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Mach's principle from the relativistic constraint equations

D. Lynden-Bell,[1] J. Katz[2] and J. Bičák[3]

[1] *Institute of Astronomy, The Observatories, Madingley Road, Cambridge CB3 0HA*
[2] *Racah Institute of Physics, Hebrew University at Jerusalem, Jerusalem 91904, Israel*
[3] *Department of Theoretical Physics, Charles University, Prague, Czech Republic*

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ABSTRACT

The history and meaning of Mach's principle are reviewed. In rotating and expanding nearly spherical distributions, the potential $\boldsymbol{\omega}(r, t)$ that governs the rotations of the local inertial frames is instantaneously related to the angular momentum distribution $\boldsymbol{J}(\leq r, t)$ by $\boldsymbol{\omega}(r, t) = 2Gc^{-2}(W\boldsymbol{J}r^{-3} + \int_r^\infty Wr^{-3}\partial \boldsymbol{J}/\partial r\, \mathrm{d}r)$, where $W(r, t)$ is a weight function related to the unperturbed metric, which is unity for flat space. In closed universes where r cannot reach infinity, only relative rotations are meaningful, but the relative angular velocities of inertial frames at different points and of inertial frames relative to matter can be found in terms of the angular momentum and the distribution and motion of the matter.

A general non-linear discussion of conservation laws yields a general proof that the total angular momentum (and the total of any conserved quantity the current of which derives from a superpotential) must be zero in any closed universe. By looking at conservation laws for angular momentum and quasi-linear momentum in perturbed Robertson–Walker universes, it is shown that Mach's principle follows from the constraint equations of General Relativity, provided that the universe is closed. Closed spaces with little mass are *not* almost flat, but exist for only short times and are always small; thus the conundrum of inertia in almost empty nearly flat spaces disappears if space is closed.

Key words: gravitation – relativity – cosmology: theory.

1 INTRODUCTION

To Newton (1686), absolute space was part of reality. To Leibniz it was a convenience, like a coordinate system, brought in to simplify calculations of the multitude of interactions between bodies that determine their relative motions. Leibniz (1712), Huygens (1690), Berkeley (1712), Mach (1883) and Einstein (1920) were all strongly critical of the idea of absolute space. Newton pointed out that absolute acceleration and absolute rotation can be detected without reference to any astronomical observations. He invoked his thought experiment on the tension in a thread joining two globes at relative rest, and his famous bucket experiment, to demonstrate that rotation is absolute. Much later, however, Mach (1883) pointed out that neither experiment shows what would happen were the rest of the Universe removed, so neither demonstrates that space is absolute rather than determined by distant masses.[1] Mach also asked the very provocative question as to whether the results of Newton's bucket experiment would have been the same 'had the bucket been massive and many leagues thick'. Einstein was much influenced by Mach, and in devising the General Theory of Relativity he hoped that he could embody the idea that 'there is no inertia of mass against space but only inertia of mass against mass'. From General Relativity, Thirring (1918, 1921) and Lense & Thirring (1918) showed that, indeed, a rotating massive bucket many leagues thick would drag around a Foucault pendulum a little. Thus Newton's experiments to determine absolutely non-rotating axes without reference to the surroundings would in fact give incorrect results, which would be different inside and far from such a massive rotating bucket. Brill & Cohen (1966) demonstrated that such dragging becomes complete when the radius of a massive rotating sphere reaches its Schwarzschild radius. Thus Mach's question is fully vindicated. It was at first believed that, as part of a system shrank within its Schwarzschild radius, it would cut itself off from the outside, so that all the inertia at the centre would arise from the collapsing subsystem itself. Cohen (1967), however, later

[1] Indeed, for universes with no total angular momentum, a purely relative classical mechanics which gives the same relative motions as Newtonian mechanics can be found (Zanstra 1922; Lynden-Bell 1992, 1994).

showed that it is the infinite stress necessary to keep the sphere nearly static that leads to the perfect dragging. This led Lindblom & Brill (1974) to investigate rotational dragging by a massive spherical shell in free fall under its own gravity. Their investigation showed the remakable result that the inertial frame inside such an infalling slowly rotating shell rotates uniformly at each moment. There is no time delay between the inertial frame's rotation just inside the sphere and that at the centre. Such a result is consistent with Wheeler's (1964) interpretation, in which Mach's principle follows from the initial value constraints in the $3+1$ ADM formulation together with the condition that the 3-space is closed.

Einstein (1918) originally invented the idea of a closed space in order to eliminate boundary conditions at spatial infinity. 'Flat at infinity' boundary conditions bring with them an inertial frame at infinity unrelated to the mass–energy content of the space. Einstein insisted that the postulate of the relativity of inertia *cannot* be satisfied by merely demonstrating that mass–energy *influences* inertia. The mass–energy must entirely cause it. While the field equations of General Relativity are not inconsistent with this idea, they have solutions that are inconsistent (such as empty flat space), so some restrictive boundary conditions are essential if the idea is to be maintained. Einstein originally hoped that his cosmical constant Λ would automatically close the space and eliminate the need for boundary conditions, but de Sitter's (1917) solution demonstrated that extra conditions were still needed. Attempts to define them by Altshuler (1967), Lynden-Bell (1967), Sciama, Waylen & Gilman (1969) and Raine (1975, 1981) all use interesting integral solutions of Einstein's equations via Green's functions. Unfortunately, they all fail to attribute any inertial influence to the energy and momentum contained in gravitational waves. Contrariwise, Einstein's (1906) thought experiment on the equivalence of mass and energy as applied to light transmitted from one body to another applies just as cogently to the energy of gravitational waves. Thus the energy of gravitational waves must contribute to inertia.

Barbour & Bertotti (1977) gave a purely relative yet non-relativistic dynamics, but one of the effects predicted by their theory is contradicted by the very accurate NMR experiments by Hughes, Robinson & Beltren-Lopez (1960) and by Drever (1961). Barbour's (1994) subsequent work on relativistic theories has led him to the conclusion that *all* solutions of Einstein's equations are Machian. We disagree because, following Mach, we insist that flat empty space is not!

At the recent Tübingen conference on Mach's principle, different participants used that title with different meanings. Here we shall adapt the beautifully clear discussion in chapter IV of Bondi's (1952) book 'Cosmology'. By Mach's principle we mean that: 'All motions, velocities, rotations and accelerations are relative. Local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages of the apparent motions'.

In this paper we show how such averages are to be taken for perturbed spherical universes. In particular, we show how the relative rotations and accelerations of the local inertial frames at different places are determined by the distribution of energy and momentum in the universe. We also show how the rotation of the local inertial frame relative to the matter distribution is determined. Lausberg (1969) gave a fine discussion for a static Einstein universe perturbed by rotating spheres.

Sections 2.1–2.4 and 2.5 give our solutions for general spherical distributions in which each sphere rotates rigidly but changes its rotation rate as it expands or contracts. Section 4 gives our solutions for general perturbations of homogeneous Robertson–Walker universes. This is considerably more general than Klein's (1993) nice result which shows how a slowly rotating shell may be incorporated into a Friedmann–Robertson–Walker background. Pfister & Braun (1985a,b) have generalized the slowly rotating sphere analysis to second order in Ω to discuss centrifugal forces. To get the Newtonian result, the space within the sphere must be flat to second order in Ω but rotating relative to infinity. In Section 3 we give a general discussion of superpotentials and conservation laws without any smallness assumptions. We do, however, imagine a mapping of the real Universe on to an idealized symmetrical one. This section gives our general proof that the angular momentum of any closed universe is zero (cf. Cohen & Wald 1972; King 1990). Embacher (1988) has demonstrated that both dragging and centrifugal effects occur with the correct ratio within systems of rotating cylinders.

Anti-Mach solutions of Einstein's equations have been found from time to time. In the paper that gave the metric of his universe, Gödel (1949) also discussed others derived by similar methods. Building on these clues, Ozsvath & Schücking (1962, 1969) discovered a closed universe in which the matter rotated everywhere relative to the compass of inertia. This was claimed to disprove Mach's principle. It may well disprove the version of Mach's principle in which the local inertial frame is to be associated only with the angular momentum distribution of the *matter* tensor but, as von Hönl & Dehnen (1966) and, more recently, Wheeler & King (in King 1990) have pointed out, while the total *matter* angular momentum of this closed universe is non-zero, it is cancelled out by the angular momentum contained in the gravity field in the form of a long-wavelength gravitational wave. Thus Mach's principle must involve all of the energy and momentum distribution, whatever its form.

2 INERTIAL FRAMES IN SLOWLY ROTATING COLLAPSING OR EXPANDING SPHERES

2.1 Systems of spheres rotating about one axis

Lindblom & Brill (1974) have given the solution to Einstein's equations for a spherical dust shell with angular momentum J (assumed small) and radius $r_s(t)$. Inside the shell, space–time is flat. Outside, the metric is perturbed Schwarzschild:

$$ds^2=(1-2m/r)\,dT^2-(1-2m/r)^{-1}\,dr^2 - r^2[d\theta^2+\sin^2\theta(d\phi-\omega\,dT)^2], \tag{2.1}$$

where the frame-dragging potential $\omega(r)$ is given by

$$\omega(r)=2Gc^{-2}J/r^3. \tag{2.2}$$

Within the sphere we could take the metric to be flat,

$$ds^2=d\bar{t}^2-dr^2-r^2(d\theta^2+\sin^2\theta\,d\bar{\phi}^2),$$

but it is better to convert to rotating coordinates so chosen that they agree with the external coordinates on the shell; to do this we write

$$\bar{t}=\bar{t}(T), \qquad \bar{\phi}=\phi-\int \omega_s \, dT, \tag{2.3}$$

where $\omega_s=\omega(r_s)$ and r_s is the T-dependent radius of the shell.

On the moving shell itself, the reduced metric can be deduced from either the outside or the inside metric, so

$$d\sigma^2=[1-2m/r_s-(1-2m/r_s)^{-1}\dot{r}_s^2]\,dT^2-r_s^2\,d\Omega_s^2$$
$$=(\dot{\bar{t}}^2-\dot{r}_s^2)\,dT^2-r_s^2\,d\Omega_s^2, \tag{2.4}$$

where the dots denote d/dT, and

$$d\Omega_s^2=d\theta^2+\sin^2\theta(d\phi-\omega_s\,dT)^2.$$

$\dot{\bar{t}}^2$ may be evaluated in terms of $\dot{r}_s^2$ by equating the coefficients of dT^2 in $d\sigma^2$.

In any metric of the form

$$ds^2=e^{2\nu}\,dt^2-e^{2\lambda}\,dr^2-r^2[d\theta^2+\sin^2\theta(d\phi-\omega\,dt)^2], \tag{2.5}$$

with ω small, the dragging of inertial frames may be determined from the potential ω by thinking of $e^{-2\nu}g_{0\phi}=e^{-2\nu}\omega r^2\sin^2\theta$ as the ϕ-component of a 3-vector potential, $\boldsymbol{A}$, with no r- or θ-components. Using the three-dimensional curl from Landau & Lifshitz (1975), the local inertial frame rotates with a locally defined angular velocity of

$$e^{-\nu}\boldsymbol{\omega}_I=\frac{1}{2}ce^{\nu}\nabla\times\boldsymbol{A}. \tag{2.6}$$

The factor $e^{-\nu}$ is the apparent increase of the rotation rate due to the gravitational slowing of local clocks relative to those at infinity. Inside the shell, e^{ν} is constant, so $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}$ are equal. Just outside the shell, these rates are still equal at the pole, but the inertial drag-rate changes with latitude and, like the magnetic field of a dipole, is reversed at the equator.

In this paper we shall be primarily concerned with determining $\boldsymbol{\omega}$ from the distribution of matter and motion. The transformations of the inertial spin rates $\boldsymbol{\omega}_I$ between different rotating axes are complicated by de Sitter and Thomas precessions, since differently moving gyroscopes define $\boldsymbol{\omega}_I$ in the different axes. In place of considering these complications, we confine our attention to the poles where $\boldsymbol{\omega}$ and $\hat{\boldsymbol{r}}$ are parallel, for there $\boldsymbol{\omega}_I=\boldsymbol{\omega}$, so the prediction of $\boldsymbol{\omega}$ itself is sufficient.

We now generalize the Lindblom–Brill problem, and discuss a system of many spherical shells with slow rotations. The unperturbed solution between any two shells is always of Schwarzschild form, and the perturbed solution can always be put in the form (2.1) with $\omega(r)$ given by (2.2) and J_i the angular momentum within and on the inner shell. Let m_i be the Schwarzschild mass parameter in the region between the ith and the $(i+1)$th shells. Then our solution in $r_i(t)\le r\le r_{i+1}(t)$ is of the form

$$ds^2=(1-2m_i/r)\,dT_i^2-(1-2m_i/r)^{-1}\,dr^2$$
$$-r^2[d\theta^2+\sin^2\theta(d\bar{\phi}_i-\bar{\omega}_i\,dT_i)^2], \tag{2.7}$$

where $\bar{\omega}_i=2Gc^{-2}J_i/r^3$. The transformation

$$T_i=T_i(t), \qquad \bar{\phi}_i=\phi-\int\Delta\omega_i\,dt$$

yields

$$ds^2=e^{2\nu}\,dt^2-e^{2\lambda}\,dr^2-r^2[d\theta^2+\sin^2\theta(d\phi-\omega_i\,dt)^2], \tag{2.8}$$

where

$$e^{2\nu}=(1-2m_i/r)\,\dot{T}_i^2=e^{-2\lambda}\dot{T}_i^2, \tag{2.9}$$

and

$$\omega_i=2Gc^{-2}J_i\dot{T}_ir^{-3}+\Delta\omega_i. \tag{2.10}$$

Outside the last shell we have $\Delta\omega_N=0$, $\dot{T}_N=1$, and dots denote d/dt. We now choose $\Delta\omega_i$ so that the dragging terms agree across r_{i+1}. Then

$$2Gc^{-2}J_i\dot{T}_ir_{i+1}^{-3}+\Delta\omega_i=2Gc^{-2}J_{i+1}\dot{T}_{i+1}r_{i+1}^{-3}+\Delta\omega_{i+1}.$$

Hence, for all i,

$$2Gc^{-2}r_{i+1}^{-3}(J_{i+1}\dot{T}_{i+1}-J_i\dot{T}_i)=-(\Delta\omega_{i+1}-\Delta\omega_i). \tag{2.11}$$

We now pass to the continuous limit of very many shells, and obtain, using (2.11), (2.9) and (2.10),

$$2Gc^{-2}r^{-3}\frac{\partial}{\partial r}(Je^{\lambda+\nu})=-\frac{\partial(\Delta\omega)}{\partial r}=-\frac{\partial}{\partial r}(\omega-2Gc^{-2}r^{-3}Je^{\lambda+\nu}). \tag{2.12}$$

Hence

$$\partial\omega/\partial r=-6Gc^{-2}e^{\lambda+\nu}J/r^4. \tag{2.13}$$

$J(\le r,t)$ is the total angular momentum within r, and $\omega(r,t)$ is the continuous perturbation of the metric replacing $\omega_i(r,t)$. Equation (2.13) has been derived by one of us directly from Einstein's equations, allowing for a more general $T_{\mu\nu}$ than that appropriate for freely falling spheres (i.e. pressures or anisotropic pressures in the spherically symmetrical unperturbed flow). If we assume an open universe and take the appropriate boundary condition that $\omega\to 0$ at infinity, we deduce, from (2.13),

$$\omega=\int_r^\infty 6Gc^{-2}e^{\lambda+\nu}r^{-4}J\,dr. \tag{2.14}$$

It is instructive to integrate this expression by parts.

Define a weight function $W(r,t)$ by

$$W=r^3\int_r^\infty 3e^{\lambda+\nu}r^{-4}\,dr. \tag{2.15}$$

Then W would be 1 for a flat space where λ and ν are zero and

$$\omega(r,t)=2Gc^{-2}\left(Wr^{-3}J+\int_r^\infty Wr'^{-3}\,\partial J/\partial r'\,dr'\right). \tag{2.16}$$

This tells us how much dragging comes from within r and how much comes from outside. $\partial J/\partial r'\,dr'$ is the angular momentum contained in the region between r' and $r'+dr'$.

2.2 Instantaneous relationship of $\boldsymbol{\omega}$ to $\boldsymbol{J}$

Notice that ω is a function of r and t only, and is directly related to the distribution of J at the same time t by equation (2.16). No time delay is involved. This very interesting conclusion was pointed out by Lindblom & Brill. Although we have picked definite coordinates to get the particular expression given above, similar expressions can be obtained with different time-slicings. There is nothing very special about this instantaneity in these particular coordinates. It follows from the superpotential which generates angular momentum conservation in General Relativity discussed in Section 3. The consistency of the angular momentum distribution and the rotations of the local inertial frames follow from the constraint equations of the initial-value problem of General Relativity, just as Gauss's theorem, relating the electric flux out of a closed surface to the charge within it, follows from the constraints of the initial-value problem in electrodynamics. The equation relating ω to J in the different 'comoving' time-slicing of the Robertson–Walker universes is given in Section 4.

2.3 Slow rotation in a closed universe

When we deal with a closed universe there is no 'infinity' to give an 'absolute' standard of rotation, and there is no place where we give a zero-point for ω. Thus (2.14) is inappropriate, and we see from (2.13) that an arbitrary constant can be added to any solution. For a closed universe, however, there will be two points where $r=0$ rather than one, and we need to be more careful in defining J because there are two regions with radius less than r. It makes things much simpler to take a new labelling $\chi(r, t)$ which runs from 0 to π, such that $\sin\chi = r/r_{\max}$. Then even when the universe is inhomogeneous we may use the two values of χ at given r to distinguish the two different spheres of radius r. Now r reaches its maximum at $r_{\max}$, so $\partial r/\partial s = 0 = e^{-\lambda}$ there and e^{λ} is infinite there. This infinity is removed by using χ, since $\partial\chi/\partial s$ is finite at $\chi = \pi/2$. (2.13) may now be rewritten

$$(\partial\omega/\partial\chi)_t = -6Gc^{-2}(e^{\lambda}\partial r/\partial\chi)\,e^{\nu}J/r^4, \tag{2.17}$$

where J is the total angular momentum in the region with χ less than some chosen value. This is the region that corresponds to the 'inside' of Lindblom & Brill's shells when we generalize to a closed universe. Admittedly, for $\chi > \pi/2$ this 'inside' is larger than the 'outside' region, but we are considering fluxes through the surfaces χ = constant so we must keep the normals continuously pointing in the direction of increasing χ through $\chi = \pi/2$ and beyond. We may integrate (2.17) just as we integrated (2.13):

$$\omega(\chi, t) - \omega(\chi_*, t) = \int_{\chi}^{\chi_*} 6Gc^{-2}e^{\lambda+\nu}r^{-4}J\frac{\partial r}{\partial\chi'}\,d\chi'. \tag{2.18}$$

We may likewise define a weight function

$$W_* = r^3\int_{\chi}^{\chi_*} 3e^{\lambda+\nu}r^{-4}\,\frac{\partial r}{\partial\chi'}\,d\chi', \tag{2.19}$$

and thence write $\omega - \omega_*$ in a form similar to (2.16):

$$\omega - \omega_* = 2Gc^{-2}\left[JW_*r^{-3} + \int_{\chi}^{\chi_*} W_*r^{-3}\,\frac{\partial J}{\partial\chi'}\,d\chi'\right]. \tag{2.20}$$

To define $\omega(r, t)$ itself, we lack an integration 'constant' $\omega_0(t)$. *Thus with no infinity to give an absolute rotation, the angular momentum distribution gives only the relative rotations of the inertial frames at different points, in full conformity with Mach's principle.*

An interesting deduction from equation (2.17) is that J must be zero when $r=0$ because $\partial\omega/\partial\chi$ should be zero there. Furthermore, if $\partial\omega/\partial r$ is regular there, J should be $O(r^5)$. While this follows naturally at $\chi=0$, it is only consistent at $\chi=\pi$, where r is again 0, if $J(\pi)=0$. This is a special case of a general theorem we shall prove presently (see also King 1990; also Barbour 1994), that the angular momentum of any closed universe is necessarily zero. This second boundary condition arises naturally in any closed universe because there are two points $r=0$ where $\partial\omega/\partial r$ must vanish, instead of only the one 'origin' of an open universe.

2.4 The relationship of inertial frames to angular velocities

For material of rest density $\rho_0(r, t)$ falling with radial velocity $v(r, t)$ and rotating slowly with angular velocity $\Omega(r, t)$, the relevant component of the energy momentum tensor is

$$T_3^0 = T_\phi^t = \rho_0 r^2\sin^2\theta(\Omega-\omega)/(1-v^2/c^2); \tag{2.21}$$

hence the angular momentum within r is given by

$$J = \int_0^r\int\int T_\phi^t\sqrt{-g}\,d\theta\,d\phi\,dr$$

$$= 2\pi\int_0^r\int_0^\pi \frac{\rho_0}{(1-v^2/c^2)}(\Omega-\omega)\,r^4e^{\lambda+\nu}\sin^3\theta\,d\theta\,dr \tag{2.22}$$

$$= \frac{8}{3}\pi\int_0^r \frac{\rho_0(\Omega-\omega)}{(1-v^2/c^2)}\,r^4e^{\lambda+\nu}\,dr.$$

Although Ω changes if we decide to measure the shells' rotations relative to rotating axes, ω also changes so that $\Omega-\omega$ remains unchanged. Thus the angular momentum is unchanged when one views the system from rotating axes. Rather, such a view shifts the relative contributions of the particle movement, Ω, and the frame dragging, ω, to the angular momentum. In Mach's principle, we wish to relate the dragging not merely to the angular momentum distribution but also to the distribution of the angular velocity of the 'stars' about us. To do this, we wish to relate ω to the Ω-distribution. Inserting expression (2.22) for J into equation (2.13), multiplying by $r^4e^{-\lambda-\nu}$ and differentiating, we obtain a differential equation for ω in terms of Ω:

$$\frac{1}{r^4}e^{-\lambda-\nu}\frac{\partial}{\partial r}\left(r^4e^{-\lambda-\nu}\frac{\partial\omega}{\partial r}\right) = -\frac{16\pi Gc^{-2}\rho_0(\Omega-\omega)}{(1-v^2/c^2)}. \tag{2.23}$$

Writing D^2 for the operator on the left and putting

$$K^2(r, t) = 16\pi Gc^{-2}\rho_0/(1-v^2/c^2), \tag{2.24}$$

this equation becomes

$$(D^2-K^2)\,\omega = -K^2\Omega. \tag{2.25}$$

This equation must be solved under the boundary condition that ω should tend to zero at infinity, assuming that there is either no matter or no rotation (Ω) there, and $\partial\omega/\partial r$ must be zero at $r=0$. Once again, when the universe is closed we may expect to be denied any 'natural' zero-point for ω, but now a remarkable thing happens. If the matter $\Omega(r, t)$ is given,

$\omega(r, t)$ *can* be determined uniquely. One boundary condition is given by $\partial\omega/\partial r \to 0$ at $r=0$, but the weighted average of ω follows from the condition that the total angular momentum of the universe be zero. Thus

$$\int_0^\pi \frac{\rho_0 r^4 \omega e^{\lambda+\nu}}{(1-v^2/c^2)} \frac{\partial r}{\partial \chi} d\chi = \int_0^\pi \rho_0 \frac{r^4 \Omega e^{\lambda+\nu}}{(1-v^2/c^2)} \frac{\partial r}{\partial \chi} d\chi, \tag{2.26}$$

and this equation effectively provides the zero-point of ω that was missing from the equation relating ω to J. Both (2.26) and (2.25) are of course invariant if the same r-independent $\omega_0(t)$ is added to *both* ω and Ω. So, at least for small rotations, any rotating axes give the same relationships between the inertial rotation rates and the motion of the mass-energy across the sky. However, this invariance arises only in a closed universe. In the open ones, different axes lead to different boundary conditions at infinity. For the very special case in which the unperturbed universe is Einstein's static universe, we may actually solve equation (2.25) and inspect the solutions. Appendix A generalizes Lausberg's (1969) treatment to vector $\mathbf{\Omega}$.

2.5 Misaligned rotations

$$ds^2 = (1-2m/r)\, dT^2 - (1-2m/r)^{-1} dr^2 - r^2(d\hat{\boldsymbol{r}} - \boldsymbol{\omega}\times\hat{\boldsymbol{r}}\, dT)^2,$$

where $\hat{\boldsymbol{r}}$ is the unit 'vector' that transforms like a Cartesian vector at ∞ and whose (x, y, z)-components are $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$. Notice that

$$d\hat{\boldsymbol{r}} = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)\, d\phi$$
$$+ (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\, d\theta,$$

so

$$(d\hat{\boldsymbol{r}})^2 = \sin^2\theta\, d\phi^2 + d\theta^2,$$

and, when $\boldsymbol{\omega} = (0, 0, \omega)$,

$$d\hat{\boldsymbol{r}}\cdot\boldsymbol{\omega}\times\hat{\boldsymbol{r}} = \omega\sin^2\theta\, d\phi,$$

so the general result reduces to Lindblom & Brill's by orientating the z-axis along $\boldsymbol{\omega}$. In our case, $\boldsymbol{\omega} = 2Gc^{-2}\boldsymbol{J}/r^3$, where $\boldsymbol{J}$ is the total angular momentum within and in the shell. Since linear perturbations may be added, we merely find vectorial forms of equations (2.10) and (2.11) which yield the vectorial form of (2.13), to wit

$$\partial\boldsymbol{\omega}/\partial r = -6Gc^{-2}e^{\lambda+\nu}\boldsymbol{J}/r^4.$$

Integrating with W defined as before,

$$\boldsymbol{\omega} = 2Gc^{-2}\left(W\boldsymbol{J}/r^3 + \int_r^\infty Wr^{-3}\, \partial\boldsymbol{J}/\partial r\, dr\right).$$

Similarly, for closed universes, there is a 'vector' form of (2.20) with $\boldsymbol{\omega}$, $\boldsymbol{\omega}_*$, $\boldsymbol{J}$ and $\partial\boldsymbol{J}/\partial\chi'$ all 'vectors'. 'Vector' forms of (2.23) and (2.25) may also be derived.

3 CONSERVATION LAWS

3.1 Varying action in Special Relativity

In exploring conservation laws derived via Noether's theorem from action principles, the physics of what is being done is not always clearly explained. This is especially true in General Relativity, when as in our case there are two spaces involved, described by metrics ds^2 and $d\bar{s}^2$. To keep the physics to the fore, we start with a Lagrangian which gives the action for a matter field in Special Relativity, but, to make the transition to General Relativity easier, we use general curvilinear coordinates and general covariance even for this. If X^α are special Minkowski coordinates of the flat space and $A_\beta(X^\alpha)$ is a vector field describing the matter, then the action is of the form

$$S_F = \int_\Omega \mathscr{L}_F\left[A_\beta(X^\alpha), \frac{\partial A_\gamma}{\partial X^\beta}\right] d^4X, \tag{3.1}$$

where $\mathscr{L}_F$ is the Lagrangian for a massive vector field A_β. We now rewrite this Lagrangian in general coordinates x^μ by taking $X^\alpha = X^\alpha(x^\mu)$.

$$\hat{\mathscr{L}}_F = \mathscr{L}_F[a_\sigma(x^\mu), \bar{D}_\nu a_\sigma, \bar{g}_{\mu\nu}]\sqrt{-\bar{g}}, \tag{3.2}$$

where

$$\bar{g}_{\mu\nu} = \eta_{\alpha\beta}\, \partial_\mu X^\alpha\, \partial_\nu X^\beta, \tag{3.3}$$

$$a_\sigma = \partial_\sigma X^\alpha A_\alpha(X^\beta), \tag{3.4}$$

and the $\sqrt{-\bar{g}}$ arises because

$$d^4X = |\partial X/\partial x|\, d^4x = \sqrt{-\bar{g}}\, d^4x. \tag{3.5}$$

In the above, $\bar{D}_\nu$ stands for the covariant derivative based on the metric $\bar{g}_{\mu\nu}$, and ∂_μ stands for $\partial/\partial x^\mu$. Our action is now

$$S_F = \int_\Omega \hat{\mathscr{L}}_F[a_\rho, \bar{D}_\mu a_\rho, \bar{g}_{\mu\nu}]\, d^4x. \tag{3.6}$$

We henceforth regard a_ρ and X^α as functions of x^μ, and consider infinitesimal changes δa_ρ and δX^α of these functions as independent; x^μ is kept unchanged. Putting

$$\xi^\nu = \partial x^\nu/\partial X^\alpha\, \delta X^\alpha, \tag{3.7}$$

the change in $\bar{g}_{\mu\nu}$ is

$$\delta\bar{g}_{\mu\nu} = \bar{D}_\mu\xi_{\bar{\nu}} + \bar{D}_\nu\xi_{\bar{\mu}} = 2\bar{\zeta}_{\mu\nu}. \tag{3.8}$$

In what follows, a hat denotes a density, so $\hat{\mathscr{L}} = \mathscr{L}\sqrt{-\bar{g}}$.

Varying a_μ and $\bar{g}_{\mu\nu}$ we have, for $\delta\bar{g}_{\mu\nu}$ of the form (3.8),

$$\delta\hat{\mathscr{L}}_F = \hat{\mathscr{L}}^\nu\delta a_\nu + \bar{D}_\mu(\hat{p}^{\mu\nu}\delta a_\nu) + \hat{\alpha} + \partial\hat{\mathscr{L}}_F/\partial\bar{g}_{\mu\nu}\,\delta\bar{g}_{\mu\nu}, \tag{3.9}$$

where

$$\hat{\mathscr{L}}^\nu = \partial\hat{\mathscr{L}}_F/\partial a_\nu - \bar{D}_\mu\hat{p}^{\mu\nu}, \tag{3.10}$$

$$\hat{p}^{\mu\nu} = \partial\hat{\mathscr{L}}_F/\partial(\bar{D}_\mu a_\nu), \tag{3.11}$$

and

$$\hat{\alpha} = -\hat{p}^{\mu\nu}\delta\bar{\Gamma}^\lambda_{\mu\nu}a_\lambda = \frac{1}{2}\hat{s}^{\lambda\mu\nu}\bar{D}_\lambda\delta\bar{g}_{\mu\nu}; \tag{3.12}$$

here

$$\hat{s}^{\lambda\mu\nu} = -\hat{p}^{(\lambda\mu)}a^\nu - \hat{p}^{(\lambda\nu)}a^\mu + \hat{p}^{(\mu\nu)}a^\lambda. \tag{3.13}$$

In the above, round brackets around indices denote symmetrization. The stress–energy tensor $T^{\mu\nu}$ generated by the vector field is given by

$$\frac{1}{2}\hat{T}^{\mu\nu} = \delta\hat{\mathscr{L}}_F/\delta\bar{g}_{\mu\nu} = \partial\hat{\mathscr{L}}_F/\partial\bar{g}_{\mu\nu} - \frac{1}{2}\bar{D}_\lambda s^{\lambda\mu\nu}, \tag{3.14}$$

so the varied field Lagrangian gives

$$\delta\hat{\mathscr{L}}_{\rm F}=\hat{\mathscr{L}}^{\nu}\delta a_{\nu}-(\bar{D}_{\mu}\hat{T}^{\mu}_{\nu})\,\xi^{\nu}+\bar{D}_{\mu}(\hat{p}^{\mu\nu}\delta a_{\nu}+\hat{T}^{\mu}_{\nu}\xi^{\nu}+\hat{s}^{\mu\nu}_{\lambda}\bar{D}_{\nu}\xi^{\lambda}). \tag{3.15}$$

We require that the action should be stationary both for all variations $(\delta a_{\mu})_{x}$ and for all ξ^{ν}, so $\hat{\mathscr{L}}^{\nu}$ is zero and also

$$\bar{D}_{\mu}\hat{T}^{\mu}_{\nu}\equiv(\partial_{\nu}a_{\mu}-\partial_{\mu}a_{\nu})\hat{\mathscr{L}}^{\mu}-a_{\nu}\partial_{\mu}\hat{\mathscr{L}}^{\mu}=0. \tag{3.16}$$

$\mathscr{L}^{\nu}=0$ gives the equations governing the evolution of the vector field, and (3.16) states that its energy–momentum tensor is conserved. The special cases when $\boldsymbol{\xi}$ is a Killing vector of the background (in this case the flat space) are especially interesting. Each such Killing vector can be expressed as $f^{a}\boldsymbol{\xi}_{a}$, where the f^{a} are 10 constants and the $\boldsymbol{\xi}_{a}$ are the 10 Killing vectors of displacement and rotation in the 4-space. Any such Killing vector obeys $\bar{D}_{\mu}\xi_{\bar{\nu}}+\bar{D}_{\nu}\xi_{\bar{\mu}}=0$, so such a $\boldsymbol{\xi}$ can be brought within the $\bar{D}_{\mu}$ in the action. Variation of the f^{a}, when $\hat{\mathscr{L}}^{\nu}=0$, then yields the conservation laws

$$\bar{D}_{\mu}\hat{J}^{\mu}_{a}=0, \tag{3.17}$$

where $\hat{J}^{\mu}=\hat{T}^{\mu}_{\rho}\xi^{\rho}_{a}$ is the conserved density of energy, momentum, angular momentum, etc., depending on whether the $\boldsymbol{\xi}_{a}$ chosen is a time displacement, spatial displacement, rotation or Lorentz rotation.

If instead of (3.3) we consider a vector field A_{β} in a *curved* space with metric $\bar{\gamma}_{\alpha\beta}$ instead of $\eta_{\alpha\beta}$, then

$$\bar{g}_{\mu\nu}=\partial_{\mu}X^{\alpha}\partial_{\nu}X_{\beta}\bar{\gamma}_{\alpha\beta}(X^{\gamma}), \tag{3.18}$$

but otherwise (3.3) to (3.8) hold and expression (3.9) for $\delta\hat{\mathscr{L}}_{\rm F}$ remains unchanged.

In General Relativity, $\mathscr{L}_{\rm F}$ depends on $g_{\mu\nu}$ in place of $\bar{g}_{\mu\nu}$. It is thus independent of X^{α}, so displacements in the background are no longer associated with conservation laws via $\mathscr{L}_{\rm F}$. We shall see how they reappear from the gravitational action.

3.2 Backgrounds

Einstein's equations for perturbed metrics inevitably involve mapping the perturbed space $g_{\mu\nu}$ on to the unperturbed background $\bar{g}_{\mu\nu}$. The ability to make this mapping slightly differently generates a gauge freedom and a corresponding gauge invariance in the equations for $g_{\mu\nu}$. When a mapping or one-to-one correspondence between the perturbed and unperturbed spaces has been chosen, it is important to make the convention that corresponding points in the two spaces are always labelled with the same coordinate values. With this convention, we are free to choose coordinates as we like on one of the spaces, but then the coordinates on the other follow. A change of coordinates in one space involves a change of coordinates in the other with the same mathematical functions. With this convention, quantities such as $\Delta g_{\mu\nu}=g_{\mu\nu}-\bar{g}_{\mu\nu}$ become tensors, as does $\Delta^{\mu}_{\nu\sigma}=\Gamma^{\mu}_{\nu\sigma}-\bar{\Gamma}^{\mu}_{\nu\sigma}$, while $\sqrt{g/\bar{g}}$ is a scalar. In all these expressions, the field functions are evaluated at the same values of the coordinates so, for example, g and $\bar{g}$ are evaluated at corresponding points. *There is no necessity for the perturbations to be small.*

3.3 Varying action in General Relativity

In an earlier note, Katz (1985) (see also Katz & Ori 1990) gave a general covariant treatment of conservation laws in a space that is mapped on to a background $\bar{g}_{\mu\nu}$. There are occasions, however, when they specialize to a flat background which we do not wish to impose. Here we shall follow the derivation of Katz & Deruelle (1994) in outline, and give the general results in detail. Two covariant derivatives are defined in the space; D_{μ} and $\bar{D}_{\mu}$ depend on $\Gamma^{\lambda}_{\mu\nu}$ and $\bar{\Gamma}^{\lambda}_{\mu\nu}$ respectively. Even non-linearly, we have the tensor quantities

$$\Delta^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}-\bar{\Gamma}^{\lambda}_{\mu\nu}=g^{\lambda\rho}\left(\bar{D}_{(\mu}g_{\nu)\rho}-\frac{1}{2}\bar{D}_{\rho}g_{\mu\nu}\right) \tag{3.19}$$

and

$$R_{\mu\nu}=\bar{D}_{\lambda}\Delta^{\lambda}_{\mu\nu}-\bar{D}_{\lambda}\Delta^{\lambda}_{\mu\nu}+\Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\sigma}-\Delta^{\rho}_{\mu\sigma}\Delta^{\sigma}_{\rho\nu}+\bar{R}_{\mu\nu}. \tag{3.20}$$

In General Relativity, we add the Lagrangian of the gravitational field $\hat{\mathscr{L}}_{\rm G}$ to $\hat{\mathscr{L}}_{\rm F}$ expressed in terms of $g_{\mu\nu}$ instead of $\bar{g}_{\mu\nu}$: $\hat{\mathscr{L}}=\hat{\mathscr{L}}_{\rm F}+\hat{\mathscr{L}}_{\rm G}$. Hereafter, hats denote multiplication by $\sqrt{-g}$, not $\sqrt{-\bar{g}}$.

Following Misner, Thorne & Wheeler (1973), we take the gravitational action to be $(1/2\kappa)\hat{R}$ plus a divergence that is so chosen as to make $\hat{\mathscr{L}}_{\rm G}$ quadratic in first-order derivatives of $g_{\mu\nu}$. (Here $\kappa=8\pi G/c^{4}$.) Thus

$$\hat{\mathscr{L}}_{\rm G}=\frac{1}{2\kappa}(\hat{R}+\partial_{\lambda}\hat{k}^{\lambda})=\frac{1}{2\kappa}\hat{g}^{\mu\nu}[\bar{R}_{\mu\nu}-(\Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\sigma}-\Delta^{\rho}_{\mu\sigma}\Delta^{\sigma}_{\rho\nu})], \tag{3.21}$$

where

$$\hat{k}^{\lambda}=-(-g)^{-1/2}\bar{D}_{\sigma}(gg^{\lambda\sigma})=\hat{g}^{\lambda\rho}\Delta^{\sigma}_{\rho\sigma}-\hat{g}^{\rho\sigma}\Delta^{\lambda}_{\rho\sigma}. \tag{3.22}$$

Notice that $\hat{k}^{\lambda}$ depends on the mapping, and is therefore gauge-dependent whereas $\hat{R}$ is not. Now the variation of $\hat{\mathscr{L}}_{\rm G}$ is given by variations of both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$:

$$2\kappa\delta\hat{\mathscr{L}}_{\rm G}=-\hat{G}^{\mu\nu}\delta g_{\mu\nu}+2\kappa\partial_{\lambda}[\hat{p}^{\lambda\mu\nu}\delta g_{\mu\nu}-\hat{\Xi}^{\lambda}], \tag{3.23}$$

where

$$\begin{aligned}2\kappa\hat{p}^{\lambda\mu\nu}=&\left(g^{\mu\sigma}g^{\nu\tau}-\frac{1}{2}g^{\mu\nu}g^{\sigma\tau}\right)\hat{\Delta}^{\lambda}_{\sigma\tau}\\&-\left(g^{\lambda(\mu}g^{\nu)\tau}-\frac{1}{2}g^{\mu\nu}g^{\lambda\tau}\right)\hat{\Delta}^{\sigma}_{\sigma\tau},\end{aligned} \tag{3.24}$$

and

$$\Xi^{\lambda}=\hat{g}^{\lambda\rho}\delta\bar{\Gamma}^{\sigma}_{\rho\sigma}-\hat{g}^{\mu\nu}\delta\bar{\Gamma}^{\rho}_{\mu\nu}; \tag{3.25}$$

here

$$\delta\bar{\Gamma}^{\lambda}_{\rho\sigma}=\bar{g}^{\lambda\nu}\left(\bar{D}_{(\rho}\delta\bar{g}_{\sigma)\nu}-\frac{1}{2}\bar{D}_{\nu}\delta\bar{g}_{\rho\sigma}\right), \tag{3.26}$$

and $\delta\bar{g}_{\rho\sigma}$ is given in terms of $\zeta_{\mu\nu}$ by (3.8). Clearly, Ξ^{λ} would be zero if the displacements were Killing displacements of the background $\bar{g}_{\mu\nu}$. We now use the identity of Katz & Ori (1990), extended to non-flat backgrounds [equation (2.6) of that paper], in a slightly modified form. The relevance of this identity follows from the fact that $\hat{\mathscr{L}}$ is a scalar density and so

$$\delta_{\xi}\hat{\mathscr{L}}+\partial_{\nu}(\hat{\mathscr{L}}\xi^{\nu})=0 \tag{3.27}$$

for any displacement field $\boldsymbol{\xi}$ that affects both $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. Equation (3.27) becomes

$$-(2\kappa)^{-1}\hat{E}^{\mu\nu}\delta g_{\mu\nu}+\partial_{\mu}(\hat{I}^{\mu}-\kappa^{-1}\hat{E}^{\mu}_{\nu}\xi^{\nu})=0, \tag{3.28}$$

where

$$\hat{E}^{\mu\nu} = \hat{G}^{\mu\nu} - \kappa \hat{T}^{\mu\nu}, \tag{3.29}$$

and so vanishes whenever Einstein's equation holds and

$$\hat{I}^{\mu} = D_{\nu}\hat{J}^{\mu\nu}, \tag{3.30}$$

where we use the notation that square brackets around indices denote anti-symmetrization and

$$\kappa\hat{J}^{\mu\nu} = D^{[\mu}\hat{\xi}^{\nu]} + \xi^{[\mu}\hat{k}^{\nu]} = -\kappa\hat{J}^{\nu\mu}. \tag{3.31}$$

Thus $\hat{I}^{\mu}$ is derived from the anti-symmetric superpotential $\hat{J}^{\mu\nu}$. A useful form of $\hat{I}^{\mu}$ comes from expanding it in $\bar{D}$ derivatives of ξ^{λ}, which gives the Katz–Ori identity

$$\hat{I}^{\mu} \equiv \partial_{\nu}\hat{J}^{\mu\nu} \equiv \hat{\Pi}^{\mu}_{\nu}\xi^{\nu} + \hat{\Sigma}^{\mu\sigma}_{\nu}\bar{D}_{\sigma}\xi^{\nu} + \hat{\Xi}^{\mu}, \tag{3.32}$$

where Ξ^{μ}, as defined by (3.25) and (3.26), is seen to be linear and homogeneous in the second $\bar{D}$ derivatives of $\boldsymbol{\xi}$, and

$$\Pi^{\mu}_{\nu} = \kappa^{-1}G^{\mu}_{\nu} + t^{\mu}_{\nu}. \tag{3.33}$$

t^{μ}_{ν} is Einstein's pseudo-tensor, which has been made into a real tensor by use of the background metric $\bar{g}_{\mu\nu}$. Specifically,

$$t^{\mu}_{\nu} = \delta^{\mu}_{\nu}\mathscr{L}_{G} + (2\kappa)^{-1}[(\Delta^{\lambda}_{\rho\nu}\Delta^{\mu}_{\lambda\sigma} + \Delta^{\mu}_{\rho\sigma}\Delta^{\lambda}_{\lambda\nu} - 2\Delta^{\mu}_{\rho\lambda}\Delta^{\lambda}_{\sigma\nu})g^{\rho\sigma} + (\Delta^{\lambda}_{\rho\nu}\Delta^{\sigma}_{\sigma\lambda} - \Delta^{\lambda}_{\lambda\nu}\Delta^{\sigma}_{\rho\sigma})g^{\rho\mu}]. \tag{3.34}$$

Finally,

$$2\kappa\Sigma^{\mu\nu}_{\lambda} = \delta^{\mu}_{\lambda}g^{\nu\rho}\Delta^{\sigma}_{\rho\sigma} + g^{\mu\nu}\Delta^{\rho}_{\lambda\rho} - 2\Delta^{\mu}_{\lambda\rho}g^{\rho\nu} - (g^{\mu\rho}\Delta^{\sigma}_{\rho\sigma} - g^{\rho\sigma}\Delta^{\mu}_{\rho\sigma})\delta^{\nu}_{\lambda} \tag{3.35}$$

is a contribution to angular momentum from the spin of the gravitational field. Since I^{μ} derives from a superpotential, its divergence is identically zero for any ξ^{ν} whatsoever. Although these have the appearance of conservation laws, they are identities of no physical content. Now define

$$\hat{J}^{\mu} = (\hat{T}^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu})\xi^{\nu} + \hat{\Sigma}^{\mu\sigma}_{\nu}\xi^{\nu} = \hat{I}^{\mu} - \kappa^{-1}\hat{E}^{\mu}_{\nu}\xi^{\nu} - \hat{\Xi}^{\mu}, \tag{3.36}$$

and notice that $\hat{J}^{\mu}$ becomes equal to $\hat{I}^{\mu}$ when Einstein's equations hold and $\hat{\Xi}^{\mu} = 0$. As noted in (3.26), the latter is true whenever $\boldsymbol{\xi}$ is a Killing vector of the background $\bar{g}_{\mu\nu}$, so the currents J^{μ} corresponding to such Killing vectors are conserved. Notice that the general Killing vector is of the form

$$\xi^{\mu} = f^{a}\xi^{\mu}_{a}, \tag{3.37}$$

where the f^{a} are constants and, here and above, a is an index that runs over the linearly independent Killing vectors. It is interesting to look at our full varied action,

$$\delta S = \int [-(2\kappa)^{-1}\hat{E}^{\mu\nu}\delta g_{\mu\nu} + \hat{\mathscr{L}}^{\nu}\delta a_{\nu}]\,\mathrm{d}^4x + \oint_{\Sigma}[\hat{p}^{\lambda\mu\nu}\delta g_{\mu\nu} + \hat{p}^{\lambda\nu}\delta a_{\nu} + (\hat{\Pi}^{\lambda}_{\nu}\xi^{\nu} + \hat{\Sigma}^{\lambda\sigma}_{\nu}\bar{D}_{\sigma}\xi^{\nu})]\,\mathrm{d}\Sigma_{\lambda}. \tag{3.38}$$

Variations with respect to $\delta g_{\mu\nu}$ and δa_{ν} give Einstein's equations and the field equation for a_{ν}. It is writing ξ^{ν} in the form (3.37) and varying with respect to the parameters f^{a} that give the conservation laws, just as in special relativity.

3.4 The angular momentum of a closed universe is zero

If we map our perturbed space on to a background with symmetries, then each continuous symmetry will generate a conservation law through a corresponding superpotential. Consider any space-like surface Σ with future-pointing vector elementary area $\mathrm{d}^3\Sigma_{\mu}$. Then the total angular momentum within Σ is given by

$$\int \hat{J}^{\mu}_{a}\,\mathrm{d}^3\Sigma_{\mu} = \int_{\Sigma}\partial_{\nu}(\hat{J}^{\mu\nu}_{a})\,\mathrm{d}^3\Sigma_{\mu} = \int_{S}\hat{J}^{\mu\nu}_{a}\,\mathrm{d}^2S_{\mu\nu}, \tag{3.39}$$

where S is the boundary of Σ and $\hat{J}^{\mu}_{a}$ is the angular momentum current. Thus, just as charge can be detected within a surface by the electric flux coming out, so can angular momentum be detected by the flux of its superpotential. Now suppose that the universe is closed, and consider for example Σ to be the interior of some surface S. Now extend Σ to cover more of the space. At first S will grow in area but, after it reaches the radius of the universe, further increase of the volume of Σ can cause the area of S to decrease. Finally, as Σ reaches all parts of the universe at one time, S has shrunk to zero. Equation (3.39) now reads $\int \hat{J}^{\mu}_{a}\,\mathrm{d}^3\Sigma_{\mu} = 0$ for any closed universe. This argument clearly holds for any quantity the conservation of which stems from its being the derivative of a superpotential. This includes all the conservation laws derived in the last section. All such conserved quantities must have zero totals!

4 CONSERVATION LAWS FOR PERTURBATIONS IN A ROBERTSON–WALKER SPACE-TIME

4.1 Simplified form of the conservation laws for linear perturbations

If we deal with small perturbations and take the unperturbed space as a reference background, the conserved vectors for these perturbations, associated with the Killing fields of the background, are defined by equation (3.36). There are, however, significant simplifications if we restrict ourselves to linear perturbations. First (an index 1 means linear or first-order approximation),

$$\theta^{\mu}_{\nu} = \bar{T}^{\mu}_{\nu} + T^{\ \mu}_{\nu(1)}, \tag{4.1}$$

since t^{μ}_{ν} is quadratic in derivatives. Secondly, the spin part is of order 2,

$$\Sigma^{\mu[\rho\sigma]}_{(1)} = 0. \tag{4.2}$$

As a result, the conserved vector is, to first order,

$$J^{\mu} = \bar{J}^{\mu} + T^{\mu}_{\nu(1)}\xi^{\nu} + l^{\rho\sigma}\bar{R}_{\rho\sigma}\xi^{\mu}, \tag{4.3}$$

in which

$$l^{\rho\sigma} \equiv \frac{1}{\sqrt{-\bar{g}}}(\sqrt{-g}\,g^{\rho\sigma} - \sqrt{-\bar{g}}\,\bar{g}^{\rho\sigma}). \tag{4.4}$$

This $l_{\rho\sigma}$ is the $-\bar{h}_{\rho\sigma}$ of Misner et al. (1973), but $\bar{h}$ is obviously not an appropriate notation here. Thus, if

$$g_{\rho\sigma} = \bar{g}_{\rho\sigma} + h_{\rho\sigma}, \qquad |h_{\rho\sigma}| \ll |\bar{g}_{\rho\sigma}|, \tag{4.5}$$

then

$$l_{\rho\sigma} = -\left(h_{\rho\sigma} - \frac{1}{2}\bar{g}_{\rho\sigma}h\right), \tag{4.6}$$

in which

$$h = \bar{g}^{\rho\sigma}h_{\rho\sigma} = \bar{g}^{\rho\sigma}l_{\rho\sigma} = l. \tag{4.7}$$

The perturbed linearized current is explicitly given by the following expression, the result of a straightforward but somewhat tedious calculation:

$$J^{\mu}_{(1)}=J^{\mu}-\bar{J}^{\mu}=[\bar{D}_{\rho\sigma}l^{\rho\sigma}\delta^{\mu}_{\lambda}+\bar{D}^{\rho}(\bar{D}_{\rho}l^{\mu}_{\lambda}-\bar{D}_{\lambda}l^{\mu}_{\rho})-\bar{D}^{\mu}(\bar{D}_{\rho}l^{\rho}_{\lambda})]\,\xi^{\lambda}$$
$$+[2l^{\mu}_{\rho}\bar{R}^{\rho}_{\lambda}-\bar{R}^{\mu}_{\rho}l^{\rho}_{\lambda}-l^{\rho\sigma}\bar{R}^{\mu}_{\rho\sigma\lambda}]\,\xi^{\lambda}. \tag{4.8}$$

Notice that both sides of equation (4.3) have no derivative of ξ^{λ} anymore, and one could have used Misner et al.'s linearized Einstein equations to arrive at equation (4.3) with (4.8). Calculations are, however, not simpler if we start that way.

4.2 Conserved currents on a Robertson–Walker space–time

We shall use standard coordinates to describe the metric (Weinberg 1972) in either this form:

$$d\bar{s}^2=dt^2-a^2(t)\left[\frac{1}{1-kr^2}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right], \tag{4.9}$$

$$k=0,\ \pm 1;$$

or this form:

$$d\bar{s}^2=dt^2-a^2(t)\left[\delta_{kl}+\frac{kx^kx^l}{1-kr^2}\right]dx^k\,dx^l, \tag{4.10}$$

$$r=\sqrt{\sum_k (x^k)^2},$$

where k, l, m, $n=1, 2, 3$. [Notice that this new r differs from that defined earlier, by the factor $a(t)$.] This space has six Killing fields which in such coordinates have only spatial components $\xi^k_a(a,\ b,\ c\ldots=1,\ 2,..,6)$. Rotations in $t=$constant subspaces are associated with three Killing vectors, which in x^k coordinates are

$$\xi^k_a=\varepsilon_{akl}x^l, \qquad a=1, 2, 3 \text{ only}. \tag{4.11}$$

The background space of metric $d\bar{s}^2$ is clearly not translationally invariant, so momentum will not be conserved, but it has both spatial homogeneity and rotational symmetry. Now, in a plane, a rotation through an angle β about a point P, followed by a rotation through an angle $-\beta$ about a displaced point Q, will give the whole space a uniform displacement. Similarly, a rotation of a sphere with a fixed centre at $\bar{O}$ about a point $\bar{P}$, followed by an opposite rotation about a displaced point $\bar{Q}$, leads to a 'quasi-translation'. This looks like a translation if $\overline{PQ}$ is much smaller than the radius of the sphere. Looked at globally, this is a rotation of the sphere about an axis nearly 90° around from $\bar{P}$ and $\bar{Q}$. The quasi-translations of Robertson–Walker spaces are the analogues of this, one dimension higher up with the sphere replaced by a hypersphere. Quasi-translations have Killing vectors of the form

$$\xi^k_a=\delta^k_{a-3}\sqrt{1-kr^2}, \qquad a=4, 5, 6 \text{ only}. \tag{4.12}$$

As a result, conserved quantities at given instants t involve only $T^0_{k(1)}$ and thus only the $(0k)$ components of Einstein's equations, i.e. the spatial 'constraints'. These equations have no time derivatives of h_{00} and h_{0k}, a crucial property for a realization of Mach's idea that local inertial frames are defined with respect to distant matter at any given global time t. The conserved currents in a Robertson–Walker space–time are identically zero. Thus not only are all globally conserved quantities identically zero, but their densities on $t=$constant or any other hypersurface are zero as well. The $J^0_{(1)}$ component in equation (4.8) contains the following components of the curvature tensor (taken from Weinberg) (dots denote derivatives with respect to t):

$$\bar{R}^0_0=-3\frac{\ddot{a}}{a}, \qquad \bar{R}^l_k=\delta^l_k\left(\frac{2}{a^2}+2\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}\right). \tag{4.13}$$

In addition, one readily finds that

$$l^{\rho\sigma}\bar{R}^0_{\rho\sigma k}=-l^0_k\frac{\ddot{a}}{a}=h^0_k\frac{\ddot{a}}{a}. \tag{4.14}$$

With these results inserted into (4.8), and for $\mu=0$, the factor of ξ^k_a becomes

$$\bar{g}^{mn}\nabla_m(\nabla_n l^0_k-\nabla_k l_{0n})-\frac{4k}{a^2}l^0_k-\mathscr{D}_k=2\kappa T^0_k. \tag{4.15}$$

Here ∇_m are 3-covariant derivatives in $t=$constant, and

$$\mathscr{D}_k=\nabla_l\left[l^l_k+\delta^l_k\left(\frac{1}{a}\{al^0_0\}^{\cdot}-\frac{\dot{a}}{a}l^m_m\right)\right]. \tag{4.16}$$

4.3 Equations for h^0_l and Mach's principle for both rotations and quasi-translations

Equation (4.15) can be checked by showing that it is gauge-invariant, i.e. that it is invariant for arbitrary non-Killing displacements in the background $\delta x^{\rho}=\zeta^{\rho}(x)$, for which

$$\delta h_{\mu\nu}=-2\bar{D}_{(\mu}\zeta_{\nu)}=-\bar{g}_{\mu\rho}\partial_{\nu}\zeta^{\rho}-\bar{g}_{\nu\rho}\partial_{\mu}\zeta^{\rho}-\zeta^{\rho}\partial_{\rho}\bar{g}_{\mu\nu} \tag{4.17}$$

and

$$\delta T^{\nu}_{\mu(1)}=\bar{T}^{\rho}_{\mu}\partial_{\rho}\zeta^{\nu}-\bar{T}^{\nu}_{\rho}\partial_{\mu}\zeta^{\rho}-\zeta^{\rho}\partial_{\rho}\bar{T}^{\nu}_{\mu}. \tag{4.18}$$

For small Killing displacements, $h_{\mu\nu}$ and $T^{\nu}_{\mu(1)}$ are invariant. It is most appealing to take the 3-gauge conditions

$$\mathscr{D}_k=0, \tag{4.19}$$

since these give three separable equations for $l^0_k=-h^0_k$. If we use the fact that (script $\mathscr{R}$s refer to curvature tensors in $t=$constant spaces)

$$(\nabla_{mk}-\nabla_{km})h^m_0=h^m_0\mathscr{R}^{\ m}_{nmk}=h^n_0\mathscr{R}_{nk}=-\frac{2k}{a^2}h^0_k, \tag{4.20}$$

then equation (4.15) can be written in the following telling form:

$$\nabla_l H-\nabla_l H^l_k=\kappa T^0_{k(1)}, \tag{4.21}$$

where

$$H_{kl}=\nabla_{(k}h^0_{l)}, \qquad H=\bar{g}^{kl}H_{kl}. \tag{4.22}$$

This shows clearly that h^0_l is defined up to a solution of $H_{kl}=0$, i.e. up to a 3-Killing displacement. Now, since our six Killing vectors have no time components, they are all 3-Killing, so

$$\nabla_k\xi_{al}+\nabla_l\xi_{ak}=0, \tag{4.23}$$

and thus the general 3-Killing displacement is of the form

$$\zeta^0=0, \qquad \zeta^k=f^a(t)\,\xi^k_a. \tag{4.24}$$

These displacements leave $T^0_{k(1)}$ invariant, as may be seen from (4.18), remembering that $\overline{T}^0_k=0$:

$$\delta T^0_{k(1)}=\overline{T}^\rho_k\partial_\rho f^a\xi^0_a-\overline{T}^0_0\partial_k f^a\xi^0_a-f^a\xi^h_a\partial_h\overline{T}^0_k=0. \tag{4.25}$$

Thus if h^k_0 is a solution of (4.21) then

$$h^k_0+\dot f^a(t)\,\xi^k_a \tag{4.26}$$

is also a solution. Notice that h^k_0 are the only metric components generated by the displacement (4.24) in which

$$\delta h_{kl}=\delta h_{00}=0, \qquad \delta h_{0k}=-\dot f^a\xi_{ak}. \tag{4.27}$$

Looking back at Section 2.3 under equation (2.20), we see that $\dot f^a$ is the six-component generalization of the arbitrary integration constant $\omega_0(t)$ of (2.20). In the open universe, these will be determined by the condition that h_{0k} tends to zero at infinity, so no problem will arise, but accelerations, rotations, etc., will be 'absolute'. For open universes, Mach's principle is not satisfied. In the closed universe the situation is exactly analogous to that discussed earlier for ω. The total quasi-momenta corresponding to the quasi-translations (which are actually angular momenta about rotation axes one-quarter of the way around the universe) come from surface integrals of superpotentials. For closed universes, these integrals vanish because the bounding surface shrinks to zero at $\chi=\pi$. The resulting six globally conserved quantities must all be zero. If the distributions of these angular momenta and quasi-momenta are given, the freedom of the $f_a(t)$ will remain, so only differences of rotation rates are determinable and no absolute rotation or acceleration exists. *Thus, at least for slow motions of axes, any axes are as good as any others, fully in accordance with Mach's ideas.* Furthermore, if the velocities of the heavenly bodies are given, then each of the six globally conserved quantities (which are each zero in total) provides just one equation like (2.26), so together they produce six equations which determine the $\dot f^a$ uniquely. Therefore motions in a *closed* universe do provide a complete determination of the h_{0k}. Thus the observable motions of the heavenly bodies do in this sense provide the inertial frame, just as Mach supposed. *THIS IS OUR PRIMARY RESULT.*

The solution of equation (4.21) is simplified if we use as our fourth gauge condition

$$H=\nabla_l h^l_0=0. \tag{4.28}$$

It then becomes

$$\nabla_l H^l_k=-\kappa T^0_{k(1)}. \tag{4.29}$$

Using (4.28), equation (4.29) is separable in spherical coordinates. We obtain an equation for h^1_0 and, using the solution of that equation, the separated equations for h^2_0 and h^3_0 result.

4.4 Rotating spheres in Friedmann–Robertson–Walker spaces

The type of perturbations described in Section 2 provide a simply integrable example in which

$$h^0_k=[0,0,-\omega(t,r)]. \tag{4.30}$$

For zero-pressure (Friedmann) universes,

$$T^0_3=-\rho_0c^2r^2\sin^2\theta(\Omega-\omega), \tag{4.31}$$

where $\rho_0(r,t)$ is the mass–energy density of the universe which rotates at the rate $\Omega(r,t)$ on the sphere of radius ar. The h^k_0 defined above already obeys the gauge condition $H=0$. Equation (4.29) for index $k=3$ reads

$$-\sqrt{1-kr^2}\,r^{-2}\partial/\partial r[\sqrt{1-kr^2}\,\partial/\partial r(r^2\omega)]+2\omega r^{-2}-4k\omega$$
$$=2\kappa\rho_0c^2a^2(\Omega-\omega); \tag{4.32}$$

for $k=0$ we find, cf. equation (2.23),

$$r^{-4}\partial/\partial r(r^4\partial\omega/\partial r)=2\kappa\rho_0c^2a^2(\Omega-\omega)=-6Gc^{-2}r^{-4}\partial J/\partial r, \tag{4.33}$$

while for $k=\pm1$ we write $\mu=\sqrt{1-kr^2}$ to obtain

$$(1-\mu^2)^{-3/2}\partial/\partial\mu[(1-\mu^2)^{5/2}\partial\omega/\partial\mu]=-2k\kappa\rho_0c^2a^2(\Omega-\omega)$$
$$=-6Gc^{-2}(1-\mu)^{-3/2}\frac{\partial J}{\partial\mu}. \tag{4.34}$$

When J is given, these equations may be solved similarly to (2.20). When instead Ω is given, our equation is of similar type to (2.23) or (2.25). The transformation

$$\omega=(1-\mu^2)^{-3/4}\varpi$$

brings it into the form of Legendre's equation with K known:

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\varpi}{\partial\mu}\right]+\left[\nu(\nu+1)-\frac{(3/2)}{(1-\mu^2)}\right]\varpi$$
$$=-2\kappa k^{-1}\rho_0c^2a^2\Omega(1-\mu^2)^{3/4}\equiv-K, \tag{4.35}$$

where

$$\left(\nu+\frac{1}{2}\right)^2=4-2\kappa k^{-1}\rho_0c^2a^2. \tag{4.36}$$

The solution of the homogeneous equation is

$$\varpi=AP^{3/2}_\nu(\mu)+BQ^{3/2}_\nu(\mu), \tag{4.37}$$

where P^m_ν and Q^m_ν are Legendre functions, which for $m=3/2$ are expressible in terms of sines and cosines (see Appendix B). The inhomogeneous equation may be solved by the method of variation of parameters and gives, for ω,

$$\omega=\frac{2(1-\mu^2)^{-3/4}}{(\nu^2-\frac{1}{4})(\nu+\frac{3}{2})}\left[P^{3/2}_\nu(\mu)\int_{-1}^{\mu}KQ^{3/2}_\nu\,\mathrm{d}\mu'\right.$$
$$\left.+Q^{3/2}_\nu(\mu)\int_{\mu}^{1}KP^{3/2}_\nu\,\mathrm{d}\mu'\right], \tag{4.38}$$

where we have used the method of Appendix A and the Wronskian:

$$[P^{3/2}_\nu\,\mathrm{d}Q^{3/2}_\nu/\mathrm{d}\mu-Q^{3/2}_\nu\,\mathrm{d}P^{3/2}_\nu/\mathrm{d}\mu]$$
$$=\frac{1}{2}\left(\nu^2-\frac{1}{4}\right)\left(\nu+\frac{3}{2}\right)\Big/(1-\mu^2). \tag{4.39}$$

As a varies, $\rho_0a^2\propto a^{-1}$, so $(\nu+\frac{1}{2})^2$ is strongly negative when a is small and $k=+1$. This implies that $\nu+\frac{1}{2}$ is pure imaginary, but the formulae given above hold for complex ν (see Appendix B). As in Appendix A, the convergence of the integrals at $\mu = \pm 1$ is assured by the fact that the total angular momentum is zero.

5 RETROSPECT

While this paper has gone some way to vindicate the idea that all motion is relative, it has in fact done so only for small motions, with the neglect of second-order quantities. Section 3 lays the foundation for conservation laws in a non-linear theory. It is also clear that even large rotations and quasi-translations in the background will merely generate gauge transformations. Thus we expect that a theory on these lines will be possible even for high frame rotation rates. Such a theory will show the gravitational wave angular momentum and quasi-linear momentum contributions that are obviously lacking from the first-order theory.

Mach and Einstein were both concerned as to how inertia would diminish if more and more mass were removed from the universe. If we add to Einstein's theory the requirement that space must be closed, this problem is beautifully resolved. Universes with less and less mass have smaller and smaller maximum radii of expansion, and exist for shorter and shorter total proper times. In the limit, a universe with no mass will never expand beyond a point and will only exist for one instant. Thus the universe shrivels up as mass is removed, so the conundrum of inertia in an almost empty, almost flat space does not occur!

Explicitly, the sum of the rest masses in the closed Friedmann universe is given by $\mathcal{M} = 2\pi^2 \rho_0 a^3$, while the maximum in the mass-energy function $M(R)$ occurs at the equator where $2GMc^{-2} = a$, where a is at maximum. Thus the maximum radius of the universe is linearly proportional to its mass, $a = (4/3\pi) G\mathcal{M}c^{-2}$, and its minimum mean density is proportional to $\mathcal{M}^{-2}$. As matter is 'removed', this becomes larger and larger, so the universe becomes less and less like Minkowski space. Similarly, the maximum time interval of any path from Big Bang to Big Crunch is $\tau = \oint \mathrm{d}r/\dot{r} = 2\pi GMc^{-3} = \pi a/c = (4/3)\, G\mathcal{M}c^{-3}$, which shrinks to zero as M tends to zero.

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APPENDIX A: SOLUTION FOR $\omega(r)$ WHEN $\Omega(r)$ IS GIVEN, FOR EINSTEIN'S STATIC UNIVERSE

Equation (2.23) now reads

$$\frac{\sqrt{1-(r/a)^2}}{(r/a)^4}\,\frac{\partial}{\partial r/a}\left[(r/a)^4\sqrt{1-(r/a)^2}\,\frac{\partial \omega}{\partial r/a}\right] - 4\omega = -4\Omega; \quad \text{(A1)}$$

putting

$$r/a = \sin\chi, \quad \text{(A2)}$$

$$\mu = \cos\chi, \quad \text{(A3)}$$

and

$$\omega = (\sin\chi)^{-3}\,\tilde{\omega}, \quad \text{(A4)}$$

our equation reduces to

$$(1-\mu^2)\partial^2\tilde{\boldsymbol{\omega}}/\partial\mu^2+\mu\partial\tilde{\boldsymbol{\omega}}/\partial\mu-\tilde{\boldsymbol{\omega}}=-4\boldsymbol{\Omega}(1-\mu^2)^{3/2}. \tag{A5}$$

We first solve the homogeneous equation with the right-hand side set to zero, and then use the method of variation of parameters to solve the inhomogeneous equation.

Clearly, $\tilde{\boldsymbol{\omega}}=\boldsymbol{A}\mu$ solves the homogeneous equation for any constant $\boldsymbol{A}$. To get the general solution, we write

$$\tilde{\boldsymbol{\omega}}=\mu\boldsymbol{f}. \tag{A6}$$

Then our homogeneous equation becomes, using primes to denote $\mathrm{d}/\mathrm{d}\mu$,

$$(1-\mu^2)(\mu\boldsymbol{f}''+2\boldsymbol{f}')+\mu^2\boldsymbol{f}'. \tag{A7}$$

Hence, considering for a moment just one vectorial component of $\boldsymbol{f}$,

$$\begin{aligned}(\ln f')'=f''/f'&=-\frac{2-\mu^2}{\mu(1-\mu^2)}=-\frac{2}{\mu}+\frac{1/2}{1+\mu}-\frac{1/2}{1-\mu}\\ &=\frac{\mathrm{d}}{\mathrm{d}\mu}\ln[(1-\mu^2)^{1/2}\mu^{-2}];\end{aligned} \tag{A8}$$

so, restoring the vectors since the equation is linear,

$$\boldsymbol{f}'=\boldsymbol{B}(1-\mu^2)^{1/2}\mu^{-2}. \tag{A9}$$

Hence

$$\boldsymbol{f}=\boldsymbol{A}-\boldsymbol{B}\int_0^\chi(\sec^2\theta-1)\,\mathrm{d}\theta=\boldsymbol{A}+\boldsymbol{B}(\chi-\tan\chi), \tag{A10}$$

$$\tilde{\boldsymbol{\omega}}=\boldsymbol{A}\cos\chi+\boldsymbol{B}(\chi\cos\chi-\sin\chi). \tag{A11}$$

This is the general solution of the homogeneous equation. To solve the inhomogeneous equation, we make $\boldsymbol{A}$ and $\boldsymbol{B}$ functions of χ. Our equation (A7) is

$$E(\tilde{\boldsymbol{\omega}})\equiv\partial^2\tilde{\boldsymbol{\omega}}/\partial\chi^2-2\cot\chi\partial\tilde{\boldsymbol{\omega}}/\partial\chi-\tilde{\boldsymbol{\omega}}=-4\boldsymbol{\Omega}\sin^3\chi. \tag{A12}$$

We now choose the gradient of $\boldsymbol{A}$ such that

$$\mathrm{d}\boldsymbol{A}/\mathrm{d}\chi\cos\chi+\mathrm{d}\boldsymbol{B}/\mathrm{d}\chi(\chi\cos\chi-\sin\chi)=0; \tag{A13}$$

then

$$E(\tilde{\boldsymbol{\omega}})=-\mathrm{d}\boldsymbol{A}/\mathrm{d}\chi\sin\chi-\mathrm{d}\boldsymbol{B}/\mathrm{d}\chi\chi\sin\chi=-4\boldsymbol{\Omega}\sin^3\chi. \tag{A14}$$

Eliminating $\mathrm{d}\boldsymbol{A}/\mathrm{d}\chi$,

$$\mathrm{d}\boldsymbol{B}/\mathrm{d}\chi=-4\boldsymbol{\Omega}\sin\chi\cos\chi; \tag{A15}$$

hence

$$\boldsymbol{B}=-\int_\chi^\pi 4\boldsymbol{\Omega}\sin\chi\cos\chi\,\mathrm{d}\chi+\boldsymbol{B}_1, \tag{A16}$$

and, from (A13),

$$\boldsymbol{A}=\int_0^\chi 4\boldsymbol{\Omega}\sin\chi(\sin\chi-\chi\cos\chi)\,\mathrm{d}\chi+\boldsymbol{A}_1, \tag{A17}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ are constants of integration. We thus obtain our general solution for $\boldsymbol{\omega}$ in the form

$$\boldsymbol{\omega}=\boldsymbol{A}(\chi)\frac{\cos\chi}{\sin^3\chi}+\boldsymbol{B}(\chi)\left(\frac{\chi\cos\chi-\sin\chi}{\sin^3\chi}\right). \tag{A18}$$

Both $\boldsymbol{B}(\chi)$ and the function that it multiplies have zero gradients at $\chi=0$; thus the boundary condition that $\boldsymbol{\omega}'(\pi)=0$ is probably most easily understood if we rewrite (A12) in its original form (2.17). In that form it is clear that we need $\boldsymbol{J}(\pi)=0$. This condition gives

$$\int_0^\pi(\boldsymbol{\Omega}-\boldsymbol{\omega})\sin^4\chi\,\mathrm{d}\chi=0. \tag{A19}$$

Using this and integrating the vector form of (2.17), we have

$$\begin{aligned}\sin^4\chi\,\partial\boldsymbol{\omega}/\partial\chi&=-4\int_0^\chi(\boldsymbol{\Omega}-\boldsymbol{\omega})\sin^4\chi\,\mathrm{d}\chi\\ &=+4\int_\chi^\pi(\boldsymbol{\Omega}-\boldsymbol{\omega})\sin^4\chi\,\mathrm{d}\chi.\end{aligned} \tag{A20}$$

In this final form it is clear that $\partial\boldsymbol{\omega}/\partial\chi=0$ at $\chi=\pi$. Hence the $\boldsymbol{J}(\pi)=0$ condition (A19) is necessary and sufficient. We therefore determine $\boldsymbol{B}_1$ from (A19). This gives

$$\boldsymbol{B}_1=-\frac{4}{\pi}\int_0^\pi\boldsymbol{\Omega}(\chi)\sin\chi(\sin\chi-\chi\cos\chi)\,\mathrm{d}\chi. \tag{A21}$$

Thus finally we find Lausberg's result generalized to vectors:

$$\begin{aligned}\boldsymbol{\omega}=&\int_0^\chi 4\boldsymbol{\Omega}\sin\chi(\sin\chi-\chi\cos\chi)\,\mathrm{d}\chi\frac{\cos\chi}{\sin^3\chi}\\ &+\left(\frac{\sin\chi-\chi\cos\chi}{\sin^3\chi}\right)\left[\int_\chi^\pi 4\boldsymbol{\Omega}\sin\chi\cos\chi\,\mathrm{d}\chi\right.\\ &\left.+\frac{1}{\pi}\int_0^\pi 4\boldsymbol{\Omega}(\chi)\sin\chi(\sin\chi-\chi\cos\chi)\,\mathrm{d}\chi\right].\end{aligned}$$

This is our explicit linear relationship of $\boldsymbol{\omega}$ to $\Omega(\chi)$ in a perturbed Einstein universe.

APPENDIX B: EXPRESSIONS FOR $P_\nu^{3/2}(\cos\theta)$ AND $Q_\nu^{3/2}(\cos\theta)$, θ REAL

$$\begin{aligned}P_\nu^{3/2}=&\frac{1}{2}(\pi/2)^{-1/2}(\sin\theta)^{-3/2}\left\{\left(\nu-\frac{1}{2}\right)\cos\left[\left(\nu+\frac{3}{2}\right)\theta\right]\right.\\ &\left.-\left(\nu+\frac{3}{2}\right)\cos\left[\left(\nu-\frac{1}{2}\right)\theta\right]\right\},\end{aligned}$$

$$\begin{aligned}Q_\nu^{3/2}=&-\frac{1}{2}(\pi/2)^{-1/2}(\sin\theta)^{-3/2}\left\{\left(\nu-\frac{1}{2}\right)\sin\left[\left(\nu+\frac{3}{2}\right)\theta\right]\right.\\ &\left.-\left(\nu+\frac{3}{2}\right)\sin\left[\left(\nu-\frac{1}{2}\right)\theta\right]\right\}.\end{aligned}$$

For $\nu+\frac{1}{2}=iN$, these become

$$P_\nu^{3/2}=\left(\frac{1}{2}\pi\right)^{-1/2}(\sin\theta)^{-3/2}[N\sin\theta\,\mathrm{sh}(N\theta)-\cos\theta\,\mathrm{ch}(N\theta)],$$

$$Q_\nu^{3/2}=-i\left(\frac{1}{2}\pi\right)^{1/2}(\sin\theta)^{-3/2}[N\sin\theta\,\mathrm{ch}(N\theta)-\cos\theta\,\mathrm{sh}(N\theta)],$$

and the Wronskian is as before and equals $\frac{1}{2}iN(N^2+\frac{1}{4})/\sin^2\theta$; thus the imaginary terms cancel in (4.38).