 1.0 foot-candle = 10.76 lux. There are many possible units of surface brightness (also known as luminance). The SI unit is the nit, which is 1.0 lumen per square metre per unit solid angle, or 1.0 cd/m^2. Another metric unit is the stilb, where 1.0 stilb = 10^4 nit. A unit that is frequently used is the lambert, where 1.0 lambert = $1/\pi$ stilb = 1.0 lumen per square centimetre for a uniformly diffusing surface. Engineers often use 1.0 foot-lambert = 1.0 lumen per square foot for a perfectly diffusing surface = 1.076×10^{-3} lambert.

For the night sky, astronomers often use magnitudes per square second or per square degree, or the number of 10th magnitude stars per square degree. Nanolamberts (1.0 nanolambert = 10^{-9} lambert) are also convenient. Most physicists would use photons per second per square metre (or centimetre) per unit solid angle. Relationships between the various units may be found in Allen (1973, p. 26) and Garstang (1986, 1989). The present paper uses illuminances in lux and surface brightnesses in nanolamberts, abbreviated nL.

2. Modifications

Some modification is needed if a source being observed is of finite size. That occurs when looking at stars through a telescope with a high power eyepiece or when the seeing is very bad. Schaefer proposed one modification, which will be discussed below. This paper suggests an alternative modification.

Threshold contrast is the ratio of the minimum increment of brightness of a source that can be detected to the value of the brightness. As a result of a major World War II project, Blackwell (1946) reported a large set of laboratory naked-eye binocular observations of threshold contrast as a function of b (about 2 million observations were made, and 450,000 analyzed). The present paper uses the final results in his Table VIII, which were based on about 90,000 observations by seven observers with an average age of about 23 years and of normal eyesight. Seven small, circular, illuminated disks of various diameters were used as stimuli against a large background whose brightness could be varied from 10^9 nL down to 10 nL. The experiments determined the threshold contrast for seeing a disk against the background. Tousey & Hulburt (1948) modified Blackwell's data by changing from threshold contrasts to absolute thresholds and by doubling the values of i to change the threshold criterion from a 50% probability of detection to a 98% probability of detection. The units of illuminance i in Tousey & Hulburt's data were changed to lux. Knoll *et al.* used nanolamberts, as did Hecht (1947). Blackwell used foot-lamberts, but Tousey & Hulburt changed his data to nanolamberts. Nanolamberts were adopted in this work because the unit has been used extensively in studies of light pollution, and it is a convenient size. All the photometry was expressed in terms of the photopic response curve. A correction was applied to the observations of Blackwell in the scotopic region to allow for the difference of colour temperatures between his sources and those of Knoll *et al.* Finally small systematic corrections were applied to Blackwell's data for each background brightness separately, so that for effectively point sources Blackwell's data would agree with the results of Knoll *et al.* The effect of such corrections is to ensure that the Knoll *et al.* data were used for point sources and the contrast ratios measured by Blackwell were used for larger sources.

It seemed desirable to continue to use equations of Hecht's form, but generalized to include the effect of source size θ as described by Blackwell's data. In astronomical applications θ is the seeing disk diameter, which Blackwell expressed in arcminutes. When the data were examined, it became apparent that the values of illuminance i increased from the values for point sources as θ increased. The increments were very nearly proportional to θ^2 for a given value of surface brightness b. It seemed appropriate to multiply Hecht's formula [equation (1) above] by a correction factor of the form $(1 + a\theta^2)$, with a different value of a for each value of b. It turned out that the values of a increased as b increased. A relationship of the type $a = \alpha + yb^z$ proved to be a good approximation.

The complete formula for the threshold illuminance is therefore:

$$i = i_o(1+kb^{1/2})^2(1+[\alpha + yb^z]\theta^2) \ , \tag{2}$$

which is used for the scotopic region. A similar formula can be used in the photopic region, with different numerical values for the constants, but it will not be needed here. There does not seem to be a simple explanation for the presence of the term in θ^2 in the formula. However, θ^2 is proportional to the area of the image on the retina. One might guess that its occurrence is probably connected with the areal density of the rods on the retina, which falls off on either side of the position of maximum sensitivity to averted vision (Cornsweet 1970, p. 11). The observed rate of fall-off is much faster than the rate of rod density fall-off, however, so that other factors, perhaps the angular sensitivity of the rods, must be involved.

3. Results

The values of the constants in equation (2) were determined by fitting the formula by least squares to the data for b = 10 and 100 nL, and θ = 0.595, 3.6, 9.68, 18.2, 55.2, and 121 arcminutes. Data for b = 1000 nL were omitted because the cones of the eye begin to contribute to the eye sensitivity at that brightness. Our results should be quite good up to at least b = 300 nL, which for the unaided eye corresponds to a moderately light polluted sky. Data for θ = 360 arcminutes were also omitted; including them would have reduced the goodness of fit significantly, and one is not usually interested in observing fields as large as 6° in apparent diameter.

The fit was made using the logarithmic form of the data. The experimental data used and all the formulae in the present paper use logarithms to the base 10. The constants obtained were $i_0 = 2.908 \times 10^{-9}$ lux, $k = 0.115$, $\alpha = 0.000154$, $y = 0.000062$, and $z = 0.276$. The root mean square (r.m.s.) uncertainty for the fit to the log i data was about ±0.033 in log i. Omission of the $y\, b^z\, \theta^2$ term in equation (2) and a subsequent least squares determination for i, k, and n produced an r.m.s. uncertainty for the fit to the same log i data of about ±0.046 in log i. The fits made use of the mean values of Knoll, Tousey and Hulburt. A comparison of the Hecht formula with the original individual experimental points of Knoll, Tousey and Hulburt gave an r.m.s. uncertainty of ±0.20 in log i. The fit obtained above is applicable to the whole range of θ. With i in lux, the visual magnitude is given by:

$$m = -13.98 - 2.5\log i \ . \tag{3}$$

The constant in the formula is quoted from Allen (1973, p. 197); it is based on very extensive detailed spectrophotometry of stars of many spectral types and on comparisons with laboratory standard lamps seen through a telescope.

Such considerations apply to the unaided eyes. Application to the visual use of a telescope requires appropriate changes in equation (2). One may follow Schaefer's detailed treatment, or one may make a few approximations and derive the results from first principles. The following substitutions are made in equation (2):

$$i_o \rightarrow \frac{\sqrt{2}\,p^2}{tD^2}\,i_o, \qquad b \rightarrow \frac{tD^2 b}{\sqrt{2}\,p^2 M^2}, \qquad \theta \rightarrow M\,\theta . \tag{4}$$

The factor of $\sqrt{2}$ converts from binocular vision with unaided eyes to telescopic observations with a single eye. The factor t is the transmission of the telescope, to allow for losses in reflections at mirrors, obstruction by the secondary mirror, and losses in the lenses by absorption and at the air-glass interfaces. D is the diameter of the objective, and p is the average diameter of the eye pupils of the observers who determined the data on which equation (2) is based. The factor p^2/D^2 compensates for the increased light gathering power of the telescope relative to the eye, and results in a decrease in the threshold illuminance and an increase in the number of background photons received. The factor M^2 accounts for the magnification M of the telescope, which reduces the surface brightness of the observed background; the factor M also makes the apparent image size larger. If we define:

$$\beta = \frac{kD\sqrt{tb}}{2^{1/4}p} \quad \text{and} \quad \gamma = \left(\frac{\beta}{k}\right)^{2z}, \tag{5}$$

then the illuminance received is given by:

$$i = \frac{\sqrt{2}p^2}{tD^2} i_o \left(1+\frac{\beta}{M}\right)^2 (1 + \alpha M^2\theta^2 + y\gamma M^{2-2z}\theta^2) \ . \tag{6}$$

There is a value for the magnification M for which the illuminance i is a minimum. Calculus shows that it is the value of M that satisfies the equation:

$$\alpha M^3\theta^2 = \beta - y\gamma(1-z)M^{3-2z}\theta^2 + \beta y\gamma z M^{2-2z}\theta^2. \tag{7}$$

This equation does not admit of a simple algebraic solution for M. In any given case, the equation can easily be solved by iteration on a personal computer, starting with the value of M given by equation (8) below. When the value of M has been found, equation (6) gives the minimum value of i and equation (3) gives the limiting magnitude. A correction for atmospheric extinction should also be applied.

Examination of many cases has shown that the term with co-efficient $\beta y \gamma z$ is very small and usually negligible, but the term with co-efficient $y\gamma$ is appreciable, typically being between roughly 0.1 β and 0.3 β for telescopes ranging in size from 15 cm up to 152 cm (the 60-inch reflector at Mount Wilson) and with sky brightnesses up to ten times the natural brightness at sunspot minimum. It is a reasonable first approximation to neglect the $y\gamma$ term, in which case:

$$M = \left(\frac{\beta}{\alpha\theta^2}\right)^{1/3} = \left(\frac{kD\sqrt{tb}}{2^{1/4}p\alpha\theta^2}\right)^{1/3} \tag{8}$$

This is the simple general formula. The value obtained for M can then be used to calculate the minimum value of i, which is [with the $y\gamma$-term in equation (6) neglected]:

$$i = \frac{\sqrt{2}\ p^2}{tD^2} i_o [1+(\alpha\beta^2\theta^2)^{1/3}]^3. \tag{9}$$

The limiting magnitude is then calculated using equation (3). It works out to be:

$$m = 7.76 + 5\log D + 2.5\log t - 7.5\log(1 + 0.000935\,D^{2/3}t^{1/3}b^{1/3}\theta^{2/3}) \ . \tag{10}$$

An extinction correction must be applied to the derived value of m. A more accurate value can be obtained using Schaefer's theory, which includes several factors that have been neglected in the above derivation, such as Stiles-Crawford corrections, star colour corrections, and an acuity correction to take into account any exceptional sensitivity of the observer's eyes. The simple formula above gives a result of acceptable accuracy for most purposes.

A useful approximation can be obtained by substituting $p = 0.7$ cm for the eye pupil, a reasonable average for the young observers who obtained the laboratory data. To convert θ to arcseconds, it is replaced by the ratio $\theta/60$. The dependence of magnification M on telescope transmission t varies as $t^{1/6}$, which lies between 0.91 and 0.99 for most telescopes. It is adequate to assume that $t^{1/6} = 0.95$, in which case:

$$M = 140\ \frac{D^{1/3}b^{1/6}}{\theta^{2/3}} \ . \tag{11}$$

The above formula is useful for making estimates as well as for perceiving the importance of various factors. Note that the background brightness is not very critical, the most important factor being the seeing.

Schaefer replaced equation (2) by:

$$i = i_o(1+kb^{1/2})^2\, g \ \cdot \tag{12}$$

For a telescope, the variables i_0 and b are replaced by the expressions given by relation (4), and the function g is taken to be $g(M) = 1$ for $(M\theta/900) < 1$ and $g(M) = (M\theta/900)^{1/2}$ for $(M\theta/900) \geq 1$; θ is in arcseconds. There is a value of the magnification M for which the illuminance i is a minimum. Define $M = 3\beta$, in which case the assumption that $k = 0.115$ and $p = 0.7$ cm leads to:

$$M_o = 0.414Dt^{1/2}b^{1/2}. \tag{13}$$

A careful calculus derivation indicates that if $M_0 < 900/\theta$ then $M = 900/\theta$ is the optimum magnification, while if $M_0 \geq 900/\theta$ then $M = M_0$ is the optimum magnification. The threshold illuminance and the limiting visual magnitude can be calculated. The results are:

$M_0 < 900/\theta$,

$$m = 7.76 + 5\log D + 2.5\log t - 5\log(1+0.000153D\sqrt{tb}\,\theta), \tag{14a}$$

$M_0 \geq 900/\theta$,

$$m = 11.31 + 3.75\log D + 1.875\log t - 0.625\log b - 1.25\log\theta. \tag{14b}$$

Although the formulae look very different from equation (10), they give results that are nearly the same.

There is another interesting formula that can be compared with the above formulae. Lewis (1913) made a study of the magnifications that were in use by 36 observers of double stars, and he showed that their practice was well represented (with D now measured in cm) by the equation:

$$M = 88D^{1/2}. \tag{15}$$

Although it is not a formula for limiting magnitude, it does represent an optimum magnification.

4. Discussion

Table I gives a few examples of results calculated using the various formulae discussed here, together with the values given by the traditional rule of magnification equals 30× per inch of aperture. The limiting magnitudes in columns (6), (7), and (8) are in close agreement except for the largest telescope under adverse sky brightness conditions.

Figure 1 illustrates the behavior of the limiting magnitude for the case $D = 15$ cm, $t = 0.9$, $b = 60$ nL, and $\theta = 1.5$ arcsecond. Curve 1 illustrates the relation between limiting magnitude and magnification for equations (6) and (3) combined. Curve 2 shows the relation given by Schaefer's formula [equations (12), (4) and (3) combined], and curve 3 plots the relation for Hecht's formula [equations (1), (4) and

(3) combined]. The optimum magnification is for a minimum in the illuminance *i*, and hence for maximum numerical value for the limiting magnitude *m*. (Note that in figure 1 the magnitude scale is inverted so that the brightest magnitude is at the top.) The curve for the maximum value of *m* and the curve for the minimum value of *i* (which is not shown in the figure) are very broad functions of *M*, so moderate changes in magnification hardly change the values of *i* and *m*. Hecht's formula has no minimum illuminance as a function of magnification. Schaefer's results only differ slightly from results based on equation (2), but the discontinuity in his curves probably does not give a proper representation of the behavior of the human eye. The Yerkes Observatory 40-inch (102 cm) refractor was included for comparison with observations by Barnard, who obtained a limiting magnitude of about 17.1 — which, after an extinction correction, is somewhat fainter than our formula predicts, probably in part because of Barnard's well-known very good eyesight. A more elaborate treatment of the same case (Garstang 1999) leads to the conclusion that Barnard must have had an eyesight capable of detecting a star with 69 per cent of the normal threshold illuminance.

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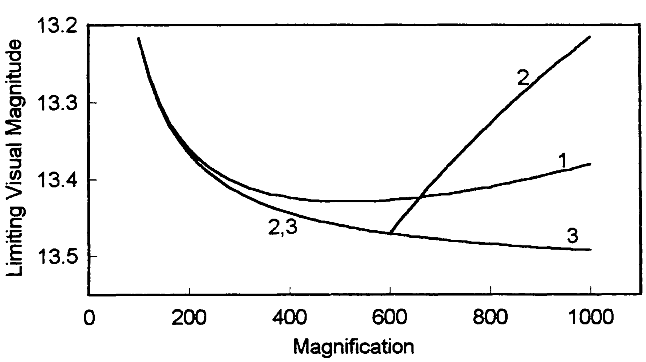

Fig. 1 — Limiting visual magnitude as a function of the magnification of the telescope, for D = 15 cm, t = 0.9, b = 60 nL, and θ = 1.5 arcsecond. The magnitude axis is plotted with the brightest at the top and faintest at the bottom, so that a minimum of a curve corresponds to the least detectable illuminance. Curve 1 is calculated from equations (6) and (3), curve 2 is from Schaefer's modification [equations (12), (4) and (3)], and curve 3 is from Hecht's formula [equations (1), (4) and (3)]. Note the very broad minimum for curve 1, the sudden change of slope for curve 2, and the absence of a minimum for curve 3.

Table I
Optimum Magnifications and Limiting Magnitudes

D (cm)	b (nL)	M (1)	M (2)	M (3)	M (4)	M (5)	m (6)	m (7)	m (8)
15	60	540	520	600	340	180	13.46	13.45	13.49
15	180	650	620	600	340	180	13.42	13.41	13.46
15	600	800	750	600	340	180	13.36	13.34	13.38
40	60	750	710	600	560	470	15.51	15.49	15.54
40	180	900	840	600	560	470	15.43	15.41	15.44
40	600	1100	1010	600	560	470	15.32	15.27	15.25
102	60	960	890	600	890	1200	16.99	16.96	16.98
152	60	1090	1000	600	1080	1800	17.73	17.68	17.67
152	180	1310	1180	640	1080	1800	17.58	17.51	17.41
152	600	1600	1420	1180	1080	1800	17.36	17.25	17.09

Notes: All the above were calculated using θ = 1.5 arcseconds. For typical 15 cm and 40 cm telescopes, a value of t = 0.92 was adopted. For D = 102 cm (the Yerkes refractor), the value of t was taken to be 0.61, and for D = 152 cm (the Mount Wilson 60-inch reflector), the value of t was taken to be 0.58. No extinction corrections have been applied.
(1) Optimum magnification, Garstang's formula, simple approximation, equation (8).
(2) Optimum magnification, Garstang's formula, accurate solution, equation (7).
(3) Optimum magnification, Schaefer's formulae, see text following equation (13).
(4) Optimum magnification, Lewis's formula for double star observers, equation (15).
(5) Optimum magnification, traditional rule, M = 30× per inch of aperture.
(6) Limiting magnitude, Garstang's formula, simple approximation, equation (10).
(7) Limiting magnitude, Garstang's formula, accurate solution, equations (6) and (3).
(8) Limiting magnitude, Schaefer's formulae, equations (14a) and (14b).

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