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Analytical Representation of Five-Minute Solar Oscillations

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Abstract—The oscillations of gravitating objects are considered. Their diagnostics consist of determining the parameters of the internal state from temporal and spatial properties of the oscillations. Here, analytically solvable cases play a major role. A new model, which considerably extends the analytical potential in terms of justification, classification, parametric analysis, and even quantitative estimates, is proposed. The problem of waves in a polytropic atmosphere with an arbitrary polytropic index is solved. In mathematical terms, the solution reduces to Laguerre's polynomial functions, while the corresponding branches of acoustic and gravity modes prove to be mutually conjugated and are described by a hydrogen-like spectrum. The fundamental mathematical properties of the oscillation equations, which are associated with isospectral deformations of the density profile, are noted. A new, very simple and physically clear proof of the existence of a hidden symmetry in this equation is given. An important consequence of this proof is the establishment of an explicit relation between the transformed eigenfunctions.

The theory of oscillations and the associated spectral apparatus form one of the bases of modern physics. In particular, studies in the field of solar physics rely heavily on the theory of solar oscillations (see Unno *et al.* 1979; Cox 1980; Vorontsov and Zharkov 1981, 1989; Leibacher *et al.* 1985; Christensen-Dalsgaard *et al.* 1985; Brown *et al.* 1986; Toomre 1986; Gough and Toomre 1991; Stix 1991; and Vorontsov 1992). The direct and inverse spectral problems consist in calculating the functions $s_i(p_1, p_2, \ldots)$ and $p_i(s_1, s_2, \ldots)$, where s_i and p_i are, respectively, a set of parameters of the oscillation spectrum and a set of parameters that specify the physical state of the Sun. The direct problem is solved at once by computations: the parameters p_i are specified, and the spectrum up to modes of rather high order is numerically calculated by the fork method. In contrast, the inverse problem is solved by fitting: the sought-for parameters p_i are adjusted to achieve the best agreement with experimental values of s_i. This problem presents some difficulties, because there are many parameters, because they are dissimilar, and because it is not easy to move blindly. A good knowledge of both experimental and numerical results, separation of the parameters into primary and secondary, etc., are required.

For orientation in parameter space and for preliminary estimates, it is desirable to develop theoretical approaches that allow analytical expressions to be derived for the functions $s_i(p_1, p_2, \ldots)$ and $p_i(s_1, s_2, \ldots)$. There is an asymptotic approach that is equivalent to a quantum-mechanical semiclassical approximation, which is asymptotically valid in the number of zeros for high-order modes. In addition, three exactly solvable cases are known: (i) isothermal, (ii) isochoric, and (iii) isentropic. The first applies to the problem of oscillations of an exponentially distributed, incompressible fluid. This problem was solved in a classical paper by Rayleigh (1883). Oscillations of a star with an unperturbed constant density were calculated in the well-known paper of Pekeris (1938); see also Cox (1980, s. 17.7). The isentropic case was analyzed by Inogamov (1977, 1984, 1985a).

In this paper we propose a new, exactly solvable model. This model does not apply to the above cases (i), (ii), and (iii). It incorporates (ii) and (iii) as special cases. According to this model, the classification of solvable cases looks as follows: there are (A) exponential distributions of entropy $s = p/\rho^\gamma$, which are exhausted by case (i), and (B) power-law entropy distributions, which are covered by the model. Thus, our model is based on the solution of a spectral problem of perturbations of an arbitrary polytrope. It makes it possible to classify oscillations, determine the positions of zeros of all eigenfunctions, and bridge the gap between the descriptions of low- and high-order modes in the asymptotic theory. A comparison with experimental data on solar oscillations shows that the model is fairly consistent with them.

It is well known [see, e.g., the monographs of Unno *et al.* (1979), Cox (1980), and Vorontsov and Zharkov (1981, 1989)] that the adiabatic approximation can be used to describe stellar oscillations. The dynamical equations that express the laws of conservation of mass,

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momentum, and energy in this approximation in differential form are

$$D_t\rho = 0, \quad (1)$$

$$\rho_t + \mathrm{div}(\rho\mathbf{v}) = 0, \quad (2)$$

$$\rho D_t\mathbf{v} + \mathrm{grad}p - \rho\mathbf{g} = 0, \quad (3)$$

$$\mathrm{div}\mathbf{v} = 0, \quad (4)$$

$$D_t s = 0, \quad (5)$$

where $D_t = \partial t + (\mathbf{v} \cdot \nabla)$ is the Lagrangian derivative. Equations (1), (3), and (4) refer to an inhomogeneous incompressible fluid [in this case, the energy equation replaces the condition of incompressibility (4)], whereas equations (2), (3), and (5) describe a compressible case. The local gravity $\mathbf{g}$ is derived from the gravitational potential φ, which obeys the Poisson equation $\Delta\varphi = 4\pi G\rho$, where G is the gravitational constant. We restrict the analysis to the widely used, fairly accurate Cowling's approximation (Unno *et al.* 1979; Cox 1980; Vorontsov and Zharkov 1989), in which the perturbations of the gravitational potential are ignored. Let us consider the problem in a plane approximation, which is valid for sufficiently high spherical harmonics l (assume that $k = \sqrt{l(l+1)}/R$, where k is the wave number, and R is the radius of the star). We shall disregard the self-gravitation of the outer envelopes, because the mass of a fairly thick convective envelope in the case of the Sun, which is adjacent to its edge and which has a thickness of approximately a third of the solar radius, is known to be about 2% of the total mass of the Sun. In the unperturbed state, the equation of hydrostatics holds: $dP_0/dr = -\rho_0 g$, where $g = |\mathbf{g}|$, and the zero subscript denotes unperturbed quantities. At equilibrium, the medium is at rest. The unperturbed state is completely defined by a single arbitrary function. Indeed, in the compressible case, there are two independent thermodynamic variables and also a hydrostatics equation. In the incompressible and compressible cases, the density and entropy distributions, respectively, are chosen for the above function.

Let us linearize the system of equations (1), (3), and (4) that refer to an incompressible fluid. We denote the density, velocity, and energy perturbations by $\rho(r)$, $u(r)$, $v(r)$, and $p(r)$, respectively. The dependence of the perturbations on the remaining variables is $f(r, x, t) = f(r)\exp(i\omega t + ikx)$. In what follows, x is the coordinate that is transverse to the radius r, and u and v are the velocity components along the x axis and the radius r, respectively. Let us consider the pressure in Lagrangian coordinates. We expand it in small displacements of the Lagrangian particles

$$P(r + \delta r, x + \delta x, t)$$
$$= P_0(r) + \frac{dP_0(r)}{dr}\frac{v(r)}{i\omega}e^{i\omega t + ikx} + p(r)e^{i\omega t + ikx}.$$

Here $\delta\mathbf{r} = \dfrac{\mathbf{v}}{i\omega}$ is the displacement of a Lagrangian particle. The linear term added to the pressure in this particle is

$$p_L(r) = -g\rho_0 v(r)/(i\omega) + p(r). \quad (6)$$

We eliminate the unknown ρ and u from the system, which arises as a result of linearization of equations (1), (3), and (4). We obtain a system of two first-order equations for v and p. If we eliminate v, the remaining equation for p will contain the second derivative d^2p_0/dr^2. Therefore, we eliminate p. The resulting equation contains the first derivative with respect to ρ_0. It is the well-known Rayleigh equation (1883)

$$H_p v''_{\eta\eta} + v'_\eta - (H_\rho + 1/\Omega^2)v = 0, \quad (7)$$

where the function $H_\rho(\eta) = 1/(\ln\rho_0)'_\eta$ gives the inhomogeneity scale, $\eta = kr$ is the dimensionless radius, and $\Omega = \omega/\sqrt{gk}$ is the dimensionless frequency. Clearly, ρ_0 enters this equation only through H_ρ.

It can be readily seen that equation (7) has the solutions

$$\Omega^2 = 1, \quad v = \exp(\eta), \quad (8)$$

$$\Omega^2 = -1, \quad v = \exp(-\eta), \quad (9)$$

which represent the gravity wave and the mode of Rayleigh–Taylor instability, respectively (Inogamov 1984, 1985a, and 1985b). The modes (8) and (9) are isobaric in the sense that the pressure in the Lagrangian particles remains constant during their motion. It thus follows that these modes are invariant with respect to the density distribution. The mode (8) is closely associated with trochoidal waves (Inogamov 1985b). All these factors suggest that it makes sense to rewrite equation (7) using p_L as an unknown.

For this purpose, we replace p by p_L in the system of two first-order equations for v and p using formula (6). As a result, the equation takes the form

$$v'_\eta - \frac{1}{\Omega^2}v - \frac{ikp_L}{\omega\rho_0} = 0,$$
$$\frac{1}{i\Omega^2}v'_\eta + iv + \frac{k}{\omega}\frac{(p_L)'_\eta}{\rho_0} = 0. \quad (10)$$

Let us eliminate v. Solving system (10) for v, we obtain

$$\left(1 - \frac{1}{\Omega^4}\right)v = \frac{ik}{\omega}\frac{\eta(p_L)'_\eta}{\rho_0} + \frac{1}{\Omega^2}\frac{ikp_L}{\omega\rho_0}. \quad (11)$$

Taking the first derivative of (11) with respect to η, we obtain the expressions for v and v'_η in terms of p_L, $(p_L)'_\eta$, and $(p_L)''_{\eta\eta}$. Substituting these expressions into the first or the second equation of system (10), we obtain the equation

$$H_\rho(p_L)''_{\eta\eta} - (p_L)'_\eta - (H_\rho + 1/\Omega^2)p_L = 0. \quad (12)$$

Let ρ_0 be such that it tends to finite values in the limit $\eta \longrightarrow \pm\infty$. It can then be easily shown that the boundary conditions for v and p_L are

$$v(\pm\infty) = 0, \quad (13)$$

$$p_L(\pm\infty) = 0. \quad (14)$$

Since the unknown quantities v and p_L are related by equation (11), the spectra (sets of eigenvalues) $(\Omega_i^2)_v$ and $(\Omega_i^2)_p$, $i = 1, 2, \ldots$, which refer to the problems (7), (13) and (12), (14), coincide:

$$(\Omega_i^2)_v = (\Omega_i^2)_p, \quad i = 1, 2, \ldots . \quad (15)$$

Let us consider the inversion transformation (Mikaelian 1982; Inogamov 1990; Kull 1991):

$$\hat{\rho}_0(\eta) = 1/\rho_0(-\eta). \quad (16)$$

We denote the variables, which are related to the inversion problem by a circumflex above the letter. In the case of the $\hat{\rho}_0$ distribution [equation (16)], the spectral problem (7), (13) takes the form

$$H_\rho \hat{v}''_{\eta\eta} - \hat{v}'_\eta - (H_\rho + 1/\Omega^2)\hat{v} = 0, \quad (17)$$

$$\hat{v}(\pm\infty) = 0. \quad (18)$$

This can easily be shown using the definition of inversion (16).

The striking hidden symmetry is that, as it turns out,

$$\hat{\Omega}_i^2 = \Omega_i^2, \quad i = 1, 2, \ldots \quad (19)$$

for any distributions of ρ_0! Here $(\hat{\Omega}_i^2)$ is the spectrum of the inversion problem (17), (18). Mikaelian (1982) put forward the hypothesis for the existence of symmetry (19). Inogamov (1990) gave a rigorous, very complicated proof of the isospectral property (19). Kull (1991) developed another approach. Equation (12) for Lagrangian pressure perturbations makes it possible to trivially prove this property. Indeed, the spectral problems for the inversion distribution (17), (18) are literally identical to those for the pressure (12), (14). The required result follows from this identity and from (15).

An important consequence of the proof "in terms of p_L" is that we see not only the invariance of the spectrum of eigenvalues, but also how the eigenfunctions are transformed as a result of inversion. Specifically, the eigenfunctions $\hat{v}$ after the inversion (16) transform into the eigenfunctions v, which are equal to the eigenfunctions p_L. This circumstance remained hidden in the previous proofs.

Let us consider the compressible case [equations (2), (3), and (5)]. The analog of the Rayleigh equation for v turns out to be cumbersome. Given the above discussion of isobaricity and p_L, it makes sense to use the analog of equation (12). We supplement system (2), (3), and (5) with equation (6). As in the incompressible case, eliminating the known quantities u, ρ, p, and v, we obtain

$$H_\rho H_s(p_L)''_{\eta\eta} - H_s(p_L)'_\eta - [H_\rho H_s + H_s/\Omega^2 + H_s/\Omega^2 + (\Omega^2 - 1/\Omega^2)(H_\rho/\gamma + H_s)]p_L = 0, \quad (20)$$

where the local scale height in entropy is $H_s = k(d\ln S_0/dy)^{-1}$.

Let us consider an arbitrary polytrope $P_0 \propto \rho_0^{1+1/n}$. The hydrostatic distribution is

$$S_0 \propto (-\Delta r)^{1-n(\gamma-1)}, \quad (21)$$

$$\rho_0 \propto (-\Delta r)^n, \quad (22)$$

where $\Delta r = r - R$.

We calculate the scale heights H_ρ and H_s using formulas (21) and (22). Substituting the derived expressions into equation (20), we obtain

$$\Omega^2\eta(p_L)''_{\eta\eta} - n\Omega^2(p_L)'_\eta - [\Omega^2\eta + n + \gamma^{-1}(n+1)(\Omega^4 - 1)]p_L = 0. \quad (23)$$

In what follows, $\eta = k\Delta r$. After the replacement $p_L = \exp(-\eta)F(\eta)$, $\eta = x/2$, equation (23) transforms into a Kummer-type equation, which is closely related to the Schrdinger equation for a field with the Coulomb potential. Its general solution is

$$p_L(\eta) = e^\eta[c_1(-\eta)^{n+1} \times F[(a+n+1), (n+2), (-2\eta)] + c_2F(a, -n, -2\eta)], \quad (24)$$

$$a = -(\Omega^2 + 1)[n\gamma + (n+1)(\Omega^2 - 1)]/(2\gamma\Omega^2),$$
$$F(a, b, x) = \sum_{i=0}^{\infty}(a_i/b_i)x^i/i!, \quad (25)$$

where $F(a, b, x)$ is a degenerate hypergeometric function, $a_i = a(a+1)\ldots(a+i-1)$, and $a_0 = 1$. An exhaustive description of this procedure is given in the monograph of Landau and Lifshitz (1974, p. 741).

In order to find the spectrum for Eq. (23), we must impose boundary conditions on the surface and in the depth of the star. We require that $p_L(0) = 0$ and $p_L(-\infty) = 0$ and impose these constraints on the general solution (24). From the conditions of hydrostatic equilibrium [equations (21) and (22)] we find that $n > -1$. From this inequality, together with an analysis of expression (24), and from satisfaction of the condition at the surface it follows that $c_2 = 0$.

In order to use the second boundary condition, we must analyze the asymptotic behavior of the function

The values of γ calculated from (27). The $p_1, p_2, p_3, \ldots$ modes refer to $m = 0, 1, 2, \ldots$, respectively

l \ m	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}
100		1.59	1.61	1.62	1.63	1.64	1.66	1.67	1.69	1.70	1.71
200	1.51	1.47	1.49	1.50	1.52	1.52	1.54	1.57	1.58	1.59	1.60
300	1.40	1.42	1.43	1.45	1.46	1.48	1.54	1.52	1.53	1.55	1.56

$F[(a+n+1), (n+2)$ for $\eta \longrightarrow -\infty$. The asymptotic solution of p_L splits into decaying and increasing exponentials. For the hydrogen atom, this corresponds to the asymptotic behavior of the electron wave function at large distances from the center. It is necessary to find conditions for the resonance between the boundary conditions to eliminate the increasing exponential function. This will give the dispersion relation between ω and k. For $-\eta \gg 1$ we have (Landau and Lifshitz 1974)

$$\frac{F(a, b, x)}{\Gamma(b)} = \frac{G(a, a-b+1, -x)}{\Gamma(b-a)(-x)^a} + \frac{e^x G(b-a, 1-a, x)}{\Gamma(a)x^{b-a}}, \quad G(a, b, x) = \sum_{i=0}^{\infty} \frac{a_i b_i}{i! x^i},$$

where $\Gamma(a)$ is the gamma function; to calculate the powers, the variables x and $-x$ must have the smallest argument in absolute value. We thus obtain

$$p_L \propto (-\eta)^{n+1}\Gamma(n-2)\left\{e^{\eta}\frac{G[a+n+1, a+4, 2\eta]}{\Gamma(-a-3)(2\eta)^{a+n+1}} + e^{-\eta}\frac{G[-a-3, -a-n, -2\eta]}{\Gamma(a+n+1)(-2\eta)^{-a-3}}\right\}, \tag{26}$$

where a is given by formula (25). To eliminate the increasing exponential in (26), we must satisfy the condition $a + n + 1 = -m$, where $m = 0,1,2,\ldots$ is an integer, because the function $\Gamma(a + n + 1)$ has a pole at this value of the argument. Substituting formula (25), we obtain a quadratic equation for Ω^2, whose solutions are

$$\{(\Omega^2)_p\}_j = \frac{n+j\gamma}{n+12} + \sqrt{\left(\frac{n+j\gamma}{n+12}\right)^2 + \frac{\theta}{n+1}} \tag{27}$$

for a family of acoustic modes and

$$\{(\Omega^2)_g\}_0 = 1,$$
$$\{(\Omega^2)_g\}_j = \frac{n+j\gamma}{n+12} - \sqrt{\left(\frac{n+j\gamma}{n+12}\right)^2 + \frac{\theta}{n+1}} \tag{28}$$

for a family of gravity modes. Here $\theta = 1 - n(\gamma - 1)$, $j = 2m + 2$, and $m = 0, 1, 2, \ldots$. Curiously, the increase in the degree of stratification instability, i.e., the increase in θ, results in a rise of the fan of acoustic modes, which corresponds to their hardening. The spectrum (28) for gravity modes is supplemented with the fundamental mode $\{(\Omega^2)_g\}_0$, for which the corresponding eigenfunction is $\{p_L\}_0 \equiv 0$. It is therefore absent in the spectrum of the spectral problem for p_L. The remaining functions that correspond to this mode are finite. From (27) and (28) we find that the eigenfunctions of the modes p and g are

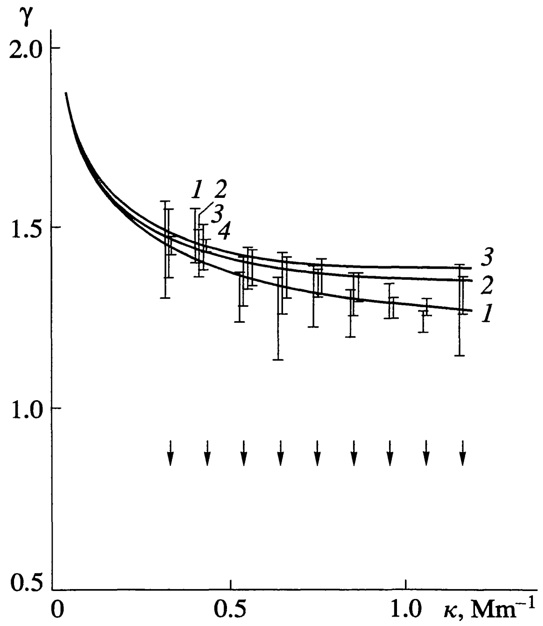
$$p_L(\eta) \propto (-\eta)^{n+1}e^{\eta}F(-m, n+2, -2\eta) = \frac{(-\eta)^{n+1}e^{\eta}m!L_m^{(n+1)}(-2\eta)}{(n+2)(n+3)\ldots(n+m+1)},$$

where $L_m^{(n+1)}$ are the generalized Laguerre polynomials.

Expressions (27) and (28) are compact analytical formulas which can be used to calculate the oscillation frequencies $\omega = \Omega\sqrt{gk}$ from the wave number k, mode number m, and the parameters n and γ. Let us check them against the results of typical numerical calculations and against experimentally measured frequencies of the best studied, five-minute, solar p oscillations. Since the entropy gradients inside a thick convection zone are small, we set $n = 1/(\gamma - 1)$ in (27) and fix k. We take from the literature the set of p-mode frequencies $\omega_1, \omega_2, \ldots$ that correspond to this k. The substitute these test values into (27) and calculate $\gamma(m)$ at various values $m = 0,1,2, \ldots$ for the quantity $\gamma(m)$ ($m = 0,1,2$, etc. refer to the modes $p_1, p_2, p_3, \ldots$, respectively). If the tested formulas are unrelated to these spectra, then varying m over a wide range would result in variations of $\gamma(m)$ over the same range.

Our results are summarized in the figure and the table. We carried out the check over a wide range of variations in k. The vertical bars in the figure indicate $\gamma(m)$ values which were obtained by recalculating the experimental data of Deubner (1975). The height of the bar is determined by the error of the frequency measurement. Since Deubner took measurements for nine values of k, there are nine groups of bars. Each group refers to its "own" value of k, which is marked with a vertical arrow under this group. Each group includes several closely spaced bars. The extreme left bar refers to the p_1 mode, the next bar refers to the p_2 mode, etc. Clearly, there is no disagreement among $\gamma(m)$ for various values of m. On the contrary, all of these values are in good agreement within the experimental error.

The recalculated dispersion relations of Ando and Osaki (1975) are indicated in the figure by the solid



The values of γ calculated from (27). The vertical bars represent experimental data, the curves indicate numerical calculations, the numbers are $m + 1$.

lines. The numbers beside the curves are p-mode numbers. It can be seen from the figure that the divergence of the curves $2|\gamma(3) - (2)|/[\gamma(3) + \gamma(2)]$ and $2|\gamma(2) - (1)|/[\gamma(2) - \gamma(1)]$ is small and does not exceed 5–7%.

The table compares our results with typical current experimental relations $\omega(l)$. We took the experimental data from Fig. 3 in the review of Harvey (1995). The table gives the values of γ as a function of harmonic number l for a sequence of $p_1, p_2, \ldots$ modes. An analysis of the data given in the table shows that the values of γ remain approximately constant when varying m over a rather wide range. This leads us to conclude that the derived formulas are in satisfactory agreement with known numerical and experimental results.

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