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An asymptotically vanishing time-dependent cosmological "constant"

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Abstract. We investigate the coupled system of gravity and a scalar field with exponential potential. The energy momentum tensor of the scalar field induces a time-dependent cosmological "constant". This adjusts itself dynamically to become in the "late" universe (including today) proportional to the energy density of matter and radiation. Possible consequences for the present cosmology are shortly discussed. We also address the question of naturalness of the model.

Key words: cosmology: theory – cosmology: dark matter – cosmology: early Universe

1. Introduction

Whenever cosmology encounters potential difficulties in the description of the present universe cosmologists revive the discussion about the cosmological constant (Einstein 1917). The discrepancy between the critical energy density expected from inflationary cosmology and lower dynamical estimates of this density has been attributed to the cosmological constant (Gunn et al. 1975; Sandage et al. 1984; Peebles 1986; Brown et al. 1987; Olson et al. 1987).

The discussion also pertains to the age of the universe and the formation of structure (Vittorio et al. 1985; Sugiyama et al. 1990; Kofman et al. 1993). In fact, a cosmological constant λ of the order of today's critical energy density in the universe ($\lambda \approx (2 \cdot 10^{-3}\,\mathrm{eV})^4$) strongly affects the present universe without altering the successful predictions of the hot big bang model at early stages of the evolution of the universe.

Despite many attempts (Weinberg 1989; Carrol et al. 1992) we have at present no satisfactory understanding why $\lambda^{1/4}$ should be much smaller than typical energy scales of the standard model or even the Planck mass M_p. For a time-independent cosmological constant it seems even harder to explain why it should be of the order of the present energy density. The latter depends on the age of the universe rather than on fundamental constants. It looks then not very natural that a constant λ should have a value which equals the energy density just at a time within the present cosmological epoch. In this work we consider a model where the cosmological "constant" varies with time such that the asymptotic solution for late times is characterized by a constant ratio $\lambda(t)/\rho(t)$ (Wetterich 1988; Ratra et al. 1988). We discuss consequences for present cosmology and various alternatives how "early cosmology" could have made a transition to this type of "late cosmology". We also briefly address the question of naturalness of an asymptotically vanishing cosmological "constant".

2. Scalar field with exponential potential

We start from the field equations for a scalar field φ coupled to gravity in a homogenous and isotropic universe (with $k = 0$ and H the Hubble parameter)

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = q^\varphi \qquad (1)$$

$$\dot{\rho} + 3H(\rho + p) + q^\varphi \dot{\varphi} = 0 \qquad (2)$$

$$H^2 = \frac{1}{6M^2}\left(\rho + \frac{1}{2}\dot{\varphi}^2 + V\right) \qquad (3)$$

The effective scalar potential potential is assumed to decrease exponentially for large φ

$$V(\varphi) = \hat{V} \exp\left(-a\frac{\varphi}{M}\right). \qquad (4)$$

where we define $M^2 = M_\mathrm{p}^2/16\pi$ and $a > 0$ is a free parameter of the model[1]. The constant $\hat{V} > 0$ is arbitrary since it can be changed by a shift in φ. The scalar potential acts as a cosmological constant (for details see later). Exponential potentials arise very naturally in all models of unification with gravity as Kaluza-Klein theories, supergravity theories or string theories. In higher dimensional theories the scalar could be associated with the volume of "internal space". The exponential form of the potential reflects here the fact that time derivatives in gravity typically involve the logarithm of length scales (Wetterich

[1] For potentials increasing exponentially with the field we can obtain positive a by changing the sign of the field.

1985). Similar general arguments can be formulated for the exponential form of the potential for the dilaton in string theories (or for some of the $SU(3)\times SU(2)\times U(1)$ singlet moduli fields). In the context of inflation exponential potentials have been discussed (Shafi et al. 1985, 1987, 1988; Luchin et al. 1985; Halliwell 1987).

We also account for possible couplings of the scalar field to matter (Wetterich 1988a)

$$q^{\varphi} = g^{-1/2}\left\langle \frac{\delta S}{\delta\varphi}\right\rangle_{\text{incoherent}} = \beta\frac{\rho}{M}. \tag{5}$$

Here β may effectively depend on the component dominating the energy density (e.g. radiation, baryons, or non-baryonic dark matter). For example, in a baryon dominated universe Eq. (5) obtains from a nucleon-scalar coupling corresponding to the Lagrangian

$$\mathscr{L}_n = -m_n^{(0)}\exp\left(-\beta_n\frac{\varphi}{M}\right)\bar{n}n \tag{6}$$

similar to the one obtained in string theories. The last term in Eq. (2) reflects the change of the effective nucleon mass induced by a change of φ. The late cosmology for a string theory where one of the scalar modes remains massless can be formulated in terms of the parameters a and β. (There are possibly different β_i for the scalar couplings of different fields and a completely flat potential is equivalent to the limit $a\to\infty$.) As a byproduct of our general discussion one obtains cosmological constraints on acceptable string theories once a and β are computed for a given massless mode.[2]

We note that the stability properties of condensed objects with a static energy density (as stars or the earth) depend on the sign of β: A static solution for φ must obey (1)

$$\frac{\partial V}{\partial\varphi} = -\frac{a}{M}\hat{V}\exp\left(-a\frac{\varphi}{M}\right) = \beta\frac{\rho}{M} \tag{7}$$

It only exists for negative β where, for homogeneous ρ

$$\varphi(\rho) = -\frac{M}{a}\ln\left(-\frac{\beta\rho}{a\hat{V}}\right) \tag{8}$$

Then also the nucleon mass (6) and therefore ρ is static. For $\beta=0$ the scalar field decouples from ordinary matter. Even though there is no static solution for φ anymore, this does not affect the stability properties of matter: The nucleon mass (or similar parameters) are independent of φ. For $\beta=0$ ordinary matter and the scalar field "know about each other" only through their common gravitational effects. In contrast, for $\beta>0$ the absence of a static solution for φ results in a time dependence of the nucleon mass. We will mainly concentrate here on the case $\beta\le 0$.

[2] For bounds on such a coupling in the more general case of a scalar field with mass see (Ellis et al. 1989).

3. Asymptotic solution

Let us now investigate cosmological solutions of the field Eqs. (1)–(5). We first look for asymptotic solutions where $\rho(t)\sim t^{-2}$. Indeed, the field equations admit the solution

$$\begin{aligned} H &= \eta t^{-1}\\ \rho &= \rho_0 t_0^2 t^{-2}\\ \varphi &= \varphi_0 + \frac{2M}{a}\ln\frac{t}{t_0} \end{aligned} \tag{9}$$

with

$$\dot{\varphi} = \frac{2M}{a}t^{-1} \tag{10}$$

$$V(t) = V_0 t_0^2 t^{-2}. \tag{11}$$

All energy densities $(\rho, V, \frac{1}{2}\dot{\varphi}^2)$ are $\sim t^{-2}$ and the energy density of the scalar field decreases simultaneously with ρ.[3] This solution can be interpreted as the asymptotic solution for $t\to\infty$ (see later). The coefficient η obtains from (2)

$$\eta = \frac{2}{n}\left(1-\frac{\beta}{a}\right) \tag{12}$$

where we use for the equation of state the convention

$$3(\rho+p) = n\rho. \tag{13}$$

A deviation from the expansion law of the standard Friedmann universe occurs only for β different from zero. (We only consider $\beta/a<1$.)

The time-independent ratios of energy densities follow from (1) and (3)

$$V + \frac{\beta}{a}\rho = \frac{2}{a^2}\frac{(3\eta-1)}{\eta^2}M^2H^2 \tag{14}$$

$$V+\rho = \left(6-\frac{2}{a^2\eta^2}\right)M^2H^2. \tag{15}$$

With the definitions

$$\begin{aligned} \rho_c &= 6M^2H^2, \quad \Omega_M = \rho/\rho_c\\ \Omega_V &= V/\rho_c, \quad \Omega_K = \frac{1}{2}\dot{\varphi}^2/\rho_c \end{aligned} \tag{16}$$

one obtains

$$\begin{aligned} \Omega_M &= 1-\frac{n-2\beta(a-\beta)}{2(a-\beta)^2}\\ \Omega_V &= \frac{n(6-n)-12\beta(a-\beta)}{12(a-\beta)^2}\\ \Omega_K &= \frac{n^2}{12(a-\beta)^2}\\ \Omega_\varphi &= \Omega_V+\Omega_K = \frac{n-2\beta(a-\beta)}{2(a-\beta)^2} = 1-\Omega_M. \end{aligned} \tag{17}$$

[3] Cosmologies with a decaying scalar field were proposed first in (Wetterich 1988b) and studied in a more general context in (Ratra et al. 1988; Peebles et al. 1988). An exponential potential without coupling to matter ($\beta=0$) was considered by (Wetterich 1988b; Ratra et al. 1988), and an extensive discussion for potentials decreasing with a power of φ can be found in (Ratra et al. 1988; Peebles et al. 1988).

Our solution exists only for positive Ω_M and Ω_V and we have to require

$$a(a-\beta) > \frac{n}{2} \tag{18}$$

$$\beta(a-\beta) < \frac{n(6-n)}{12}. \tag{19}$$

For the radiation dominated epoch ($n = 4$) β presumably vanishes. Then our solution only exists for $a^2 > 2$ (18). Equation (19) requires for $n=3$ (non-relativistic matter) $\beta(a-\beta) < \frac{3}{4}$. We emphasize that the scalar field contributes to the energy density of the universe even for $\beta = 0$. It is also interesting to note that the scalar field fulfils in our case the "cosmon condition" (Peccei et al. 1987) for the relation between the dilatation anomaly Θ^μ_μ and the trace of the energy momentum tensor T^μ_μ

$$\Theta^\mu_\mu \sim T^\mu_\mu. \tag{20}$$

A scalar field with an exponential potential may therefore be called a cosmon.

4. Modifications of late cosmology

Our scenario has several important consequences for cosmology at the present epoch ($n = 3$):

1. For $\beta \neq 0$ the age of the universe is modified

$$t_0 = \eta H_0^{-1} = \frac{2}{3} H_0^{-1} \left(1 - \frac{\beta}{a}\right) \tag{21}$$

For given H_0 it is larger than the standard age if β is negative.

2. The scalar field contributes to the cosmological energy density today. In units of the critical energy density this contribution is given by Ω_φ (17). The cosmon is therefore a potential dark matter candidate.

3. The cosmon mediates a new attractive long range force. The scalar mass m_φ (inverse of the range) is time-dependent and proportional to the Hubble parameter

$$m_\varphi^2 = V''(\varphi) = \frac{a^2}{M^2} V(\varphi) = \frac{9a^2 - 12\beta a^2(a-\beta)}{2(a-\beta)^2} H^2 \tag{22}$$

For all purposes except cosmology the cosmon is effectively massless.[4] Its coupling to nucleons $\sim \beta_n / M$ (6) is of gravitational strength (or weaker). At distances small compared to H^{-1} the effective Newton's constant for nucleons has a contribution from the scalar force

$$G_N = \frac{1}{M_{\rm p}^2}(1 + 4\beta_n^2) \tag{23}$$

General relativity distinguishes between the tensor and scalar component of the attractive force. For distances much smaller than H^{-1} the mass term, or more general the influence of the potential V, can be neglected. For the matter-dominated epoch the coupled system of gravity and small scalar fluctuations around the cosmological value $\varphi(t)$ (9) is in this range identical to the standard Brans-Dicke (Brans et al. 1961) theory with

$$\omega = \frac{1}{8\beta_n^2} - \frac{3}{2} \tag{24}$$

(By an appropriate Weyl scaling the Brans-Dicke theory appears as a gravity theory with fixed Newton's constant plus a massless scalar field with universal coupling to matter (Wetterich 1988a).) By this analogy we infer the bound (from $\omega > 500$)

$$|\beta_n| < 0.016 \tag{25}$$

For a of the order one or larger the modification of the age of the universe (21) can therefore only be minor and is far smaller than the observational uncertainties in the foreseeable future. We also note that a possible non-universal coupling of the scalar field to nucleons (for example proportional to baryon number) must be orders of magnitude smaller than the bound (25). We finally remark that for a uniform density of condensed objects the effective range of the scalar interaction is modified according to (8), i.e. $V''(\varphi(\rho)) = -\beta a \rho / M^2$. A typical range for the density of the earth $\rho_{\rm E} = 5\,{\rm g\,cm^{-3}}$ is $(M^2/\rho_{\rm E})^{1/2} = 7.32 \cdot 10^{10}$m. This exceeds by far the radius of the earth and justifies the neglection of the potential term for observations in our solar system.

4. The scalar field induces a time-dependent cosmological constant. As we have seen the ratio V/ρ approaches asymptotically the constant Ω_V/Ω_M (17). The kinetic term of the scalar field also contributes to the total energy density. There is an ambiguity in the definition of the cosmological constant λ. One may either put the emphasis of the fact that the energy density of the scalar field does not participate in gravitational collapse on scales much smaller than the horizon (Peebles 1984) and associate the cosmological constant with Ω_φ (17). Alternatively, one may concentrate on the dynamics of the universe as a whole and take the contribution of the cosmological constant to the energy momentum tensor proportional to the metric. Then part of the potential and kinetic scalar energy may be treated as a new form of nonrelativistic or relativistic "matter". If we adopt for the energy momentum tensor the definition

$$\begin{aligned} T_{00} &= V + \frac{1}{2}\dot\varphi^2 + \rho = \lambda + \rho_\varphi + \rho \\ T_{ij} &= (V - \frac{1}{2}\dot\varphi^2 - p) g_{ij} = (\lambda - p_\varphi - p) g_{ij} \\ p_\varphi &= (\frac{1}{3}n - 1)\rho_\varphi \end{aligned} \tag{26}$$

one obtains the splitting

$$\begin{aligned} \rho_\varphi &= \frac{3}{n}\dot\varphi^2 \\ \lambda &= V + (\frac{1}{2} - \frac{3}{n})\dot\varphi^2 = -\frac{6\beta}{a-\beta} M^2 H^2 \end{aligned} \tag{27}$$

In this language λ vanishes for $\beta = 0$ (compare also (21)), whereas for $\beta \neq 0$ we find $\dot\lambda \sim H^3$ and recover one of the general solutions for a varying cosmological "constant" discussed in (Reuter et al. 1987).

[4] This distinguishes the present cosmon model from the model of Peccei et al. 1987.

We should emphasize that the point of view of a time-varying cosmological "constant" is only one way of looking at our model. Alternatively, our solution can be interpreted as a modification of the standard cosmological scenario by the presence of scalar field contributions to the energy momentum tensor. The combined energy momentum tensor of matter, radiation and the scalar field remains conserved. In this language the energy density of the scalar field is just one of the contributions to the total energy density and the latter obeys all usual conservation laws. We also note that in our formulation Newton's constant is kept fixed and sets the mass scale. A possible time variation of other fundamental parameters such as particle masses, the fine structure constant or the Fermi constant, can in principle arise if these constants depend on the value of the scalar field φ. For $\beta_n \neq 0$ a cosmological time dependence of the nucleon mass is indeed predicted by Eq. (6) which implies $m_n(t) = m_n^{(0)} \exp\left(-\frac{\beta_n}{M}\varphi(t)\right)$. The modifications of late cosmology for $\beta \neq 0$ are closely related to this time dependence. (Possible consequences for nucleosynthesis of the cosmological time dependence of m_n/M_p are discussed in Sect. 7.) The cosmological time dependence of other fundamental constants depends on details of the particle physics model for the coupling of φ to the other elementary particles. A discussion can be found in (Ellis et al. 1989). A cosmological time dependence of fundamental parameters does not imply a time dependence of these parameters on earth or in the stars. In condensed objects where the energy density is much higher than the cosmological energy density the value of the scalar field is determined by the static solution (7). This implies static fundamental constants, but the fundamental parameters become now density-dependent. This density dependence can be used to derive bounds for the coupling of φ to the different sorts of elementary particles (Ellis et al. 1989).

5. Stability analysis

The solution (9) is not the most general homogeneous and isotropic universe consistent with (1)–(5). In order to establish that a large class of solutions is attracted for large t towards this particular solution we investigate first the general behaviour of small deviations from (9)[5]

$$\begin{aligned} H &= \bar{H}(t)(1+h(t)) \\ \rho &= \bar{\rho}(t)(1+r(t)) \\ \varphi &= \bar{\varphi}(t) + \delta\varphi(t) \end{aligned} \tag{28}$$

Here $\bar{H}$, $\bar{\rho}$ and $\bar{\varphi}$ are given by (9), and we linearize in the small quantities h, r and $\delta\varphi$. We insert (3)

$$h = \frac{1}{2}(\Omega_M r - a\Omega_V \frac{\delta\varphi}{M} + 2\Omega_K \frac{\delta\dot{\varphi}}{\dot{\bar{\varphi}}}) \tag{29}$$

into Eqs. (1), (2) and introduce the variables

$$\begin{aligned} \tau &= \eta \ln(t/t_0) \\ \delta\varphi' &= \frac{d}{d\tau}\delta\varphi, \quad r' = \frac{d}{d\tau} r \\ y &= \frac{\delta\varphi}{M}, \quad z = \frac{\delta\varphi'}{M} \end{aligned} \tag{30}$$

This gives a coupled system of three linear first-order differential equations for $x = (z, y, r)$

$$x' = -Gx \tag{31}$$

with

$$G = \begin{pmatrix} A & B & C \\ -1 & 0 & 0 \\ D & E & F \end{pmatrix} \tag{32}$$

$$\begin{aligned} A &= 3 + 3\Omega_K - \frac{1}{\eta}, \qquad B = (6a^2 - \frac{3}{2}a\gamma)\Omega_V \\ C &= (\frac{3}{2}\gamma - 6\beta)\Omega_M, \qquad D = \frac{n}{\gamma}\Omega_K + \beta \\ E &= -\frac{n}{2}a\Omega_V, \qquad F = n + \frac{n}{2}\Omega_M - \frac{2}{\eta} + \beta\gamma \end{aligned} \tag{33}$$

where

$$\gamma = \sqrt{12\Omega_K} = \frac{n}{a-\beta} \tag{34}$$

The general solution depends on three initial values $r_0 = r(t_0)$, $y_0 = \delta\varphi(t_0)/M$, $z_0 = t_0\delta\dot{\varphi}(t_0)/\eta M$, namely

$$x(t) = \exp\{-\eta \ln\left(\frac{t}{t_0}\right) G\} x_0 \tag{35}$$

The asymptotic behaviour for large t depends on the eigenvalues g_i of the matrix G. If $\mathrm{Re}(g_i) > 0$, an arbitrary small r or $\delta\varphi$ vanishes asymptotically with a negative power of t. In addition, one finds oscillatory behaviour if $\mathrm{Im} g_i \neq 0$. For a large range in a and β one has indeed $\mathrm{Re}(g_i) > 0$ and the solution(9) is reached asymptotically for any small enough initial value x_0.

In particular, for $\beta = 0$ and $n = 3$ one finds

$$\begin{aligned} A &= \frac{3}{2} + \frac{9}{4a^2}, \qquad B = \frac{9}{2} - \frac{27}{8a^2} \\ C &= \frac{9}{2a}(1 - \frac{3}{2a^2}), \qquad D = \frac{3}{4a} \\ E &= -\frac{9}{8a}, \qquad F = \frac{3}{2} - \frac{9}{4a^2} \end{aligned} \tag{36}$$

and

$$\det G = \frac{27}{4}\left(1 - \frac{3}{2a^2}\right) \tag{37}$$

[5] We can restrict the discussion to isocurvature fluctuations where $H(t)$ can be computed from $\rho(t)$ and $\varphi(t)$ (3). For the zero curvature cosmologies investigated here one can always take the radius of the universe arbitrarily large. It is then easy to see that the adiabatic fluctuations (which grow as usual) decouple and need not to be considered explicitly. This reduces the stability analysis to a three-dimensional linear system instead of the five-dimensional system investigated by Ratra et al. 1988.

We observe that the stability requirement $\det G > 0$ coincides with the condition (18). For large values of $a (a \to \infty)$ the matrix G becomes block diagonal in the first two and the third indices, $C, D, E \to 0$. Then r decays exponentially with τ such that

$$\rho(t) = \bar{\rho}(t)\left(1 + \frac{r_0 t_0}{t}\right) \tag{38}$$

and the general solution for the scalar field is described by a decaying oscillation

$$\varphi(t) = \bar{\varphi}(t) + \delta\varphi_0 \left(\frac{t_0}{t}\right)^{1/2} \cos\left(\sqrt{\frac{7}{4}} \ln\frac{t}{t_0} + \alpha_0\right) \tag{39}$$

The asymptotic solution ($t \to \infty$) for $a \to \infty$ corresponds to standard cosmology since $\Omega_V, \Omega_K \to 0$ (17). For finite values of a within the range of stability of the solution (9) the general solution has qualitatively similar properties as Eqs. (38), (39), with oscillatory behaviour now also present in the time dependence of ρ (for $a^2 > \frac{12}{7}$). The eigenvalues of G are $\frac{3}{2}$ and $\frac{3}{4}(1 \pm i\sqrt{7 - \frac{12}{a^2}})$ and agree with the analysis of Ratra et al. 1988.

6. Transition to the asymptotic solution

On the other hand, it is obvious that for initial values $\varphi(t_0)$ and $\dot\varphi(t_0)$ such that $V(t_0) \ll \rho(t_0), \dot\varphi^2(t_0) \ll \rho_0(t_0)$ the scalar field does not influence the time evolution of the universe for a certain period after t_0. Within a good approximation one obtains for this epoch the standard Friedmann cosmology

$$\begin{aligned} H &= \frac{2}{n}t^{-1} = \eta t^{-1} \\ \rho &= \rho_c = 6M^2H^2 \end{aligned} \tag{40}$$

As long as $V(t)$ remains small compared to $\rho(t)$, the solution for φ is approximately[6].

$$\varphi(t) = \begin{cases} \varphi_0 + \frac{6\eta^2}{3\eta - 1}\beta M \ln\frac{t}{t_0} & \text{for} \quad \beta \neq 0 \\ \varphi_0 + \delta\varphi_0 \left(\frac{t}{t_0}\right)^{1-3\eta} & \text{for} \quad \beta = 0 \end{cases} \tag{41}$$

The contribution of the scalar potential to the total energy density of the universe is then

$$\Omega_V(t) = \begin{cases} \frac{V_0}{6M^2\eta^2} t^2 \left(\frac{t}{t_0}\right)^{-\frac{6\eta^2\beta a}{3\eta-1}} & \text{for} \quad \beta \neq 0 \\ \frac{V_0}{6M^2\eta^2} t^2 \exp\left(-a\frac{\delta\varphi_0}{M}\left(\frac{t}{t_0}\right)^{1-3\eta}\right) & \text{for} \quad \beta = 0 \end{cases} \tag{42}$$

We observe that Ω_V increases with a power of t for large values of t. After a possible short period where the kinetic energy is damped rapidly, the scalar field essentially remains constant. It "sits there and waits" until Ω_V has reached a value of the order one. Then cosmology makes a transition to the behaviour described by the solution (9). We conclude that the solution (40)–(41) is unstable at late times since Ω_V does not remain small. Instead of making a transition to the De Sitter or anti-De Sitter universe as in the case of a constant cosmological "constant" we find in our model a much smoother transition to the "late cosmology" given by (9).

There is a characteristic time of transition t_{tr} when the ρ-dominated universe turns over to "late cosmology", where Ω_V reaches a constant asymptotic value. This transition time depends strongly on the initial value of the cosmological constant as given by the potential energy density $V(t_0) = V_0$, i.e.

$$t_{tr} \approx M V_0^{-\frac{1}{2}} \tag{43}$$

One may have the prejudice that $V_0 \sim M^4$ and therefore $t_{tr} \sim M^{-1}$. In this case our modified cosmology would describe the universe at all times after a possible short initial period as, for example, inflation. We have, however, no real knowledge on the initial value $\varphi(t_0)$. Inflation is most likely driven by a field different from φ in order to assure sufficient heating of the universe. One may imagine that the system of fields describing inflationary cosmology ends with a value V_0 many orders of magnitude smaller than M^4. We therefore should also conceive the possibility that the universe (in the past-inflationary period) is first described by Friedman cosmology. Our modified cosmology becomes then relevant only at some unknown critical time t_{tr}.

As an alternative to the transition from a ρ-dominated universe to the cosmology with constant V/ρ we may also look at the case where the energy density of the universe is dominated at some early epoch by contributions from the scalar field (φ-dominated universe). Looking for solutions of the field equations (1)–(3) with $\rho = 0$ we first investigate the approximation where the damping term $3H\dot\varphi$ in Eq. (1) can be neglected[7]. In this regime the universe undergoes an exponential expansion similar to scenarios of the inflationary universe (Starobinski 1980; Guth 1981; Linde 1982, 1983; Albrecht et al. 1982):

$$\begin{aligned} V + \frac{1}{2}\dot\varphi^2 &= E = \text{const} \\ H^2 &= \frac{E}{6M^2} \end{aligned} \tag{44}$$

Here $\varphi(t)$ is determined implicitly from $V(t)$ via

$$\frac{\sqrt{E} - \sqrt{E - V(t)}}{\sqrt{E} + \sqrt{E - V(t)}} = C_0 \exp(-\sqrt{12} a H t) \tag{45}$$

with C_0 a free integration constant. Such an exponential expansion lasts as long as $3H\dot\varphi$ can be neglected compated to $\ddot\varphi$. The approximate solution (45) implies the ratio

$$\frac{3H\dot\varphi}{\ddot\varphi} = \frac{1}{a}\left(\frac{3E(E - V)}{V^2}\right)^{1/2} \tag{46}$$

[6] For $\beta \neq 0$ we have not given the most general solution. It looks similar to the solution for $\beta = 0$ at early times and makes then a transition to the logarithm evolution.

[7] Notice that $\rho = p = q_\varphi = 0$ is always a solution of the field equations.

which becomes of the order one once $V(t)$ has decreased sufficiently according to (45). The number of e-foldings of the length scale of the universe during the period of exponential expansion is roughly given by $1/(\sqrt{12}a)$.

Another solution for the pure scalar-gravity system ($\rho = 0$) is given by

$$\varphi = \varphi_0 + \frac{2M}{a}\ln\frac{t}{t_0}$$
$$H = \frac{1}{a^2}t^{-1}, \quad V = \frac{2M^2}{a^2}\left(\frac{3}{a^2} - 1\right)t^{-2} \tag{47}$$

It exists for $a^2 < 3$ and is a stable attractor. (Small deviations of φ from this solution die out with linear combinations of $t^{-\lambda_i}, \lambda_1 = 1, \lambda_2 = \frac{3}{a^2} - 1$.) As long as ρ is small compared to V this solution approximately determines the cosmology. During this epoch ρ decays as

$$\rho = \rho_0\left(\frac{t}{t_0}\right)^{-\delta}$$
$$\delta = \frac{n}{a^2} + 2\frac{\beta}{a} \tag{48}$$

For $a(a-\beta) > \frac{n}{2}$ (cf. (18)) one finds $\delta < 2$ and any initial nonzero ρ becomes comparable to V at late enough time. The asymptotic solution is then given by Eq. (9). In the opposite case $a(a-\beta) < \frac{n}{2}$ the energy density ρ decreases faster than V. The universe remains φ-dominated at late times. Even an initially ρ-dominated universe (40,41) will in this case turn over to a φ-dominated universe (47,48) at the transition time $t_{\rm tr}$. We observe that the age of the universe is given by

$$t_0 = \frac{1}{a^2}H_0^{-1} \tag{49}$$

if we live today in a φ-dominated universe. The universe is older than for standard cosmology if $a^2 < \frac{3}{2}$. The present lower bound on Ω_M indicates, however, that the universe cannot have been φ-dominated a long time before the present epoch if $a(a-\beta)$ is significantly smaller than $\frac{3}{2}$. This case therefore necessitates additional fine-tuning in the initial condition and is in this respect similar to the power-law potentials discussed by Peebles et al. 1988, Ratra et al. 1988.

7. Constraints from nucleosynthesis

Let us next ask what type of constraints must be imposed on the parameter a from the successful description of nucleosynthesis in the standard big bang model. These constraints depend strongly on the transition time $t_{\rm tr}$ when the asymptotic "late cosmology" with constant V/ρ begins. Let us first assume that $t_{\rm tr}$ is before the characteristic time for nucleosynthesis. The standard nucleosynthesis scenario is then modified by a different speed of the "gravitational clock", i.e. the ratio between the Hubble parameter and the temperature. We express temperature and time in units of the (possibly time-dependent) nucleon mass and look at the time evolution of

$$\tilde{T} = T/m_n, \quad \tilde{t} = tm_n \tag{50}$$

The critical quantity (corresponding to H/T in standard cosmology) is given by

$$X = -\tilde{T}^{-2}\frac{\mathrm{d}\tilde{T}}{\mathrm{d}\tilde{t}} = \left(1 + \frac{\beta_r}{2a\eta} - \frac{2\beta_n}{a\eta}\right)\left(1 - \frac{2\beta_n}{a}\right)^{-1}\frac{H}{T} \tag{51}$$

(Here we have included for completeness a possible coupling β_r of the scalar field to radiation. We will concentrate on $\beta_r = 0$ where $\eta = 1/2$.) Due to the scalar energy density the ratio H/T is increased by a factor $\Omega_M^{-1/2}$ (cf. (17)) as compared to the radiation-dominated universe. Another factor $(1+4\beta_n^2)^{-1/2}$ arises from the renormalization of the Planck mass (23). If we require X^2 to be modified by less than 10 % as compared to standard cosmology (more precise bounds can easily be formulated by noting the equivalence of X^2 with additional or missing neutrino species) one finds ($\beta_r = 0$)

$$\left|\frac{\left(1 - 4\frac{\beta_n^2}{a^2}\right)^2 a^2}{\left(1 - 2\frac{\beta_n^2}{a^2}\right)^2(1+4\beta_n^2)(a^2-2)} - 1\right| < 0.1 \tag{52}$$

or, for very small $|\beta_n/a|, a^2 > 22$. In contrast, if $t_{\rm tr}$ is after the end of nucleosynthesis the only constraint on a is given by the existence of the asymptotic solution (18).

For a scenario where the scalar energy density dominates the present universe and a is time-independent, we conclude that $t_{\rm tr}$ should be later than nucleosynthesis. This type of cosmology is characterized by an initial value $\varphi(t_0)$ (e.g. for t_0 at the end of inflation) in a range where $t_{\rm tr}$ (43) comes out between the end of nucleosynthesis and today. This does not require a fine-tuning since $V(t_0)$ can vary over many orders of magnitude. It necessitates, however, a small ratio V_0/M^4 which, given our lack of knowledge on details of the scalar field dynamics during inflation, waits for an explanation - somewhat similar to the small initial value of $\Omega - 1$ in Friedman cosmology. As an alternative, we may consider the case where the parameter a varies as a function of the curvature scalar R or the value of the scalar field. As we will see below the quantity a is closely related to the breakdown of dilatation symmetry and there is no reason why it should be expected to be exactly constant (Wetterich 1988b). In the course of the cosmological evolution a dependence of a on R, H or φ would lead to an effective time dependence of a. Interesting cosmological scenarios with time-dependent a can be imagined: As a first example a may depend on φ in a way that the ratio V/ρ decreases for $n = 4$ but increases for $n = 3$. An appropriate φ-dependence of a is not very difficult to construct since the exact exponential potential (4) seems to be the boundary case between potentials leading to an asymptotic φ-dominated universe (for example power law potentials or exponential potentials with small a) and potentials implying a ρ-dominated asymptotic universe (see Wetterich 1988b for an example). Consistency between small Ω_φ during nucleosythesis and an important Ω_φ in the latest period of the universe can be achieved in this way. As a different possibility we mention modifications of the field equations induced by a curvature or field dependence of a which are mild enough not to disturb substantially the cosmological picture for constant a. In this case our asymptotic solution applies with the modification that Ω_φ changes slowly. Imagine that a decreases from a value of about twenty during nucleosynthesis to three today[8]. This factor of seven can be achieved either by a logarithmic dependence of a on R or by a power law dependence with a very small power. It is pointless here to discuss details of all these possible scenarios. Our main conclusion is that for practical purposes we can simply omit the constraint from nucleosynthesis (52) since it is based on the assumption of exact constancy of a during many orders of magnitude in the cosmological time. A value of a somewhat above $\frac{3}{2}$ which makes the proposed late cosmology interesting for observation is perfectly consistent with the idea that an equilibrium between matter or radiation and the scalar energy density was established shortly after the end of inflation and prevailed until today. The initial value $V(t_0)$ would then be close to its "natural value" $H^2(t_0)M^2$.

8. Naturalness of the cosmon model

To close these arguments and elucidate the connection with the fate of dilatation symmetry (Wetterich 1988b) we briefly present our model in a somewhat different language. By an appropriate Weyl scaling the scalar-gravity model with exponential potential (4) can be obtained from the action (Wetterich 1988a, 1988b).

$$S = -\int d^4x g^{1/2} \times \left\{\frac{1}{12}\chi^2 R - \frac{1}{2}(z-1)\partial_\mu\chi\partial^\mu\chi + \frac{1}{8}\lambda(\chi)\chi^4 + \mathcal{L}_M\right\} \tag{53}$$

$$\lambda(\chi) = c\chi^{-A} \tag{54}$$

with the substitution

$$\varphi = \sqrt{12z}\ln\frac{\chi}{\sqrt{12}M} \tag{55}$$

If the "matter Lagrangian" $\mathcal{L}_M$ is dilatation-invariant (for example, the nucleon mass term is proportional χ), the effective coupling of φ to matter vanishes ($\beta = 0$) (Wetterich 1988b). A value $\beta \neq 0$ therefore reflects dilatation symmetry breaking in the matter sector and β may naturally be small. For $\beta = 0$ the ratios of particle masses to the Planck mass are independent of the value of χ and therefore constant in time (Wetterich 1988b). If in addition A vanishes, the action is scale-invariant. Stability requires $z \geq 0$ and for the boundary value $z = 0$ the gravity-scalar system is conformally invariant. We assume that quantum effects lead to a breakdown of classical dilatation symmetry and induce $A \neq 0$. The quantities z, A and λ can be partially absorbed by field-redefinitions and the only observable combination turns out to be the parameter a in the exponential potential for φ

$$a = (12z)^{-\frac{1}{2}}A \tag{56}$$

[8] Also the effective value of β_n may change with time and could have been larger during nucleosynthesis.

We note that even for small $|A|$ the parameter a can be large if z is small enough. This requires no fine-tuning since $z = 0$ is singled out by an enhanced (conformal) symmetry.

The quantity A is related to the scale dependence of the coupling λ which can be cast into the form of an evolution equation with anomalous dimension

$$\chi\frac{\partial}{\partial\chi}\lambda = -A\lambda + B\lambda^2 + \ldots \tag{57}$$

(The term $\sim B\lambda^2$ and higher order terms would give corrections which are suppressed by powers of $V(t)/M^4$ for large t.) We emphasize that an evolution equation for λ with a nonvanishing anomalous dimension would not only lead to a phenomenologically interesting cosmology but solve the whole cosmological constant problem in a natural way! In the language with fixed Newton's constant, a constant cosmological "constant" appears precisely if $V(\chi) \sim \chi^4$. For any potential increasing slower than χ^4 for large χ the cosmological constant vanishes[9] asymptotically! In this language χ grows with increasing time. Since the Planck mass ($\sim \chi$) grows faster than $V^{1/4}$ the observable ratio between the cosmological constant and the Planck mass decreases (Wetterich 1988b). A natural solution of the cosmological constant problem therefore arises whenever the asymptotic behavior of $V(\chi)$ differs from χ^4! This is independent (for $A > 0$) of the exact form of $V(\chi)$ for small χ. Additional dilatation symmetry breaking mass terms $\sim m^2\chi^2$ or constant terms in the potential would not affect the asymptotically vanishing cosmological "constant".

It seems therefore worthwhile to ask if an evolution equation of the type (57) makes sense. Unfortunately, the answer is not completely straightforward since we have to deal with the nonrenormalizable theory of gravity coupled to the cosmon. In this context we emphasize that the scalar field φ may actually not be a "fundamental field". It could also correspond to a scalar degree of freedom contained in the metric if the effective action for gravity is not the pure Einstein action. Similar possibilities arise from other geometrical objects in generalized gravity theories. At long distances these different options all reduce to an effective field theory for the graviton-cosmon system. It is instructive to recast the evolution equation in the language with field independent Newton's constant where it reads[10]

$$\frac{d}{d\varphi}V(\varphi) = -\frac{A}{M}V(\varphi) \tag{58}$$

Let us look what momenta of the quantum fluctuations contribute if we change the background field φ infinitesimally. The second derivative of V acts as an effective infrared cutoff for the quantum fluctuations of the scalar field. This cutoff is changed

[9] This also holds for $V(\chi)$ increasing faster than χ^4, with φ (55) replaced by $-\varphi$.

[10] This equation is to be interpreted here as a renormalization group equation in the sense of Coleman and Weinberg (Coleman et al. 1973) and not as a field equation.

by a shift in φ and the modes contributing dominantly[11] have presumably momentum squared of the order V''. For φ taking values close to those required by the proposed late cosmology, typical momenta are of the order of the Hubble parameter. The problem of the cosmological constant is therefore essentially decoupled from what happens at higher momentum scales and only involves the extreme infrared properties of the theory! For example, the QCD degrees of freedom typically lead to condensates and will influence the behavior of $V(\chi)$. These strong interaction effects arise from quantum fluctuations with momenta of the order of a typical QCD scale. They affect the evolution equation for values of χ much smaller than those relevant for late cosmology. Electroweak fluctuations or Planck-length fluctuations contribute to the evolution equation for the potential at even smaller values of χ. For the issue of an asymptotically vanishing cosmological constant the relevant question concerns the role that quantum effects with length scales much larger than QCD length scales or even the inverse electron mass play for the scale dependence of the effective potential[12]. The only modes which are expected to contribute to the flow equation are then the graviton and the cosmon. (We neglect here the photon and possible massless neutrinos since they do not couple directly to φ.) We emphasize that the infrared physics of concern here has strictly nothing to do with possible ultraviolet divergences of the theory. A suitable method for studying the problem could be the concept of the average action (Wetterich 1991, 1993; Reuter et al. 1993a, 1993b, 1994a, 1994b) generalized to gravity. This tests directly the infrared physics by variation of an effective infrared cutoff or averaging over larger and larger distances.

9. Conclusions

Waiting for results of this or another method it remains open if the present approach can lead to a natural solution of the cosmological constant problem. In the meanwhile we take the cosmon model as an interesting phenomenological approach. As far as we can see, the model presented here is compatible with the observational constraints. In addition to the interesting properties of a time-decaying cosmological "constant" it offers a new dark matter candidate – the cosmon. We conclude that the model seems motivated well enough to merit detailed studies of its implications for structure formation or related topics. Such investigations should lead to bounds on acceptable values of a for present cosmology, similar to those already obtained for power law potentials (Yoshii et al. 1992; Sugiyama et al. 1992). In absence of a complete theory of the fate of the cosmological "constant" observation may give important hints!

[11] It is not completely excluded that the contribution of high momentum modes is also affected by a shift of the infrared cutoff. We discard this possibility here.

[12] For a smooth transition to late cosmology we assume that the short-distance fluctuations do not generate local minima of $V(\varphi)$.

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