EPICYCLES, FROM HIPPARCHUS TO CHANDRASEKHAR

Vittorio Banfi
Centro di Astrodinamica "G. Colombo", Torino

ABSTRACT - In this paper the epicycle, introduced by Chandrasekhar (1942) and concerning the one-body problem in newtonian Celestial Mechanics, is examined and studied. A simple derivation, in alternative to the Chandrasekhar treatment, is here developed.

1. Introduction

As it is well known, "the epicycle method" was used in Ptolemaic astronomy to account for the observed periodic irregularities in planetary motions. The epicycle is a small circle, the center of which moves around the circumference of a larger circle called the deferent. Introduced for the first time by Hipparchus, it was used by Ptolemy, for instance, to explain Sun's motion round the Earth. Figure 1 shows how epicycles work.

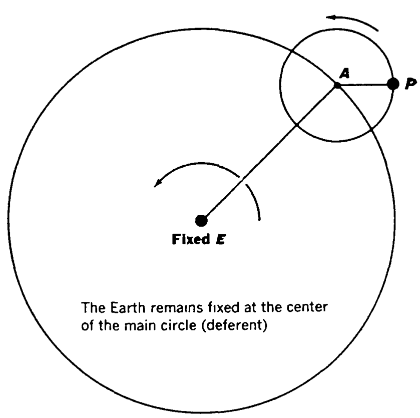


Figure 1. - The Earth remains fixed at the center of the main circle (deferent)

Epicycles survived, in explications of heavenly bodies motions, also in heliocentric theory of the Solar-System (Reference 1) until 1627, i.e. the year of publication by Kepler of "Tabulae Rudolphinae" (Reference 2). After this year classical epicycles went to science museum, since planetary orbits were recognized as ellipses. Newton completed the formulation of the fundamental principles of mechanics, and applied them with unparalleled success in the solution of mechanical and astronomical problems.

2. - Chandrasekhar's epicycle

In the textbook "Principles of Stellar Dynamics" (Reference 3), S. Chandrasekhar renewed the epicycle concept in the modern context of newtonian Celestial Mechanics. However this mathematical model does not offer exactly Hipparchus' epicycle, but with some modifications. The complete mathematical development of the theory is contained in previously cited textbook, but in this paper a more semplified version is carried on. Following rigorous methods this analysis treats newtonian ellipse as "perturbation" of a circular orbit and develops the consequences in order to explain the planetary motions. The same results of Chandrasekhar's textbook are obtained; they are applicable, in any way, in the case of small orbital eccentricities (0,01 ÷ 0,12).

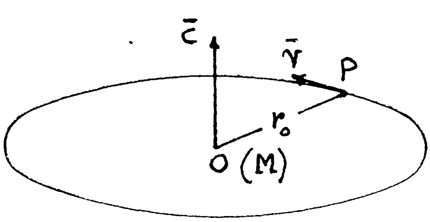
3. - Analytical development of the model

In the one-body problem approximation, let us consider a circular orbit (Figure 2) with radius r_0 and primary mass M at the fixed center 0. When P is the unit mass moving on circular orbit, the angular momentum conservation states:

$$(P - 0) \wedge \bar{v} = \bar{c} \,, \tag{1}$$

in which (P - 0) is the position vector, $\bar{v}$ the velocity vector of P and $\bar{c}$ the constant vector angular momentum (for unit mass). Assuming v_0 the constant velocity on circular orbit Eq. (1) becomes

$$r_0 \ v_0 = c \,, \tag{2}$$



$(P - 0) \wedge \bar{v} = \bar{c}$ vectorial form

$r_0 \ v_0 = c$ scalar form

Angular momentum conservation

Figure 2. - Motion on circular orbit

and the centripetal acceleration of P

(3) $$a = \frac{v_o^2}{r_o} = \frac{c^2}{r_o^3} ,$$

which will be equal to gravitation force (for unit mass) a_g. Then we have

(4) $$a_g = \frac{G\,M}{r_o^2} = \frac{c^2}{r_o^3} \text{ [G = gravitation constant] .}$$

In order to complete the circular motion description we have angular velocity

(5) $$\omega_o = \frac{v_o}{r_o} = \frac{c}{r_o^2}$$

and the corresponding period

(6) $$T_o = \frac{2\pi}{\omega_o} .$$

Let us apply to P a small instantaneous displacement δP, lying in a perpendicular plane with respect to original circular orbit (Figure 3). Vector δP is splitted in two components: δr (radial) and δz (axial).

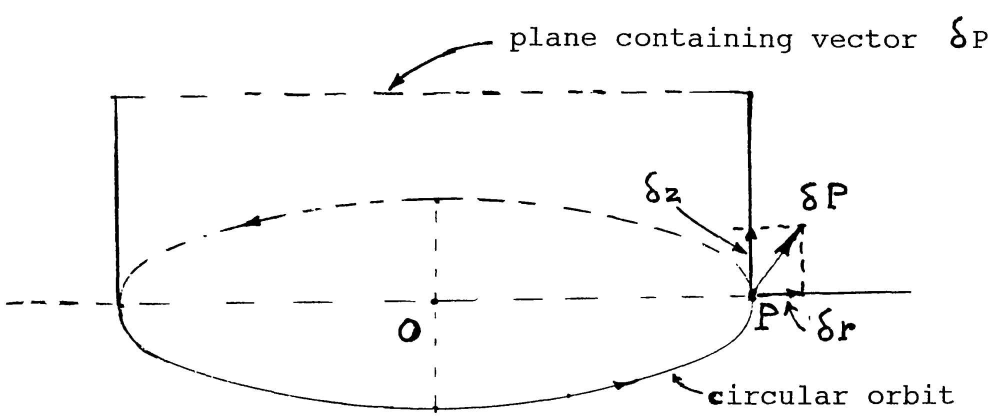


Figure 3. - Instantaneous perturbation of original circular orbit

Now we calculate the changed values a'_c and a'_g on account of variation δr. We obtain:

$$a'_c = \frac{c^2}{r^3} = \frac{c^2}{(r_o + \delta r)^3} \cong \frac{c^2}{r_o^3} - 3\,\frac{c^2}{r_o^4}\,\delta r$$

(7)

$$a'_g = \frac{G\,M}{r^2} = \frac{G\,M}{(r_o + \delta r)^2} = \frac{G\,M}{r_o^2} - 2\,\frac{G\,M}{r_o^3}\,\delta r\,,$$

considering the first order approximation only. On the basis of Eq. (7), of the complete variation δa_P of point P is

(8) $$\delta a_P = -\left(\frac{3\,c^2}{r_o^4} - 2\,\frac{G\,M}{r_o^3}\right)\delta r\,.$$

Remembering Eq. (4), after multiplication by $\frac{1}{r_o}$, we have

$$\frac{G\,M}{r_o^3} = \frac{c^2}{r_o^4} = A_o = \text{positive constant.}$$

Then Eq. (8) becomes

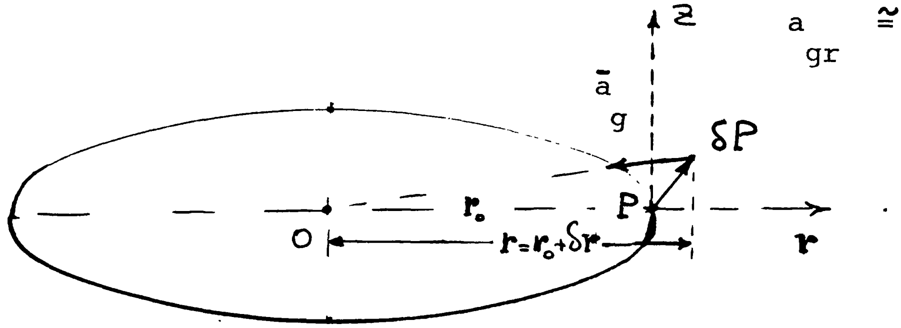
(9) $$\delta a_r = -A_o\,\delta r\,.$$

On account of an outward displacement $\delta r > 0$, P experiences a restoring force for unit mass proportional to δr, in which A_o acts as the usual spring constant. In fact when $\delta r > 0$, $\delta a_P < 0$. We recognize that there is an equation of simple harmonic motion, with angular frequency

$$\omega_o = \sqrt{A_o} = \frac{c}{r_o^2},$$ the same value of angular velocity (5)

of original circular motion.

Considet now the effects caused by axial component δz. With reference to Figure 4, owing to the smallness of δP, vector $\vec{a}_g$ (gravitational force for unit mass) has radial component $a_{gr} \cong -\frac{G\,M}{r^2}$; furthermore we must calculate a_{gz}.



$$a_{gr} \cong - \frac{G\,M}{r^2}$$

Figure 4. - Schematic sketch useful for calculation of a_{gz}

Since we can write:

$$(10) \qquad \mathrm{div}\ \bar{a}_g = \frac{1}{r}\frac{\partial}{\partial r}(r\,a_{gr}) + \frac{\partial a_{gz}}{\partial z} = 0,$$

and also at point P $(r_0, 0)$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,a_{gr})\right]_{r=r_0} = \left[\frac{1}{r}\frac{G\,M}{r^2}\right]_{r=r_0} = \frac{G\,M}{r_0^3},$$

we solve Eq. (10) with respect to $\frac{\partial}{\partial z} a_{gz}$:

$$\frac{\partial a_{gz}}{\partial z} = - \frac{1}{r}\frac{\partial}{\partial r}(r\,a_{gr})$$

and finally

$$\left[\frac{\partial a_{gz}}{\partial z}\right]_{r=r_0} = - \frac{G\,M}{r_0^3}\ .$$

From the preceeding relationships we obtain

$$(11)\ \delta a_{gz} = \left(\frac{\partial a_{gz}}{\partial z}\right)_{r=r_0} \delta z = - \frac{G\,M}{r_0^3}\,\delta z = - A_0\,\delta z\ .$$

Comparing Eqs. (9) and (11) we note that, also in vertical direction z, there is an harmonic motion with angular frequency ω_0 as before.

We can conclude that the new motion, after very short impulse at P (figure 4), is the superposition, on original circular motion, of two harmonics motions in r and z directions, calculated before.

Let us consider now the new position, in the space, of the orbital plane with regard to the original plane of circular motion. Starting from (1) we have

$$(P - 0) \wedge \bar{v} = \bar{c} \;;$$

if $\Delta \bar{c}$ is the small increment of angular momentum caused by the impulse, we can write

$$(12) \qquad \bar{c} + \Delta \bar{c} = (P - 0) \wedge \left(\bar{v} + \frac{\delta P}{\delta t}\right)$$

which holds for the transient duration δt. From Eqs. (1) and (12) we get

$$\Delta \bar{c} = (P - 0) \wedge \frac{\delta P}{\delta t}$$

This small vector $\Delta \bar{c}$ is lying in original plane and also is tangent to the circumference of radius r_0 at point P (Figure 5) and in opposite direction to the vector velocity. The subsequent orbital motion will lie in a tilted plane with respect to the previous one. The inclination angle will be $i \cong \tan i = \frac{\Delta c}{c}$ (Figure 5). We have considered Δc a small quantity of the first order with respect to c.

From Celestial Mechanics we know that the new orbit of P is an elliptic one with focus at point 0 and angular momentum pratically equal to $\bar{c}$.

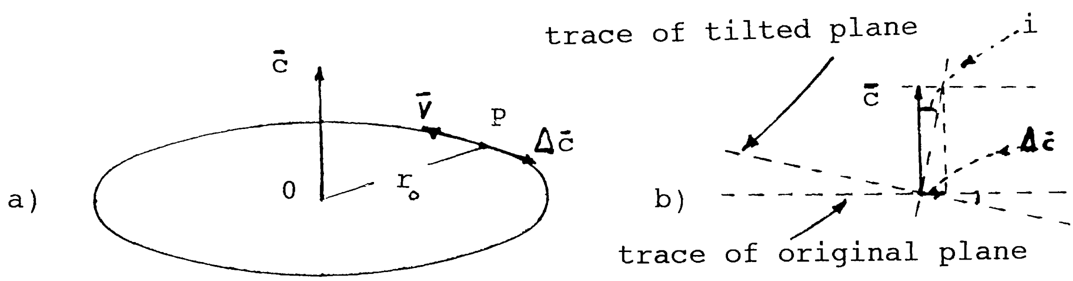


Figure 5. a) Circular orbit perturbated by impulse with angular momentum $\Delta\bar{c}$; b) inclination angle i between original plane and tilted plane.

The subsequent keplerian motion is illustrated in Figure 6 in which we note the dashed circular orbit of secondary body (before the impulse).
The elliptical trajectory, in thick trace, is the projection of real ellipse on original plane of circular orbit (the ellipse plane is almost coincident with the last plane since i is small).
Furthermore the ellipse equation

$$r = \frac{p}{1 + e \cos \theta} ,$$

in which (Figure 6):

r = radius vector,
$\theta = \psi - \psi_0$ = true anomaly,
e = eccentricity
$p = a\,(1 - e^2)$ = semi-latus rectum,

owing to the smallmess of eccentricity (0,01 ÷ 0,12) is rewritten so

$$(13) \qquad r = r_0\,(1 - e \cos \theta) \quad [p = a\,(1 - e^2) \cong a = r_0]$$

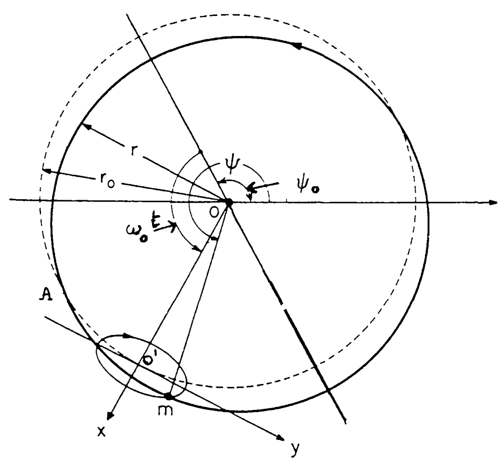


Figure 6. - The primary body is at the origin 0, which is the center of the dashed circle with radius r . Let the origin 0 of a coordinate system (x,y,z) move with constant velocity along the dashed circle. In this coordinate system the point mass m (secondary body) moves in an "epicycle" which is an ellipse with the axis ratio 2:1. The epicycle motion is retrograde.

Let us imagine that the secondary body follows around 0', which is orbiting around 0 with constant angular velocity ω_0 (Figure 6), a small closed trajectory and let us study if that is possible. The radius vector (m - 0') of this particle has true anomaly ϑ, whereas 0' has mean anomaly ω_0 t (at general epoch t, being t = 0 the instant of passage through pericenter). The we get the small angle

(14) $$\varepsilon = \psi - \psi_0 - \omega_0 t \ ,$$

and also we assume cos $\varepsilon \cong 1$, sin $\varepsilon \cong 0$. Let us take a frame of reference in uniform circular motion on the circumference of radius r_0, with origin 0' and axes x, y, z as in Figure 6. With good approximation we have:

(15) $$x = r - r_0 \ ,$$

and then by Eqs. (13), (14)

(16) $$r = r_0 \ (1 - e \cos \vartheta) = r_0 \ [\ 1 - e\cos (\psi - \psi_0)\ ] =$$
$$= r_0 \ [\ 1 - e\ (\cos \omega_0 t + \varepsilon\ )\ ] \cong r_0 \ (1 - e \cos \omega_0 t) \ .$$

The second Kepler's law gives:

$$r^2 \ \frac{d\psi}{dt} = c$$

and by (16)

(17) $$\frac{d\psi}{dt} = \frac{c}{r^2} \cong \frac{c}{r_0^2} \ (1 + 2\ e \cos \omega_0 t) .$$

Multiplying Eq. (14) by r_0 we get

(18) $$\varepsilon r_0 = r_0 \psi - r_0 \psi_0 - r_0 \ \omega_0 t \ ;$$

now, if we assume $\frac{dy}{dt} = \frac{d}{dt} (\varepsilon\ r_0)$, we can write by

Eqs.(5), (17) and (18)

(19) $$\frac{dy}{dt} = r_0 \left\{ \frac{d}{dt} - \omega_0 \right\} = r_0 \left\{ \frac{c}{r_0^2} + 2\ \frac{c}{r_0^2}\ e \cos \omega_0 t - \frac{c}{r_0^2} \right\} =$$
$$= 2\ e\ v_0 \ \cos \omega_0 t \ .$$

The simple differential equation (19) is readily integrated giving

(20) $$y = 2\, e\, r \sin\omega_0 t \;,$$

with integration constant assumed to be zero. From (16) we have also

(21) $$x = r - r_0 = -\, e\, r_0 \cos\omega_0 t \;.$$

Starting from Eqs. (20) and (21) we can affirm that the particle moves on a small ellipse, with the axis ratio 2:1, in retrograde sense (in the plane of Figure 6); but this is not the only motion, since we have also recognized the motion along z - axis provided by differential equation (11).

We can conclude that the particle motion on keplerian orbit is decomposable in the following way:

a) point $0'$ moves, on the circumference of radius r_0, with angular velocity $\omega_0 = \dfrac{c}{r_0^2}$,

b) the particle follows, in the x - y plane (the same of point a)), an elliptical epicycle having the same angular velocity ω_0,

c) the particles performs an harmonic oscillation along z - axis with angular frequency ω_0.

On account of point c), at two apsides, the values z_{max} and z_{min} will be

$$z_{max} = (1 + e)\, r_0\, i \cong i\, r_0$$

$$z_{min} = -\,(1 - e)\, r_0\, i \cong -i\, r_0 \;,$$

always considering i and e as small quantities.
At general epoch t we will have

(22) $$z(t) = i r_0 \sin\omega_0 (t - t_0).$$

In the relationship (22) t_0 is the instant of passage at point A (Figure 6).

4. - Concluding remarks

The elliptic epicycle or Chandrasekhar's epicycle is then completely explained. Let us consider now three applications of this model to some topics of astronomy and astrophysics. The first of them, on the part of same Chandrasekhar, concerns the von Weizsäcker theory on the origin of the Solar System (Reference 4). The improvement of this theory, regarding the stability of vortexes array within the gaseous primeval disk, is very interesting. Another application deals with the intragalactic motion of a star (Reference 5). Third application is connected to the Alfvén's "jet streams theory" (Reference 6). As discovered by Hirayama, there are families of asteroids with almost the same values of a, i, and e. Arnold (Reference 7) has confirmed the existence of all the Hirayama families; asteroids orbits almost coincide and these heavenly bodies are said to be members of a "jet stream". The explication of forming of these streams is based on Chandrasekhar epicycle.

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