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THE SOBOLEV OPTICAL DEPTH FOR TIME-DEPENDENT RELATIVISTIC SYSTEMS

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ABSTRACT

A general Sobelov optical depth formula and some related results are derived for time-dependent spherically symmetric relativistic systems. This formula and the other results are specialized for the case of homologous expansion. The homologous expansion results will be useful for supernova calculations. Remarkably, the Sobolev optical depth formula for relativistic homologous expansion is angle independent. This new formula corrects an erroneous angle-dependent formula given in an earlier paper.
Subject headings: radiative transfer — relativity — supernovae: general

1. INTRODUCTION

A relativistic Sobolev method has been developed by Hutsemékers & Surdej (1990, hereafter HS) and Hutsemékers (1993). The expression they give for the relativistic Sobolev optical depth (appearing in their first paper) is, however, only valid for time-independent systems. In this paper, we will generalize their expression for the time-dependent systems. We assume spherical symmetry.

We should first make clear that using a time-dependent optical depth does not imply that a full time-dependent calculation is needed. We are not aware of any full time-dependent formulation of the Sobolev method. Such a formulation may be possible, but for calculations requiring full time-dependent treatments it is probably best to use the comoving frame formalism in which time dependence is treated naturally (e.g., Mihalas 1978, p. 490, and references therein). Nonetheless, we will briefly consider the needs of a full time-dependent Sobolev formulation in our developments. A Sobolev method calculation for a time-dependent system is generally a snapshot calculation, i.e., a calculation in which all or most of the physical conditions are evaluated at a specific time and where the calculation proceeds as if or almost as if the system were time-independent. A snapshot calculation yields a sort of time-average result. The time-dependent optical depth we derive is just another of the physical conditions to be evaluated at the time of the snapshot. The introduction of the time-dependent optical depth into snapshot calculations is a formal improvement since some of the time dependence will be included in the calculations. The practical improvement may be modest in many cases.

The main reason for the generalization of the relativistic optical depth formula for time-dependent systems is to improve supernova radiative transfer calculations done with HS's relativistic Sobolev method. The highest observed velocities for supernovae are 40,000 km s^{-1} ($\beta \approx 0.13$) (e.g., Jeffery 1993, hereafter J93, and references therein), and typical velocities are $\sim$5000–15,000 km s^{-1} ($\beta \approx 0.017$–0.05). At these kinds of velocities, relativistic effects on line radiative transfer can be expected to be rather small. By applying the relativistic Sobolev method to homologously expanding systems, which include supernovae after early times (see below), J93, however, showed that the relativistic effects will not be vanishingly small in the case of the highest supernova velocities. Thus, J93 verified the value of a relativistic treatment of line radiative transfer for supernovae. Now homologous expansion is a particularly simple form of time-dependent motion (see below). J93 took account of this time-dependence in his prescriptions for the relativistic Sobolev method in the homologous expansion case, except by oversight in his formula for the relativistic homologous expansion optical depth and related formulae. In this paper we correct this error and show that the correction is in fact a simplification.

In homologous expansion the matter elements move with a range of velocities that are constant in time. Additionally, the matter elements were effectively at a point at time $t = 0$. Thus, the radius of a matter element at any time t after $t = 0$ is given by

$$r = vt , \qquad (1)$$

where v is the matter element's constant velocity. Of course, if we fix r in the observer frame instead of fixing it on a matter element, we find that the velocity at r is decreasing like t^{-1}. Moreover, even if we follow a matter element, we find that its density is decreasing like t^{-3}. Therefore, homologous expansion is a time-dependent velocity field and unlike the stationary velocity fields considered by HS.

For supernovae, the time $t = 0$ of homologous expansion corresponds to the explosion epoch. The initial radius of the exploding star is negligible compared to the radii of the mass elements after 1 day at most in the case of most Type II (and perhaps Types Ib and Ic) supernovae and after $\sim$100 s in case of Type Ia supernovae (according to the standard theory). Thus, one can assume that the matter elements started from a point at $t = 0$ for all supernovae after appropriate time intervals. Ordinarily, the forces on the matter elements vanish, and the matter elements go into uniform motion after time intervals comparable to those that make the initial matter element radii negligible. Thus, all the conditions for homologous expansion will be met by most supernovae after at most 1 day. In some cases, very large initial radius and/or interaction of the ejecta with circumstellar matter may delay the onset of homologous expansion for a few days (e.g., Höflich, Langer, & Duschinger 1993).

In § 2, we derive the general formula for the time-dependent relativistic Sobolev optical depth and some related formulae. In § 3, we specialize the results of § 2 for the case of homologous expansion. Conclusions appear in § 4.

2. THE GENERAL RESULTS

Consider a spherically symmetric moving atmosphere with a large nonzero velocity gradient everywhere. We will measure velocity in units of c and use β for the radial velocity. For the general velocity (i.e., the velocity in any direction), we use

$$\beta_\mu = \mu\beta \,, \tag{2}$$

where μ is the cosine of the angle from the radial direction. Both β and β_μ are observer frame velocities. We will distinguish comoving frame quantities from observer frame quantities with a subscript or superscript 0 where there is any ambiguity. Note the following transformations:

$$\mu = \frac{\mu_0 + \beta}{1 + \mu_0 \beta} \,, \quad \nu_0 = \eta\nu \,, \quad \chi_0 = \eta^{-1}\chi \,, \tag{3}$$

where ν is frequency, χ is the monochromatic opacity, and

$$\eta = \gamma(1 - \mu\beta) \,. \tag{4}$$

The γ factor is, of course, given by

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \,. \tag{5}$$

The above transformations are derived by, e.g., Mihalas (1978, pp. 495–496).

Consider a beam path measured by the coordinate s in the observer frame: s increases in the direction of photon propagation. The rate of change with respect to s of the general velocity encountered by a photon propagating along the beam path is given by

$$\frac{d\beta_\mu}{ds} = \mu \frac{\partial\beta}{\partial s} + \frac{\partial\mu}{\partial s}\beta + \mu \frac{\partial\beta}{\partial t}\frac{\partial t}{\partial s} \,. \tag{6}$$

The last term in equation (6) includes the effect of the time dependence of the velocity field. From simple geometry we know that

$$\frac{\partial\beta}{\partial s} = \mu \frac{\partial\beta}{\partial r} \,, \quad \frac{\partial\mu}{\partial s} = \frac{1 - \mu^2}{r} \,, \tag{7}$$

where r is the radial coordinate. For a photon,

$$\frac{\partial t}{\partial s} = \frac{1}{c} \tag{8}$$

of course, Using these results, we obtain

$$\frac{d\beta_\mu}{ds} = \mu^2 \frac{\partial\beta}{\partial r} + (1 - \mu^2)\frac{\beta}{r} + \mu \frac{\partial\beta}{\partial t}\frac{1}{c} \,. \tag{9}$$

For the Sobolev optical depth we need formulae for $d\eta/ds$ and $d\nu_0/ds$. From the expressions given above, it is straightforward to show that

$$\frac{d\eta}{ds} = -\gamma^3\left[\mu(\mu - \beta)\frac{\partial\beta}{\partial r} + (1 - \mu^2)(1 - \beta^2)\frac{\beta}{r} + (\mu - \beta)\frac{\partial\beta}{\partial t}\frac{1}{c}\right] , \tag{10}$$

$$\frac{d\nu_0}{ds} = \nu_0 \eta^{-1}\frac{d\eta}{ds} \,. \tag{11}$$

The optical depth into a line from negative infinity to a point s is given by

$$\zeta(s) = \int_{-\infty}^{s} ds' \, \chi(s') \,. \tag{12}$$

Now

$$\chi(s) = \eta\chi_0(s) = \eta\alpha_0 \, \phi[\nu_0(s) - \nu_l^0] \,, \tag{13}$$

where α_0 is the comoving frame integrated line opacity, ν_l^0 is the line center frequency, and $\phi[\nu_0(s) - \nu_l^0]$ is the line absorption profile. Substituting equation (13) into equation (12) and transforming the integration variable from s' to ν_0' using equation (11) yields

$$\zeta(\nu_0) = \pm\tau \int_{\mp\infty}^{\nu_0} d\nu_0' \, \phi(\nu_0' - \nu_l^0) \,, \tag{14}$$

where τ, which is the Sobolev optical depth of the line, is defined by

$$\tau = \eta\alpha_0 \left|\frac{d\nu_0}{ds}\right|^{-1} = \frac{\alpha_0}{\nu_l^0}\eta^2 \left|\frac{d\eta}{ds}\right|^{-1} \,. \tag{15}$$

We have assumed that the quantities used to construct τ can be extracted from the integral and evaluated at the resonance point: i.e., at the point s where $\nu_0 = \nu_l^0$. This assumption, which is the usual assumption of the Sobolev method, implies that these quantities do not vary significantly over the region where $\phi(\nu_0 - \nu_l^0)$ is signficantly different from zero. This region is called the resonance region. From equations (10) and (11) we derive the expression for the characteristic width of the resonance region, Δs:

$$\Delta s = \frac{\Delta\nu_0}{\nu_l^0}\eta\left|\frac{d\eta}{ds}\right|^{-1} = \frac{\Delta\nu_0}{\nu_l^0}\frac{1}{\gamma^2} \times \frac{(1 - \mu\beta)}{|\mu(\mu - \beta)(\partial\beta/\partial r) + (1 - \mu^2)(1 - \beta^2)(\beta/r) + (\mu - \beta)(\partial\beta/\partial t)(1/c)|} \,, \tag{16}$$

where $\Delta\nu_0$ is the frequency width of the line. In equation (14) the upper and lower case results are for when $d\nu_0/ds > 0$ and $d\nu_0/ds < 0$, respectively: i.e., for when the photons progressively blueshift and redshift, respectively, in the comoving frame. Since $\phi(\nu_0 - \nu_l^0)$ is normalized, τ is simply the total optical depth of the line.

Using equations (4), (10), and (15), we find the Sobolev optical depth to be given by

$$\tau = \frac{\alpha_0}{\nu_l^0}\frac{1}{\gamma}\frac{(1 - \mu\beta)^2}{|\mu(\mu - \beta)(\partial\beta/\partial r) + (1 - \mu^2)(1 - \beta^2)(\beta/r) + (\mu - \beta)(\partial\beta/\partial t)(1/c)|} \,. \tag{17}$$

Equation (17) is our general expression for the time-dependent relativistic Sobolev optical depth. As a function of the comoving frame angle cosine μ_0, the general expression is

$$\tau = \frac{\alpha_0}{\nu_l^0}\frac{1}{\gamma^3}\frac{1}{|\mu_0(\mu_0 + \beta)(\partial\beta/\partial r) + (1 - \mu_0^2)(1 - \beta^2)(\beta/r) + \mu_0(1 + \mu_0\beta)(\partial\beta/\partial t)(1/c)|} \,. \tag{18}$$

If the time derivative term is dropped from equation (17), we recover the time-independent expression of HS (see their eqs. [25]–[26]). If we drop all but the lowest order terms in β from equation (17) (or from eq. [18] with the μ_0's changed to μ's), we have

$$\tau_{\rm clas} = \frac{\alpha_0}{v_l^0} \frac{1}{|\mu^2(\partial\beta/\partial r) + (1-\mu^2)(\beta/r) + \mu(\partial\beta/\partial t)(1/c)|}, \quad (19)$$

the time-dependent classical Sobolev optical depth.

If for some directions $d\eta/ds$ goes to zero, the Sobolev optical depth goes to infinity (see eq. [15]). Of course, the real optical depth does not go to infinity, but is probably very large. In these directions, at least one of the physical conditions (e.g., $d\eta/ds$ or α_0) that the Sobolev method assumes are constant across the resonance region cannot be approximated as constant. In practice, radiative transfer may be dominated by the directions where $d\eta/ds \neq 0$, and infinite Sobolev optical depths may be no problem at all. A case where $d\eta/ds$ goes to zero, but does not change sign, is in the inward and outward radial directions of the outermost part of hot star winds where radial velocity is a constant with respect to radius and time (e.g., Pauldrach, Puls, & Kudritzki 1986). A case where $d\eta/ds$ goes to zero and does change sign is in very relativistic examples of the exponentially expanding atmospheres of J93.

In some radiative transfer problems it may be useful to have the comoving frame inverse angle-averaged optical depth $\bar{\tau}$ (e.g., Jeffery 1995). Using equation (18) with the assumption that the quantity in the absolute value signs never changes sign (or, equivalently, that $d\eta/ds$ never changes sign), we find $\bar{\tau}$ to be given by

$$\bar{\tau} = \langle \tau^{-1} \rangle^{-1}$$

$$= \frac{\alpha_0}{v_l^0}\left(\frac{3}{\gamma^3}\right) \frac{1}{|(\partial\beta/\partial r) + 2(1-\beta^2)(\beta/r) + \beta(\partial\beta/\partial t)(1/c)|}, \quad (20)$$

where the angle brackets indicate angle averaging in the comoving frame. However, if $d\eta/ds$ does change sign for some directions, the angle average must be done by segments.

The formal Sobolev solution for radiative transfer across a resonance region is a time-independent result and therefore requires the optical depth τ and the line source function to be constant in time. In a full time-dependent Sobolev calculation one can imagine taking time steps sufficiently short to guarantee the constancy of τ and the line source function over the time step. However, the formal Sobolev solution would be inapplicable if the time step guaranteeing constancy were shorter than the (characteristic) photon crossing time for the resonance region (i.e., $\Delta s/c$). Thus, for a full time-dependent Sobolev calculation to be realizable in a straightforward way, there must be time steps longer than $\Delta s/c$ over which τ and the line source function can be considered constant. In § 3, we show that this constancy condition is met in the case of supernovae. Of course, for snapshot calculations with the time-dependent optical depth we make the gross assumption that time variation in almost all physical quantities can be neglected. Obviously, we are not neglecting the effect of the time dependence of the velocity on the optical depth.

3. THE HOMOLOGOUS EXPANSION CASE RESULTS

For homologous expansion we have the following expressions derived from equation (1) (see § 1):

$$\beta = \frac{r}{ct}, \quad \frac{\partial\beta}{\partial r} = \frac{\beta}{r} = \frac{1}{ct}, \quad \frac{\partial\beta}{\partial t} = -\frac{\beta}{t}. \quad (21)$$

Using these expressions, the formulae for $d\beta_\mu/ds$, $d\eta/ds$, dv_0/ds, and Δs become

$$\frac{d\beta_\mu}{ds} = \frac{1}{ct}(1-\mu\beta), \quad (22)$$

$$\frac{d\eta}{ds} = -\frac{\gamma^3}{ct}(1-\mu\beta)^2 = -\frac{\gamma\eta^2}{ct}, \quad (23)$$

$$\frac{dv_0}{ds} = -\frac{v_0\gamma^2}{ct}(1-\mu\beta) = -\frac{v_0\gamma\eta}{ct}, \quad (24)$$

and

$$\Delta s = \frac{\Delta v_0\, ct}{v_l^0} \frac{1}{\gamma^2(1-\mu\beta)}, \quad (25)$$

respectively.

We note that dv_0/ds is always negative even in the limit of β going to 1 with $\mu = 1$. Thus photons always redshift in the comoving frame as they propagate in homologous expansion. There are relativistic atmospheres in general expansion (i.e., in which the velocity derivative along any path is always greater than zero) where comoving frame blueshifts are possible at very relativistic velocities (HS; J93). This can never happen in classical atmospheres in general expansion although both comoving frame redshifts and blueshifts can happen in the same classical atmosphere if the velocity derivative changes sign (Rybicki & Hummer 1978).

For the relativistic homologous expansion optical depth we obtain

$$\tau = \frac{\alpha_0\, ct}{v_l^0}\frac{1}{\gamma} = \frac{\alpha_0\, ct}{v_l^0}\sqrt{1-\beta^2}\,. \quad (26)$$

The classical homologous expansion optical depth without the time derivative term (which is the optical depth used in most supernova Sobolev calculations) is given by

$$\tau^*_{\rm clas} = \frac{\alpha_0\, ct}{v_l^0}, \quad (27)$$

where the asterisk superscript is used to indicate the lack of the time derivative term. Remarkably, τ is angle independent and does not differ from $\tau^*_{\rm clas}$ to first order in β (not considering the hidden β dependence in $\alpha_0\, ct/v_l^0$). The erroneous expression for τ given by J93 (which was obtained from eq. [17] without the time derivative term) had angle dependence and a first-order term in $\mu\beta$. Because τ and $\tau^*_{\rm clas}$ differ by at most $\sim$1% for known supernova cases where $\beta \lesssim 0.13$ (see § 1) and uncertainties in α_0, for instance, are at least as large as 1%, it is clear that we have proven that the use of $\tau^*_{\rm clas}$ is far from being the limiting error in past supernova calculations.

Some insight into the effect of time dependence on the optical depth is given by considering the (full) classical homologous expansion optical depth which we derive from equation (19):

$$\tau_{\rm clas} = \frac{\alpha_0\, ct}{v_l^0}\frac{1}{(1-\mu\beta)}. \quad (28)$$

As a photon propagates it Doppler shifts in the comoving frame because of the spatially and time-varying velocity. Eventually, it encounters a line and over some spatial region (i.e., the resonance region) can interact with the line. The time variation of the velocity field changes the spatial length over which the interaction occurs relative to the analogous stationary situation and therefore changes the optical depth relative to that situation. The sign and magnitude of the change are angle dependent. For example, for outward radial propagation the velocity increases with radius but decreases with time. The decrease with time stretches out the region of interaction and increases the optical depth. As equation (26) shows, relativistic effects cancel all the classical angular dependence of the optical depth.

We note that the relativistic optical depth τ goes to zero as β goes to 1. To understand this result, let us write τ in the general form given by equation (15):

$$\tau = \eta\alpha_0 \left|\frac{dv_0}{ds}\right|^{-1} . \tag{29}$$

From equations (4) and (24) for the case of $\mu \neq 1$, we see that the η factor (which comes from the transformation of the opacity) goes to infinity like γ and that $|dv_0/ds|^{-1}$ goes to zero like γ^{-2} when β goes to 1. Thus, in the limit of β going to 1, the Doppler shift through the frequency band where the line is significant is rapid enough to kill the line's effect on a beam even though the line opacity is going to infinity. But from the same equations for the case of $\mu = 1$, we find that the η factor goes to zero like $\sqrt{1-\beta}$ and that $|dv_0/ds|^{-1}$ goes to a constant when β goes to 1. Thus, in the forward direction the vanishing of the opacity kills the line's effect when β goes to 1.

The relative photon crossing time of the resonance region is given by

$$\frac{\Delta t}{t} = \frac{\Delta s}{ct} = \frac{\Delta v_0}{v_l^0} \frac{1}{\gamma^2(1-\mu\beta)} . \tag{30}$$

In homologous expansion the density at any comoving point (i.e., point of fixed velocity) declines as t^{-3}. If α_0 is assumed to vary like density (which is roughly true), then τ declines roughly as t^{-2}. The approximate relative change in τ in a photon crossing time is thus

$$\frac{\Delta\tau}{\tau} \approx 2\frac{\Delta t}{t} = 2\frac{\Delta v_0}{v_l^0} \frac{1}{\gamma^2(1-\mu\beta)} . \tag{31}$$

If the frequency width of the line is only the thermal Doppler width, then the relative photon crossing time and the relative change in τ in a photon crossing time usually will be very small, i.e., usually $\lesssim 4 \times 10^{-5}$ and $\lesssim 8 \times 10^{-5}$, respectively, for a temperature of $\sim 10^4$ K (e.g., Mihalas 1978, p. 110), which is characteristic of supernovae in the photospheric epoch. Since the line source function depends on nearly the same atomic conditions as τ, the relative variation in the line source function in a photon crossing time will be comparable to that of τ. Clearly, one can take time steps longer than a photon crossing time in a time-dependent calculation over which τ and the line source function can be regarded as constant. Thus, at least one of the conditions needed for full time-dependent Sobolev calculations for supernovae is met. Since β can be at least as large as 0.13 in supernovae (see § 1), it is clear that the relativistic time-dependent correction to $\tau^*_{\rm clas}$ can be much larger than the relative variation in τ in a photon crossing time. Thus, using the relativistic homologous expansion optical depth instead of $\tau^*_{\rm clas}$ in the still hypothetical full time-dependent Sobolev calculations for supernovae is certainly sensible.

As we pointed out in § 1, J93 verified that relativistic effects in supernova line radiative transfer will not be vanishingly small at the highest supernova velocities. Thus, the use of the relativistic Sobolev method for Sobolev supernova calculations is a distinct improvement over the use of the classical Sobolev method. In relativistic Sobolev supernova calculations, it is obviously conceptually better, more consistent, and hardly more work to use the correct relativistic homologous expansion optical depth, τ, than to use $\tau^*_{\rm clas}$. Because of the coincidental closeness of the τ and $\tau^*_{\rm clas}$ formulae, however, $\tau^*_{\rm clas}$ is scarcely less accurate than τ for Sobolev supernova calculations. The incorrect formula of J93 for the relativistic homologous optical depth is distinctly less accurate and more complex than the correct formula given here and thus should not be used.

4. CONCLUSIONS

The general expression for the time-dependent Sobolev optical depth for spherically symmetric relativistic systems is given by

$$\tau = \frac{\alpha_0}{v_l^0}\frac{1}{\gamma}\frac{(1-\mu\beta)^2}{|\mu(\mu-\beta)(\partial\beta/\partial r) + (1-\mu^2)(1-\beta^2)(\beta/r) + (\mu-\beta)(\partial\beta/\partial t)(1/c)|} . \tag{32}$$

Equation (18) gives the general expression as a function of the comoving frame angle cosine μ_0. The expression for the case of homologous expansion is

$$\tau = \frac{\alpha_0\, ct}{v_l^0}\sqrt{1-\beta^2} . \tag{33}$$

Remarkably, the homologous expansion expression is angle independent and lacks a first-order term in β.

Equation (33) corrects the homologous expansion optical depth formula given by J93 (his eq. [54]) and its first-order in β approximation (his eq. [55]). The general optical depth expression, equation (32), and its classical limit, equation (19) (see § 2), should replace the corresponding correct, but time-independent, formulae in J93: his equation (51) (HS's eqs. [25]–[26]) and his equation (52) (the usual Sobolev optical depth formula). J93's expressions for the characteristic width of the resonance region in the general and homologous expansion cases (his eqs. [57] and [58]) should be replaced by equations (16) and (25), respectively. The demonstration calculations reported by J93 in his § 6 would only be slightly affected by the use of the correct optical depth formula of this paper, except for the extremely relativistic calculation illustrated in his Figure 6. The extremely relativistic calculation must now be regarded as incorrect. All the other results given by J93, in particular his expressions for the homologous expansion common direction and common point frequency surfaces (see his §§ 2 and 3), still appear to be entirely correct. In fact, it was the inconsistency between his homologous expansion common point frequency surface and optical depth expressions that led to the discovery of the error in the homologous expansion optical depth expression.

An important aspect of this paper is the conceptual clarification of time-dependent and relativistic effects yielded by the expressions derived. This clarification may lead to further improvements in Sobolev method techniques, such as the inclusion of more time-dependent effects. The clarification also allows a valid appreciation of the classical homologous expansion optical depth without the time derivative term, $\tau^*_{\rm clas}$. We could not have known that $\tau^*_{\rm clas}$'s lack of angle-dependence and first-order term in β were good qualities until the correct relativistic homologous expansion optical depth expression had been worked out.

The relativistic Sobolev method of HS provides a modest improvement over the classical Sobolev method for supernova calculations as shown by J93. This improvement could not be fully realized without the correct relativistic homologous expansion optical depth formula given in this paper. Similarly, Sobolev or semi-Sobolev calculations for other time-dependent relativistic systems will benefit from the use of the general time-dependent relativistic optical depth formula we have given in place of HS's corresponding time-independent formula.

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