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High order f and g power series for orbit determination*

J. Bem and B. Szczodrowska-Kozar

Astronomical Observatory, ul. Kopernika 11, 51-622 Wrocław, Poland
E-mail: jbem @ ii.uni.wroc.pl (Internet)

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Abstract. — The table of coefficients and exponents of the power series f and g up to derivatives of the 20-th order is the main achievement of this paper. The accuracy of the calculation of orbits has been tested by tracing the motion of all planets of the Solar System. Special attention has been paid to high-eccentricity orbits in the two- or three-body problem. Power series from 10th up to 20th order have been used instead of other integration methods.

Key words: celestial mechanics — numerical methods — solar system

1. Introduction

Initial orbit determination in Gauss' and similar methods is based on the expansion of the solution of Newton's laws into the Taylor's series known in celestial mechanics as the f and g power series. These series can also be used to integrate the equations of motion for two or more bodies. We present here some remarks about application of the high-order terms of these series, the table of terms, and the results of some numerical experiments.

It is too big a task to mention here all the papers related to this problem, so we have decided to mention only such as have helped us in constructing the program of tracking the motion of N bodies. Some examples of the 2- and 3-body orbit calculations with orbital eccentricity near to or greater than unity have been exposed, because the accessible literature leaves it difficult to formulate an undisputed opinion about usefulness of these series when $e > 1$.

Theoretical formulae for the radius of convergence in time interval are given by Moulton (1903) and Taff (1985). The highest-order series (up to 8th) given explicitly are found in Escobal (1965). Tables demonstrating the convergence of the f and g series as functions of: the time interval t, the eccentricity e, the mean anomaly M, and the number of terms (up to 13th order) were given by Taff (1985). The recurrence formulae for power series adjusted to the restricted 3-body problem were given by Steffensen (1956). Next, modified versions were applied by Rabe (1961) to explore the periodic Trojan orbits; by Deprit & Price (1965) to the restricted 3-body problem; by Broucke (1971) and Black (1973) to the N-body problem; by Sitarski (1979) to investigate the motion of comets; and by Guyader (1993) to the Solar System planets. We used tables from the latter paper to test our program.

2. Calculation of high order terms

To resolve the Newtonian equation

$$\ddot{\mathbf{r}} = -u\mathbf{r} \tag{1}$$

in Taylor's series form

$$\mathbf{r} = \mathbf{r}_0 + \dot{\mathbf{r}}_0\tau + \frac{1}{2}\ddot{\mathbf{r}}_0\tau^2 + \frac{1}{6}\mathbf{r}^{\mathrm{III}}\tau^3 + ... \tag{2}$$

it is necessary to calculate the derivatives $\mathbf{r}^{\mathrm{III}}, \mathbf{r}^{\mathrm{IV}}, \mathbf{r}^{\mathrm{V}}, ..., \mathbf{r}^n$ of sufficiently high order up to n. Introducing Lagrange's variables

$$u = \frac{\mu}{\mathbf{r}_0^3}, \qquad p = \frac{\mathbf{r}_0 \cdot \dot{\mathbf{r}}_0}{\mathbf{r}_0^2}, \qquad q = \frac{\dot{\mathbf{r}}_0^2}{\mathbf{r}_0^2} - u, \tag{3}$$

and taking into account the following relationships (Escobal, 1965):

$$\dot{u} = -3up, \qquad \dot{p} = q - 2p^2, \qquad \dot{q} = -up - 2pq, \tag{4}$$

the successive terms of Taylor's series (2) of order n take the form:

$$\tau^n \left(\mathbf{r}_0 \sum_1^L R_L u^i p^j q^k + \dot{\mathbf{r}}_0 \sum_1^{L'} R_{L'} u^i p^j q^k \right) /n! \tag{5}$$

and instead of (2), we can write the following equation:

$$\mathbf{r} = \mathbf{r}_0 f + \dot{\mathbf{r}}_0 g,$$

*Table 2 is available electronically at the CDS via anonymous ftp 130.79 128.5

where f and g are sums of terms (5) to a given order n. In Eq. (5) R are integer numbers, the exponents the integers $0 < i, j, k < n - 2$, τ denotes the time interval, and L and L' are related to n. The main series (2) was divided into two series including terms connected with the position vector $\mathbf{r}$ and the velocity vector $\dot{\mathbf{r}}$. The variables u, p, q, are also functions of $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, as seen from Eq. (3).

Direct calculation of the derivative $\mathbf{r}^{n+1}$ of series (2) is difficult for the higher orders only because of the rapidly increasing number of terms. Instead of the traditional algorithm, (for example given by Escobal (1965) between the formulae 3.213 and 3.229), we propose another algorithm suitable for computers. The essence of the method consists in calculation of the derivative of each term of form $(R\, u^i p^j q^k \mathbf{r}^m)$ belonging to the derivative $\mathbf{r}^n$ according to the unified scheme. The derivatives $\ddot{\mathbf{r}}, \dot{u}, \dot{p}, \dot{q}$ are eliminated using as usual Eqs. (1) and (4). Such an operation leads to the formal change of the 5-element vector $\mathbf{W}$:

$$\mathbf{W}(R, i, j, k, m) \tag{6}$$

into the 4×5 matrix $\mathbf{M}$:

$$\begin{pmatrix} -R(3i+2j+2k) & i & j+1 & k & m \\ Rj & i & j-1 & k+1 & m \\ -Rk & i+1 & j+1 & k-1 & m \\ R & i & j & k & m+1 \end{pmatrix} \tag{7}$$

The element m of the vector $\mathbf{W}$ or the matrix $\mathbf{M}$ is a symbolic exponent of the position vector $\mathbf{r}$, where $\mathbf{r}^m$ denotes $\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}$ when m = 1,2,3 respectively. If, in the matrix $\mathbf{M}$, the exponent m+1=3, then with Eq. (1) the last line becomes:

$$R \qquad i+1 \qquad j \qquad k \qquad m-1.$$

Generally, to obtain the derivative r^{n+1}, we start from the terms belonging to the derivative of order n given by expression (5). To each term of this expression corresponds the appropriate vector W. Transforming $\mathbf{W}$ into $\mathbf{M}$ according to (6) and (7), we get at most four new terms described by the lines of matrix $\mathbf{M}$, belonging to the derivative of order $n+1$.

To obtain the third-order derivative of $\mathbf{r}$, we start from Eq. (1) written in the form of expression (5):
$\ddot{\mathbf{r}} = -1\, u^1 p^0 q^0 \mathbf{r}^1$, to which corresponds the single vector $\mathbf{W}(-1, 1, 0, 0, 1)$. Thus $\mathbf{r}^{\mathrm{III}}$ corresponds to the single matrix $\mathbf{M}$. Omitting in $\mathbf{M}$ the lines containing $R = 0$ or $i, j, k < 0$, we obtain:
$\mathbf{r}^{\mathrm{III}} = +3u^1p^1q^0\mathbf{r}^1 - 1u^1p^0q^0\mathbf{r}^2 = 3up\mathbf{r} - u\dot{\mathbf{r}}$.

There are two vectors corresponding to this derivative: $\mathbf{W}_1(3, 1, 1, 0, 1)$ and $\mathbf{W}_2(-1, 1, 0, 0, 2)$. The matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ associated with them give the terms of the derivative $\mathbf{r}^{\mathrm{IV}}$. And so on until the desired $\mathbf{r}^n$.

Note that in Eq. (2), the coefficient $1/n!$ grows rapidly with n and some of R in expression (5) even faster. So, for high-order derivatives, one can exceed the maximal integer permissible for a given computer and obtain false coefficients of series (2). In our case, the maximal integer is 2147483647 $\approx 2\ 10^9$, and we can get derivatives with integer numbers R only up to the eleventh order. To obtain the derivative of order $(n+1)$ high enough, it is best to change the integer numbers R into real $R' = R/n!$ in expressions (5) and (6), and then in the matrix $\mathbf{M}$ instead of R substitute the real numbers $R'' = R'/(n+1)$.

The number of terms included in (5) grows with the order n of the derivatives like 4^{n-2}. The simple use of the algorithm enclosed in conversion of (6) to (7) leads to a gigantic quantity of numbers which must be saved in computer memory: for example, if $n = 16$, then $5\ 4^{14} = 1.3\ 10^9$ numbers. After every conversion of (6) to (7), it is necessary to use a procedure eliminating zero terms and gathering similar terms, to minimize the growth of the series. Even so, the number of terms in (5) increases undesirably rapidly, see Table 1.

Table 1. The number of terms of the form $R\, u^i p^j q^k$ in the f and g series as a function of a given order n of the derivative used in Eq. (2). Columns Np and Nt are partial and total number of terms, respectively

n	Np	Nt	n	Np	Nt	n	Np	Nt
—	—		11	30		21	110	
2	1		12	36		22	121	
3	2		13	42		23	132	
4	4		14	49		24	144	
5	6	13	15	56	308	25	156	1378
6	9		16	64		26	169	
7	12		17	72		27	182	
8	16		18	81		28	196	
9	20		19	90		29	210	
10	25	95	20	100	715	30	225	2360

The coefficients R'' and the exponents i, j, k, included in expression (5) for a given derivative of the order n up to 20th are included in Table 2. Numbers of this table permit writing both series f and g explicitly, and their time derivatives F and G . According to this table these series up to 4th order are:

$$f = 1 - 0.5u\tau^2 + 0.5up\tau^3 + (-0.625up^2 + 0.125uq + 0.04167u^2)\tau^4,$$
$$g = \tau - 0.1667u\tau^3 + 0.25up\tau^4,$$
$$F = -u\tau + 1.5up\tau^2 + 4(-0.625up^2 + 0.125uq + 0.04167u^2)\tau^3,$$
$$G = 1 - 0.1667u \cdot 3\tau^2 + 0.25up \cdot 4\tau^3,$$

where the modified time $\tau = \sqrt{Gm}(t - t_0)$, t and t_0 expressed in days (Escobal 1965).

The partial sums in Table 2 are ordered according to the last column only, what leads to an approximate or-

Table 2. (Sample page) The table contains: order of derivative n, number of terms $L + L'$ belonging to this derivative, and in next $L + L'$ lines the coefficients R'', and exponents i, j, k, of each term. In the last column the number 1 or 2 denotes the term to f or g series, respectively

2	1			
−.5000000000000000D+00	1	0	0	1
3	2			
.5000000000000000D+00	1	1	0	1
−.1666666666666667D+00	1	0	0	2
4	4			
−.6250000000000000D+00	1	2	0	1
.1250000000000000D+00	1	0	1	1
.4166666666666666D−01	2	0	0	1
.2500000000000000D+00	1	1	0	2
5	6			
.8750000000000000D+00	1	3	0	1
−.3750000000000000D+00	1	1	1	1
−.1250000000000000D+00	2	1	0	1
−.3750000000000000D+00	1	2	0	2
.7500000000000000D−01	1	0	1	2
.8333333333333333D−02	2	0	0	2
6	9			
−.1312500000000000D+01	1	4	0	1
.8750000000000000D+00	1	2	1	1
−.6250000000000000D−01	1	0	2	1
.2916666666666667D+00	2	2	0	1
−.3333333333333333D−01	2	0	1	1
−.1388888888888889D−02	3	0	0	1
.5833333333333334D+00	1	3	0	2
−.2500000000000000D+00	1	1	1	2
−.4166666666666666D−01	2	1	0	2
7	12			
.2062500000000000D+01	1	5	0	1
−.1875000000000000D+01	1	3	1	1
.3125000000000000D+00	1	1	2	1
−.6250000000000000D+00	2	3	0	1
.1750000000000000D+00	2	1	1	1
.1250000000000000D−01	3	1	0	1
−.9375000000000000D+00	1	4	0	2
.6250000000000000D+00	1	2	1	2
−.4464285714285714D−01	1	0	2	2
.1250000000000000D+00	2	2	0	2
−.1071428571428571D−01	2	0	1	2
−.1984126984126984D−03	3	0	0	2
8	16			
−.3351562500000000D+01	1	6	0	1
.3867187500000000D+01	1	4	1	1
−.1054687500000000D+01	1	2	2	1
.1289062500000000D+01	2	4	0	1
.3906250000000000D−01	1	0	3	1
−.6093750000000000D+00	2	2	1	1
.2745535714285715D−01	2	0	2	1
−.5468750000000000D−01	3	2	0	1
.2901785714285714D−02	3	0	1	1
.2480158730158730D−04	4	0	0	1
.1546875000000000D+01	1	5	0	2
−.1406250000000000D+01	1	3	1	2
.2343750000000000D+00	1	1	2	2
−.3125000000000000D+00	2	3	0	2
.7500000000000000D−01	2	1	1	2
.3125000000000000D−02	3	1	0	2
9	20			
.5585937500000000D+01	1	7	0	1
−.7820312500000000D+01	1	5	1	1
.3007812500000000D+01	1	3	2	1
−.2606770833333333D+01	2	5	0	1
−.2734375000000000D+00	1	1	3	1
.1776041666666667D+01	2	3	1	1
−.2049851190476190D+00	2	1	2	1
.1814236111111111D+00	3	3	0	1
−.3013392857142857D−01	3	1	1	1
−.7027116402116402D−03	4	1	0	1
−.2606770833333333D+01	1	6	0	2
.3007812500000000D+01	1	4	1	2
−.8203125000000000D+00	1	2	2	2
.7161458333333334D+00	2	4	0	2
.3038194444444444D−01	1	0	3	2
−.3072916666666667D+00	2	2	1	2
.1138392857142857D−01	2	0	2	2
−.1822916666666667D−01	3	2	0	2
.6696428571428571D−03	3	0	1	2
.2755731922398589D−05	4	0	0	2
10	25			
−.9496093750000000D+01	1	8	0	1
.1564062500000000D+02	1	6	1	1
−.7820312500000000D+01	1	4	2	1
.5213541666666667D+01	2	6	0	1
.1203125000000000D+01	1	2	3	1
−.4692187500000000D+01	2	4	1	1
−.2734375000000000D−01	1	0	4	1
.9428571428571428D+00	2	2	2	1
−.5213541666666666D+00	3	4	0	1
−.2353670634920635D−01	2	0	3	1
.1653273809523809D+00	3	2	1	1
−.4151785714285714D−02	3	0	2	1
.5820105820105820D−02	4	2	0	1
−.1372354497354497D−03	4	0	1	1
−.2755731922398589D−06	5	0	0	1

Table 3. Value of D indicator for Mercury near perihelion and near aphelion for time steps 0.05 to 20 days using 10, 14 and 18th order f and g series

Days	0.05	0.5	1.0	4.0	8.0	15.0	20.0
perihelion							
fg10	$6\ 10^{-17}$	$5\ 10^{-17}$	$2\ 10^{-13}$	$2\ 10^{-07}$	$2\ 10^{-04}$	$1\ 10^{-01}$	1.4
fg14	$6\ 10^{-17}$	$3\ 10^{-16}$	$1\ 10^{-14}$	$5\ 10^{-10}$	$8\ 10^{-06}$	$4\ 10^{-02}$	1.8
fg18	$6\ 10^{-17}$	$5\ 10^{-17}$	$4\ 10^{-17}$	$1\ 10^{-12}$	$3\ 10^{-07}$	$2\ 10^{-02}$	2.4
aphelion							
fg10	$4\ 10^{-17}$	$5\ 10^{-17}$	$3\ 10^{-16}$	$6\ 10^{-11}$	$5\ 10^{-08}$	$2\ 10^{-05}$	$2\ 10^{-04}$
fg14	$4\ 10^{-17}$	$5\ 10^{-17}$	$3\ 10^{-16}$	$4\ 10^{-16}$	$7\ 10^{-11}$	$5\ 10^{-07}$	$2\ 10^{-05}$
fg18	$4\ 10^{-17}$	$5\ 10^{-17}$	$4\ 10^{-18}$	$3\ 10^{-17}$	$2\ 10^{-14}$	$5\ 10^{-10}$	$6\ 10^{-06}$

dering of the R column. To see the beauty of the simple symmetry among the exponents, the table should be ordered according to i or k exponents, separately for each order n and f and g terms. This symmetry enables us to control the creation of these power series. Thus for the derivative of order n, the relationships between the exponents should be the following:

For the odd n :
i grows from 1, with step 1, up to $(n-1)/2$,
k grows from 0, with step 1, up to $(n-1)/2-i$,
and $j=n-2(i+k)$ or $j=n-1-2(i+k)$ for the f and g series, respectively.
For the even n :
i grows up to $n/2$ for the f series and up to $n/2-1$ for the g series,
k grows up to $n/2-1-i$ for both series.
The j exponents follow the same formulae.

3. Some properties of the f and g series

Using these series as an alternative method of integration of the equations of motion of two or more bodies, it is necessary to pay attention to some of their properties.

First, it must be stated that, for a sufficiently short time interval, these series converge independently of the value of eccentricity, position on the orbit and distance from the central body. We use here the celestial mechanical meaning of "convergence" defined by Poincaré (1993, p. 317).
These series fulfil an interesting relationship $f\cdot G-g\cdot F=1$, - see Taff (1985, p.260). So a good indicator of the truncation error (which depends on the order of series) and the accuracy of calculations (at any instant for a given time interval) is the identity:

$$D=f\cdot G-g\cdot F-1=0 \quad (8)$$

where F and G denote the time derivatives of f and g. When we use the double precision mode of computer calculations, the appropriate order of series and a sufficiently short time interval, then fluctuations of D should not exceed 10^{-16}. As seen in Table 3, a one-day step at perihelion is too large to use the 10th and the 14th order series because $D>10^{-16}$, whereas at aphelion this step is nearly acceptable. We do not improve accuracy by using a step shorter than 0.5 day for both distances and different lengths of series. In most cases, we have used the 14th order series .

The time step has been adjusted by the condition:

$$(F_n+F_{n-1})/\ F<CR \quad (9)$$

The criterion CR has been fixed before calculations on such a level that also $D<10^{-16}$. Instead of the last partial sum F_n, it is better to use as numerator F_n+F_{n-1}, because often the absolute values $F_n>F_{n-1}$,as can be seen for the sequence of $f_2,...,f_{18}$ in Table 4. This saves the trouble connected with calculation of the additional series f,g,G when the time step is too large. The time step is shortened when formula (9) is not satisfied, but the program tries to make the step longer after several (up to 9) shorter steps. When we proceed with the calculations in the positive and then in the negative direction of the arrow of time, this procedure for changing the step permits us to avoid duplicate calculations for a given instant. For nearly-parabolic orbits, Taff (1985, p. 265) gives the expression necessary for convergence:

$$\mu t^2 r^{-3}<1 \quad (10)$$

which is well fulfilled by the above CR criterion, even for extremely high differences between time steps and distances. In example b) given below, for the distances $r_1=0.000422$ AU and $r_2=19.975$ AU, the $CR=10^{-20}$ criterion chooses the steps $t_1=0.0000119$ and $t_2=100.0$ days , which gives nearly equal values for expression (10): 0.032 and 0.022, respectively.

Table 4 contains the partial sums $f_2,f_3,...,f_{18}$, and the total values f,g,F,G and D, for the Mercury orbit near aphelion and perihelion for the 4-, 14-, and 20-day time steps. It is easily seen that near perihelion the f series converge very slowly and the $D<10^{-16}$ criterion is not fulfilled. It can be estimated from the rate of decrease of the perihelion terms for a 4-day step that the order of series should be about 25 for the last term to be equal to the term f_{18} in aphelion. This supports the 4-day interval

Table 4. The rates of change of the partial sums of f series for Mercury near aphelion and near perihelion for different time steps

	aphelion		perihelion		
days	4	20	4	14	20
f2	$-2.6\ 10^{-02}$	$-6.6\ 10^{-1}$	$-8.1\ 10^{-02}$	$-1.0\ 10^{0}$	$-2.0\ 10^{0}$
f3	$7.5\ 10^{-04}$	$9.3\ 10^{-2}$	$5.1\ 10^{-05}$	$2.2\ 10^{-3}$	$6.4\ 10^{-3}$
f4	$3.5\ 10^{-05}$	$2.2\ 10^{-2}$	$1.8\ 10^{-03}$	$2.7\ 10^{-1}$	$1.1\ 10^{0}$
f5	$-4.1\ 10^{-06}$	$-1.3\ 10^{-2}$	$-3.4\ 10^{-06}$	$-1.8\ 10^{-3}$	$-1.0\ 10^{-2}$
f6	$6.3\ 10^{-07}$	$9.9\ 10^{-3}$	$-4.7\ 10^{-05}$	$-8.6\ 10^{-2}$	$-7.3\ 10^{-1}$
f7	$-4.9\ 10^{-08}$	$-3.8\ 10^{-3}$	$1.7\ 10^{-07}$	$1.1\ 10^{-3}$	$1.3\ 10^{-2}$
f8	$3.6\ 10^{-09}$	$1.4\ 10^{-3}$	$1.5\ 10^{-06}$	$3.4\ 10^{-2}$	$5.8\ 10^{-1}$
f9	$-1.4\ 10^{-10}$	$-2.7\ 10^{-4}$	$-7.9\ 10^{-09}$	$-6.2\ 10^{-4}$	$-1.5\ 10^{-2}$
f10	$-6.3\ 10^{-12}$	$-6.2\ 10^{-5}$	$-5.2\ 10^{-08}$	$-1.4\ 10^{-2}$	$-5.1\ 10^{-1}$
f11	$2.3\ 10^{-12}$	$1.1\ 10^{-4}$	$3.7\ 10^{-10}$	$3.6\ 10^{-4}$	$1.8\ 10^{-2}$
f12	$-3.2\ 10^{-13}$	$-7.8\ 10^{-5}$	$2.0\ 10^{-09}$	$6.6\ 10^{-3}$	$4.8\ 10^{-1}$
f13	$3.2\ 10^{-14}$	$4.0\ 10^{-5}$	$-1.7\ 10^{-11}$	$-2.1\ 10^{-4}$	$-2.1\ 10^{-2}$
f14	$-2.5\ 10^{-15}$	$1.6\ 10^{-5}$	$-7.6\ 10^{-11}$	$-3.2\ 19^{-3}$	$-4.7\ 10^{-1}$
f15	$1.3\ 10^{-16}$	$3.9\ 10^{-6}$	$8.2\ 10^{-13}$	$1.2\ 10^{-4}$	$2.5\ 10^{-2}$
f16	$2.5\ 10^{-18}$	$3.9\ 10^{-7}$	$3.1\ 10^{-12}$	$1.6\ 10^{-3}$	$4.7\ 10^{-1}$
f17	$-1.7\ 10^{-18}$	$-1.3\ 10^{-6}$	$-3.9\ 10^{-14}$	$-6.9\ 10^{-5}$	$-2.9\ 10^{-2}$
f18	$2.6\ 10^{-19}$	$1.0\ 10^{-6}$	$-1.3\ 10^{-13}$	$-7.9\ 10^{-4}$	$-4.8\ 10^{-1}$
f	$9.7\ 10^{-01}$	$4.5\ 10^{-1}$	$9.2\ 10^{-01}$	$2.1\ 10^{-1}$	$-6.0\ 10^{-1}$
g	$6.8\ 10^{-02}$	$2.8\ 10^{-1}$	$6.7\ 10^{-02}$	$1.8\ 10^{-1}$	$2.3\ 10^{-1}$
F	$-7.3\ 10^{-01}$	$-2.8\ 10^{0}$	$-2.3\ 10^{0}$	$-5.3\ 10^{0}$	$-2.0\ 10^{+1}$
G	$9.8\ 10^{-01}$	$4.6\ 10^{-1}$	$9.2\ 10^{-01}$	$3.2\ 10^{-1}$	$1.8\ 10^{0}$
D	$3.0\ 10^{-17}$	$5.6\ 10^{-6}$	$1.1\ 10^{-12}$	$5.2\ 10^{-3}$	$2.4\ 10^{0}$

and the 25th order recurrent power series used by Guyader (1993). As seen from the last column, the series do not converge for a 20-day time interval.

4. Examples

The power series f and g are usable in many cases also when the eccentricity e of an orbit is near to 1 or even much greater. It is hard to find such extreme cases in the scientific literature, therefore some examples are presented below. The tests have been carried out by using $CR = 10^{-20}$ and the 14th order series in examples a) and e) and the 20th order series in other cases.

a) The N-body problem has been tested on the Solar System. The movements of all planets with reciprocal gravitational action have been calculated from epoch t_0 to $t_0 + 4000$ days and then backwards to t_0. This took 54000 steps, varied chiefly due to Mercury from 0.5 to 0.0625 d. The difference in coordinates for Mercury was 15 10^{-9} AU and 1 10^{-9} AU/d, whereas for the other planets the differences were ten times smaller. Comparison of the results with the Astronomical Almanac are less favorable. Starting with coordinates of the planets given for JD 2446080.5 (1985), after 3840 days the differences in x, y, and z between the Astronomical Almanac and our calculations are collected in Table 5. The backward calculation to the initial epoch leads to beginning coordinates with full accuracy.

Table 5. The differences between the Astronomical Almanac (1995) positions and coordinates calculated by f and g series on the basis of 1985 positions

Planet	Δx	Δy	Δz
Mercury	+0.0000633	−0.0000012	−0.0000073 AU
Venus	−0.0000497	+0.0000084	−0.0000007
Earth	+0.0000162	+0.0000087	+0.0000036
Mars	+0.0000221	+0.0000223	−0.0000110
Jupiter	+0.000001	+0.000000	−0.000001
Saturn	−0.000002	+0.000006	+0.000004
Uranus	−0.00000	+0.00000	+0.00000
Neptune	−0.00000	+0.00000	+0.00000
Pluto	−0.00001	−0.00000	+0.00000

b) The above N-body program acting for $N > 2$ was used to test the drop of accuracy for a high-eccentricity elliptic orbit with $e = 0.999983$. The indispensable third body had very small mass, so it introduced negligible perturbations. The Jupiter-size point-mass test body, with the position vector and velocity vector $\mathbf{r}(0,0,50)$ AU, $\dot{\mathbf{r}}(0.00001,0,0)$ AU/d was tracked during the entire period $P = 45650$ d. The steps at the beginning equaled 400 d, near perihelion (inside the Sun, $r = 63300$ km) decreased to 0.5 s. After the time interval $8P \approx 1000$ years, requiring over 900 steps, the differences were $\Delta r = 3\ 10^{-9}$ AU,$\Delta \dot{r} = 2\ 10^{-13}$ AU/d and $\Delta e < 5\ 10^{-13}$.

c) Changing in case b) the initial vectors into $\mathbf{r}(0,0,500)$ and $\dot{\mathbf{r}}(0.000003,0,-0.00111)$, we obtained $e = 1.00000062$ and $a = -6157.6$ AU. The beginning step was 5000 days, but after 300000 days, near perihelion ($r = 570000$ km), it shortened to 13 s. After 500000 days from the beginning, when $r = 375$ AU, the motion was reversed to the initial point. After 1 million days and 1800 steps: $\Delta r = 7\ 10^{-9}$ AU, $\Delta \dot{r} = 3\ 10^{-14}$ AU/d and $\Delta e < 5\ 10^{-13}$.

d) A fast moving object. If we use in the above example $\dot{z} = -0.11111$, the initial velocity grows from 1.92 to 192.4 km/sec. This gives $e = 1.1476$ and $a = -0.024$ AU. The body passes perihelion after 4500 days at distance 531000 km with step equal to 16.5 s. After 500000 days, the body reaches 55000 AU with maximal step 6400 days. At the starting point, after the reverse calculation, the differences are: $\Delta r = 6\ 10^{-9}$ AU, $\Delta \dot{r} < 1\ 10^{-12}$ AU/d, $\Delta e = 4\ 10^{-12}$. There appear some fluctuations of eccentricity of order 10^{-12} at distance 500 AU and 10^{-10} at distance 50000 AU. These fluctuations do not vanish even when the maximal step is limited to 100 days.

e) A small body placed near the $L4$ libration point, at beginning is traced in the rotating coordinate system. Adding to the velocity vector, which arises from the rotation of the coordinate system, any small back components calculated according to Rabe's (1961) algorithm, the body can be overtaken by Jupiter after 20000 days. The detailed tracing of the path of the body has begun before the encounter with Jupiter in the jovicentric, nonrotating coordinate system. The influence of the four Galilean satellites on the path of the body, tested in only a few configurations, nevertheless the body passes between Io and Europa, is not very large. So we neglect these detailed investigations and for clarity take into account only the Sun as a perturbing body. The shape of the path of the small body in the vicinity of Jupiter is seen in Fig. 1. For some of calculated orbits, take place a collision with Jupiter during the second encounter. A similar problem described in detail by Sekanina et al. (1993) is connected with the motion of C/Shoemaker-Levy 9 on its collisional orbit.

On the trajectory shown in Fig. 1, one may distinguish several significant dates. Counting from the beginning signed on the curve by the arrow, at 415 days the osculating orbits (counted after every ten steps) are transformed from hyperbolic orbits into elliptic ones. At 1265 days, there takes place the first encounter with Jupiter at distance 0.0088 AU, and at 1700 days the apojove is 0.4016 AU. This is the top of the inner loop, where the eccentricity reaches 0.999999999954. Next, at 2268 days, the second encounter with Jupiter (0.0036 AU) takes place. At 2900 days, the elliptic progression of osculating orbits transfer into hyperbolics ones.

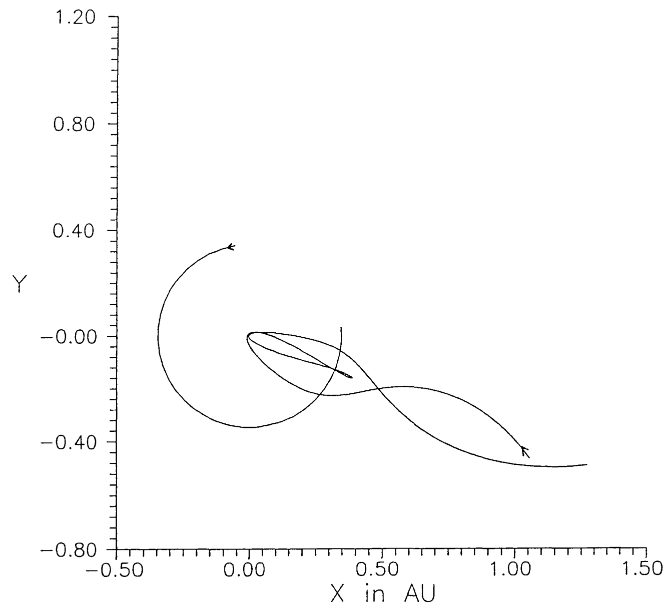

Fig. 1. The jovicentric motion of a body in the vicinity of Jupiter during a time interval of 3200 days. The eccentricity of osculating orbits changes very fast along the curve in a range $0.80 < e < 3.80$ due only to perturbations from the Sun. The integration step cannot exceed 0.5 day

After 3200 days, the calculations are carried out in the opposite direction. The influence of the four Jupiter satellites, in the absence of very close encounters, change the above dates by less than ten days.

The position of the inner libration point $L1$ during these 3200 days is marked in the figure as a circle of radius 0.3475 AU. The position of the Sun during this time should be seen on an analogous circle of radius 5.2 AU.

The eccentricities of the osculating orbits along this curve are in the range 0.80 to 3.80. The differences between the position vectors and velocity vectors in the beginning and after $2 \cdot 3200$ days in 15000 steps are: $\Delta r = 1\ 10^{-8}$ AU, $\Delta \dot{r} < 0.5\ 10^{-9}$ AU/d, $\Delta e = 1\ 10^{-7}$.

The step changes automatically from 11 minutes to the a priori limited value of 0.5 day. This limit is necessary. The final error grows rapidly when we use longer steps. For example, a 1-day step leads to $\Delta r = 15\ 10^{-8}$ AU. At maximum distance from Jupiter, the CR criterion demands a maximal step that is over 10 days, so such a step is wrongly adjusted to the rate of changes of the osculating orbit elements. Even in the short time interval of $2 \cdot 1000$ days, the free step causes an error $\Delta r = 4\ 10^{-6}$ AU. The loss of accuracy takes place also at the top of the inner loop. The use of the 20th-order series with the same $CR = 10^{-20}$ and 0.5 day maximal step leads to a smaller number of steps and to worsening of the result ($\Delta r = 1 \cdot 10^{-6}$ AU). This effect is not obvious in the variant in which the four Galilean satellites also perturb the body, together with the Sun. They, especially Io, restrict the step to 0.5 day or less on the basis of the CR criterion. Thus far, we have not been able to restrict automatically the step from the upper limit. Such a limit seems to be very important when the attractive force of a central body is approximately equaled by the forces of perturbing bodies.

5. Conclusions

I) As stated by Taff (1985), the f and g series are competitive to other methods of integration in the investigation of the motions of celestial bodies.

II) A sufficiently small time step permits avoiding the problem of convergence depending on the eccentricity and the position of a body on its orbit.

III) When we use these series to the N-body problem, the maximal step must be limited and adjusted to the rates of change of the elements of osculating orbits of all bodies, whereas the minimal step should be generated according to the minimal distance of two bodies.

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