Astron. Astrophys. 297, 588–605 (1995)

ASTRONOMY AND ASTROPHYSICS

TASS1.6: Ephemerides of the major Saturnian satellites

A. Vienne and L. Duriez

Université des Sciences et Technologies de Lille, Laboratoire d'Astronomie, 1 Impasse de l'Observatoire, F-59000 Lille, France

Received 12 July 1994 / Accepted 16 September 1994

Abstract. We recall the main features of TASS, a new theory of high precision (about ten kilometers), for the satellites Mimas, Enceladus, Tethys, Dione, Rhea, Titan and Japetus. It is analytical with respect to the dynamical parameters of the Saturnian system, allowing to adjust them by fitting TASS to observations. In this paper TASS is compared to the Earth based observations of these satellites found in the catalogue compiled by Strugnell & Taylor (1990). These observations have been put in a form usable by the theory. We have built the corresponding equations of condition ($\approx 50,000$). A least square procedure has been done with a discussion about the estimation of the errors on the dynamical parameters which we have determined. The determination of the physical parameters of the dynamical Saturnian system is in good agreement with other determinations when they exist. Furthermore, the precisions of the oblateness coefficients J_2 and J_4 reach those based on Pioneer and Voyager spacecraft. The position of the equatorial plane is also found in good agreement with other determinations. Besides, the root-mean-square residuals between theory and observations are slightly improved. So, we can consider now that the reduction of the future observations of high precision (mutual events in 1995-1996 and spacecraft observations) should be done with this new tool to improve again our knowledge of the dynamical parameters of the system. TASS is now ready to give accurate ephemerides of the major saturnian satellites.

Key words: celestial mechanics, stellar dynamics – Satellites of Saturn – planets and satellites

1. Introduction

Recently Vienne & Duriez (Duriez & Vienne 1991; Vienne & Duriez 1991, 1992) have developed a new theory which we called TASS (as "Théorie Analytique des Satellites de Saturne"). These three papers are referred to as Paper I, II and III respectively. This theory concerns the motion of the following satellites : Mimas (1), Enceladus (2), Tethys (3), Dione (4), Rhea (5), Titan (6) and Japetus (8). Hyperion is not considered in this work because its theory is still in progress. Inclusion of Hyperion is expected nextly. Nevertheless, because of its small mass, it has no significant effects on the motion of other satellites.

Send offprint requests to: A. Vienne

First we recall the main features of TASS : it is analytical with respect to the dynamical parameters of the Saturnian system. There are 55 parameters :

- $\{m_i\}_{i=1\cdots 6,8}$ masses of the satellites
- J_2, J_4 and J_6 the oblateness coefficients of Saturn
- 6 initial conditions per satellite
- i_a and Ω_a the inclination and node of the equatorial plane of Saturn on the ecliptic plane in the J2000 system
- M_S the mass of Saturn

Then we present the observations used and the transformations done on them. After some remarks about the least square procedure, we present the results. They concern the physical parameters of the Saturnian system and the residuals of the observations.

At last, the adjustment leads to a new version of the solution : TASS1.6. This version is presented in the Tables 1 to 8 and allows to produce ephemerides of the major Saturnian satellites (the series and the fortran routines are available by E-mail: Vienne@gat.univ-lille1.fr or Duriez@gat.univ-lille1.fr).

2. TASS: a precise theory of motion for the major satellites of Saturn

In TASS, all motions are referred to a cartesian coordinate system whose origin is the center of Saturn, and referred to the equatorial plane of Saturn and the node of this plane with the ecliptic 2000. Each satellite is then located in this reference system by the osculating elliptic elements p, λ, z and ζ :

$$\begin{aligned} p &= \frac{n}{N} - 1 \\ \lambda &= Nt - \sqrt{-1}\, q \\ z &= e \exp \sqrt{-1}\varpi \\ \zeta &= \sin\frac{i}{2} \exp \sqrt{-1}\Omega \end{aligned} \tag{1}$$

where N is the mean mean motion such that the linear part in time of the mean longitude λ is exactly Nt, and n, λ, e, i, ϖ, Ω are the classical elliptic elements corresponding to the constant

Table 1a. Solution for the variable $p_{\rm I}$ (mean motion of Mimas). The series is expressed in cosinus

$p_{o{\rm I}}$ n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0051969	180.000	0.00000000			644.5

Table 1b. Solution for the variable $\lambda_{\rm I}$ (mean longitude of Mimas). $\lambda_{\rm I} = 0.1822485 + 2435.14429644 \times t + \delta\lambda_{\rm I} + \Delta\lambda_{\rm I}$. The series is expressed in sinus

$\delta\lambda_{\rm I}$ n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.7574073	39.325	0.08904538	70.56	$\omega_{\rm I}$	140892.1
2	0.0124330	117.974	0.26713613	23.52	$3\omega_{\rm I}$	2312.8
3	0.0022664	126.606	10.19765304	0.62	$\phi_{\rm I}$	421.6
4	0.0010599	267.281	10.10860767	0.62	$\phi_{\rm I} - \omega_{\rm I}$	197.2
5	0.0010228	165.931	10.28669842	0.61	$\phi_{\rm I} + \omega_{\rm I}$	190.3
6	0.0007266	78.649	0.17809075	35.28	$2\omega_{\rm I}$	135.2
7	0.0005061	259.757	0.05765338	108.98	$\phi_3 + 2\Phi_{\rm I} + \omega_{\rm I}$	94.1
8	0.0003590	196.624	0.44522688	14.11	$5\omega_{\rm I}$	66.8
9	0.0002628	3.120	0.06492496	96.78	$\phi_{\rm I} + 4\Phi_{\rm I} + \omega_{\rm I}$	48.9
10	0.0002459	178.892	0.12043737	52.17	$-\phi_3 - 2\Phi_{\rm I} + \omega_{\rm I}$	45.7
11	0.0002237	47.956	10.01956229	0.63	$\phi_{\rm I} - 2\omega_{\rm I}$	41.6
12	0.0002097	205.255	10.37574380	0.61	$\phi_{\rm I} + 2\omega_{\rm I}$	39.0
13	0.0001970	255.529	0.11316579	55.52	$-\phi_{\rm I} - 4\Phi_{\rm I} + \omega_{\rm I}$	36.6
14	0.0001276	312.080	5.02184135	1.25	$-2\Phi_{\rm I} - \omega_{\rm I}$	23.7
15	0.0001164	210.730	5.19993211	1.21	$-2\Phi_{\rm I} + \omega_{\rm I}$	21.7
$\Delta\lambda_{\rm I}$	(rad)	(deg)	(rad/year)	(days)		(km)
16	0.0001456	13.921	2428.76308172	0.94	$\lambda_{o{\rm I}} + \rho_{\rm I} - \phi_{\rm I}$	27.1

$GM_s(1+m/M_s)$ (G is the gaussian constant of gravitation, M_s and m the masses of Saturn and of the satellite).

In Paper I, we have given the general method used to resolve the equations in these variables. It consists in a separation of the short-period perturbations (which have been presented in Paper II) from the long-period ones, solutions of a critical system describing the long term evolution of the motions. This system has been built in an exhaustive way: the critical terms (secular, resonant and solar terms) have been expanded up to the degree 6 in eccentricities and inclinations at the first order in the masses and in the oblateness coefficients, up to the degree 4 at the second order and to the degree 1 at the third order in J_2. A numerical evaluation of each term allows us to retain only those which lead to perturbations greater than one kilometer. In Paper III we have presented the numerical integration of the complete critical system. Because it contains no short period terms, such a computation is possible for a long time (several centuries) with a rather large integration step (several days). To obtain a formal solution from this discrete numerical solution, we have used the techniques of Fourier analysis, and then we have identified the frequencies as integer combinations of the fundamental frequencies of the system. This solution (Tables 1 to 4 and 7 to 9 of Paper III) depends on the numerical values adopted for the constant physical parameters (masses, J_2, J_4 and J_6) and for the initial conditions; these are also given in Paper I (Table 1). It is called "nominal solution". The first order variations of this solution with respect to each constant have been obtained by integrating the variational equations in the same way. The nominal solution and its variations form the theory which we called TASS.

In Tables 1 to 8, we present the solution of the elliptic elements p, λ, z and ζ obtained after the present adjustment. To compute the corresponding saturnicentric cartesian coordinates X, Y and Z, we have used the masses of Saturn and its satellites given in Tables 10. This solution is referred to as TASS1.6. The corresponding Fortran routine which computes the coordinates X, Y and Z in the J2000 system for a given satellite and a given date, is available from the authors. In the Tables 1 to 8, each solution is presented as a series of periodics terms (expressed with sinus functions for λ, with cosinus for p, and with complex exponential for z and ζ). We present here only terms whose amplitude is greater than 20 km (the complete series grows up to include terms greater than 1 km). The series are given with the long period terms in first place, followed by the short period terms (for example we have $z_{\rm I} = z_{o{\rm I}} + \Delta z_{\rm I}$ with $z_{o{\rm I}}$ and $\Delta z_{\rm I}$ for long period and short period terms respectively). The time t is expressed in Julian years from J1980 (t = (Julian date −2444240)/365.25). In most cases we have identified the argument of each term as integer combinations of fundamental arguments. For these fundamental arguments, we have used the following notation :

Table 1c. Solution for the variable z_1 (eccentricity and pericenter of Mimas). The series is expressed in complex exponential

n° z_{o1}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0159817	356.521	6.38121472	0.98	$-\rho_1+\phi_1$	2972.9
2	0.0073147	137.197	6.29216934	1.00	$-\rho_1+\phi_1-\omega_1$	1360.7
3	0.0071114	35.846	6.47026010	0.97	$-\rho_1+\phi_1+\omega_1$	1322.9
4	0.0015115	277.872	6.20312396	1.01	$-\rho_1+\phi_1-2\omega_1$	281.2
5	0.0014622	75.171	6.55930547	0.96	$-\rho_1+\phi_1+2\omega_1$	272.0
6	0.0003336	58.547	6.11407859	1.03	$-\rho_1+\phi_1-3\omega_1$	62.1
7	0.0003307	114.496	6.64835085	0.95	$-\rho_1+\phi_1+3\omega_1$	61.5
8	0.0001607	229.918	−3.81643833	1.65	$-\rho_1$	29.9
Δz_1	(rad)	(deg)	(rad/year)	(days)		(km)
9	0.0026027	10.442	2435.14429644	0.94	λ_{o1}	484.2

Table 1d. Solution for the variable ζ_1 (inclination and node of Mimas). The series is expressed in complex exponential

n° ζ_{o1}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0118896	234.213	−6.37188169	0.99	$-\rho_1+\Phi_1$	2211.7
2	0.0053177	14.888	−6.46092707	0.97	$-\rho_1+\Phi_1-\omega_1$	989.2
3	0.0053017	273.538	−6.28283631	1.00	$-\rho_1+\Phi_1+\omega_1$	986.2
4	0.0010922	155.563	−6.54997244	0.96	$-\rho_1+\Phi_1-2\omega_1$	203.2
5	0.0010741	312.862	−6.19379094	1.01	$-\rho_1+\Phi_1+2\omega_1$	199.8
6	0.0002328	352.187	−6.10474556	1.03	$-\rho_1+\Phi_1+3\omega_1$	43.3
7	0.0002224	296.239	−6.63901782	0.95	$-\rho_1+\Phi_1-3\omega_1$	41.4

- λ_{oi} $i=1,8$ the long period part of λ_i. We have $\lambda_{oi}=N_i\times t+\lambda_{oi}^{(o)}+\delta\lambda_i$, the linear part is given in the title of the table of the corresponding mean longitude.
- $\rho_1=\lambda_{o1}-2\lambda_{o3}$ (resonance Mimas-Tethys)
- $\rho_2=\lambda_{o2}-2\lambda_{o4}$ (resonance Enceladus-Dione)
- ϕ_1, so that $\phi_1-\rho_1$ is close to the proper pericenter of Mimas.
- Φ_1, so that $\Phi_1-\rho_1$ is close to the proper node of Mimas.
- ω_2 is the libration argument of the resonance Enceladus-Dione. This argument takes the place of ϕ_2 corresponding to the proper pericenter of Enceladus whose frequency is zero (see the Table 4 of Paper I).
- Φ_2, so that $\Phi_2-\rho_2$ is close to the proper node of Enceladus.
- ϕ_3, so that $\phi_3-\rho_1$ is close to the proper pericenter of Tethys.
- ω_1 is the libration argument of the resonance Mimas-Tethys. This argument takes the place of Φ_3 corresponding to the proper node of Tethys because $\dot\Phi_1+\dot\Phi_3=0$.
- ϕ_4, so that $\phi_4-\rho_2$ is close to the proper pericenter of Dione.
- Φ_4, so that $\Phi_4-\rho_2$ is close to the proper node of Dione.
- ϕ_i and Φ_i $i=5,6$ and 8 which are close to the proper pericenters and the nodes respectively for Rhea, Titan and Japetus.
- λ_9, ϖ_9, and Ω_9 are respectively the saturnicentric mean longitude, the longitudes of the pericenter and of the node of the Sun.
- μ is the fundamental argument of JASON84 (Simon & Bretagnon 1984).

We recall that in JASON84, 19μ corresponds to the great inequality between Jupiter and Saturn $-2\,\lambda_J+5\lambda_9$, 880μ corresponds to the synodic inequality between Jupiter and Saturn $\lambda_J-\lambda_9$, and 287μ corresponds to the inequality $\lambda_J-2\lambda_9$. So, the argument μ in our series represents indirect perturbations from Jupiter.

The arguments ρ_1 and ρ_2 are redundant. Because of the resonances, they correspond to long period arguments. So we have preferred not to replace them by the corresponding combination of mean longitudes in order to recognize easily the pure short period terms. For example, the argument $\lambda_{o1}+\rho_1-\phi_1$ (the last term in the mean longitude of Mimas) can be written also as $2\lambda_{o1}-2\lambda_{o3}-\phi_1$. But, according to the d'Alembert rule, the inequality $2\lambda_{o1}-2\lambda_{o3}$ has a characteristic $C_I=0$, associated to monomes in z and ζ with characteristic $C_M=0$ (see Paper I). We then deduce that the argument $2\lambda_{o1}-2\lambda_{o3}-\phi_1$ cannot correspond to the inequality $2\lambda_{o1}-2\lambda_{o3}$. It corresponds in fact to the inequality λ_{o1} for which the main contribution comes from the monome proportional to z_1; this is why the best writing is $\lambda_{o1}+\rho_1-\phi_1$. In this example, only one writing has a sense but it is not always the case; for example, in the solution z_2, consider the inequalities λ_{o2} and $2\lambda_{o4}$. The characteristic of the first is 1, so the main contribution has the argument λ_{o2}. The second has the characteristic 2, so it can be associated with the variable z_2; in that case, according to Table 2c, the development of the

Table 2a. Solution for the variable p_2 (mean motion of Enceladus). The series is expressed in cosinus

p_{o2} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0031471	180.000	0.00000000			500.2

Table 2b. Solution for the variable λ_2 (mean longitude of Enceladus). $\lambda_2 = 0.7997717 + 1674.86729850 \times t + \delta\lambda_2 + \Delta\lambda_2$. The series is expressed in sinus

$\delta\lambda_2$ n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0044964	134.242	0.56590952	11.10	ω_2	1072.0
2	0.0033546	263.436	1.61701655	3.89	$-\phi_4$	799.8

Table 2c. Solution for the variable z_2 (eccentricity and pericenter of Enceladus). The series is expressed in complex exponential

z_{o2} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0048038	182.741	2.15444222	2.92	$-\rho_2$	1145.3
2	0.0001098	316.982	2.72035174	2.31	$-\rho_2 + \omega_2$	26.2
Δz_2	(rad)	(deg)	(rad/year)	(days)		(km)
3	0.0015768	45.824	1674.86729850	1.37	λ_{o2}	375.9

Table 2d. Solution for the variable ζ_2 (inclination and node of Enceladus). The series is expressed in complex exponential

ζ_{o2} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0001281	113.626	−2.65919659	2.36	$-\rho_2 + \Phi_2$	30.5

inequality $2\lambda_{o4}$ would produce the argument $\rho_2 + 2\lambda_{o4}$ equal to λ_{o2}. In fact, the argument of the third term in Table 2c has been written λ_{o2} because this inequality is far more significant than the other. However, the cases of redundance are very few in our series, and in these cases there is generally no ambiguity to recognize the inequality.

The fundamental arguments given above are linear with respect to the time t, except the mean mean longitudes λ_{oi} (also present in ρ_1 and ρ_2). The non linear part of λ_{oi} is noted $\delta\lambda_i$. To compute the position of a satellite at a given date from these tables, the amplitude, the phase and the frequency of each term do not suffice. In fact each term of the series is defined by its amplitude (second column), and its argument of the form :$w\,t + \varphi + \sum_{i=1}^{8} k_i\delta\lambda_i$ where φ and w are given in the third and fourth columns respectively. The integers k_i are taken from the identification column. Thus the procedure to compute the saturnicentric elements of the satellite k is :

- computation of $\delta\lambda_i$ $i = 1,8$ for the given date.
- computation of p_{ok}, λ_{ok}, z_{ok}, ζ_{ok}, Δp_k, $\Delta\lambda_k$, Δz_k, $\Delta\zeta_k$, taking into account the combination of $\delta\lambda_i$ present in each argument.

For example, the 16^{th} term in $\Delta\lambda_1$ (Table 1b) with argument $\lambda_{oi} + \rho_1 - \phi_1$, must be computed by

$$0.0001456 \times \sin(13^\circ\!.921 + 2428.76308172 \times t \\ \delta\lambda_1 + (\delta\lambda_1 - 2\delta\lambda_3))$$

(the contribution of $-\phi_1$ in the argument is already in the amplitude and the phase of the term), where

$$\delta\lambda_1 = 0.7574073 \times \sin(39^\circ\!.325 + 0.08904538 \times t) \\ +0.0124330 \times \sin(117^\circ\!.974 + 0.26713613 \times t) \\ +\ldots$$

and where $\delta\lambda_3$ is computed in the same way from Table 3b.

Taking into account the combination of $\delta\lambda_i$ is important specially for $\delta\lambda_1$ and $\delta\lambda_8$ in which the main terms have a great amplitude. Because of the truncature level (20 km), the precision obtained over one century by using the Tables 1 to 8 does not exceed one hundred kilometers, even two hundreds kilometers for Mimas and Japetus. So, it is better to use the routines, available from the authors, which use the complete series.

The partial derivatives of the solution with respect to the initial conditions and to the physical parameters do not appear in these tables because they are too voluminous. But they are present in the complete series of TASS. If we note σ the ampli-

Table 3a. Solution for the variable p_3 (mean motion of Tethys). The series is expressed in cosinus

n° p_{o3}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0020480	180.000	0.00000000			402.7

Table 3b. Solution for the variable λ_3 (mean longitude of Tethys). $\lambda_3 = 5.2391094 + 1215.66392906 \times t + \delta\lambda_3 + \Delta\lambda_3$. The series is expressed in sinus

n° $\delta\lambda_3$	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0359719	219.325	0.08904538	70.56	ω_I	10610.8
2	0.0005892	297.974	0.26713613	23.52	$3\omega_I$	173.8
3	0.0001050	306.606	10.19765304	0.62	ϕ_I	31.0

Table 3c. Solution for the variable z_3 (eccentricity and pericenter of Tethys). The series is expressed in complex exponential

n° z_{o3}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0001565	261.754	1.26305641	4.97	$-\rho_I + \phi_3$	46.2
2	0.0000868	42.429	1.17401103	5.35	$-\rho_I + \phi_3 - \omega_I$	25.6
3	0.0000817	340.403	1.44114716	4.36	$-\rho_I + \phi_3 + 2\omega_I$	24.1
4	0.0000810	183.104	1.08496566	5.79	$-\rho_I + \phi_3 - 2\omega_I$	23.9
5	0.0000708	301.078	1.35210179	4.65	$-\rho_I + \phi_3 + \omega_I$	20.9
Δz_3	(rad)	(deg)	(rad/year)	(days)		(km)
6	0.0010264	300.179	1215.66392906	1.89	λ_{o3}	302.8

Table 3d. Solution for the variable ζ_3 (inclination and node of Tethys). The series is expressed in complex exponential

n° ζ_{o3}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0079790	225.618	−1.26099496	4.98	$-\rho_I - \Phi_I$	2353.6
2	0.0035868	6.293	−1.35004034	4.65	$-\rho_I - \Phi_I - \omega_I$	1058.0
3	0.0035786	264.943	−1.17194958	5.36	$-\rho_I - \Phi_I + \omega_I$	1055.6
4	0.0007456	146.969	−1.43908571	4.37	$-\rho_I - \Phi_I - 2\omega_I$	219.9
5	0.0007269	304.268	−1.08290421	5.80	$-\rho_I - \Phi_I + 2\omega_I$	214.4
6	0.0001634	343.592	−0.99385883	6.32	$-\rho_I - \Phi_I + 3\omega_I$	48.2
7	0.0001629	287.644	−1.52813109	4.11	$-\rho_I - \Phi_I - 3\omega_I$	48.1

tude, the phase or the frequency of any term in these tables, we have in fact :

$$\sigma = \sigma_o + \sum_{k=1}^{55} \left(\frac{\partial\sigma}{\partial x_k}\right)_o \delta x_k \quad (2)$$

where σ_o is the value given in Tables 1 to 8 and where x_k is one among the parameters defined above (Sect. 1). This representation is very different from what is done in previous theories (Dourneau 1987; Harper & Taylor 1993). In these works, the parameters are not linked directly to physical constants and are adjusted independently from each other. These parameters are for example, the semi major axes, the mean mean motions (see Sect. 5.2), and the precession rates of each orbit : these values are not independent from each other; the same thing occurs for the amplitudes and phases of some terms such as libration terms. Thus, we emphasize that in the present work the parameters we have adjusted are independent from each other : the coherent representation of the satellites' motions is one of the main features of TASS.

Another feature of TASS is the internal precision which reaches a few kilometers. This precision has been estimated by comparing the positions of the satellites computed with TASS, with positions issued from a direct numerical integration. The physical model used in the numerical integration is the same as that we have used to build TASS, that is : Saturn's oblateness, intersatellites mutual perturbations and solar perturbations. But in the numerical integration, the equations have not been developed, so they are not affected by any truncations. In Table 9,

Table 4a. Solution for the variable p_4 (mean motion of Dione). The series is expressed in cosinus

n° p_{o4}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0012450	180.000	0.00000000			313.5

Table 4b. Solution for the variable λ_4 (mean longitude of Dione). $\lambda_4 = 1.9945926 + 838.51087036 \times t + \delta\lambda_4 + \Delta\lambda_4$. The series is expressed in sinus

n° $\delta\lambda_4$	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0001253	314.242	0.56590952	11.10	ω_2	47.3
2	0.0000947	83.436	1.61701655	3.89	$-\phi_4$	35.8

Table 4c. Solution for the variable z_4 (eccentricity and pericenter of Dione). The series is expressed in complex exponential

n° z_{o4}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0022034	279.304	0.53742567	11.69	$-\rho_2 + \phi_4$	832.1
2	0.0001172	153.666	0.00893386	703.30	ϕ_6	44.3
Δz_4	(rad)	(deg)	(rad/year)	(days)		(km)
3	0.0006246	114.282	838.51087036	2.74	λ_{o4}	235.9

Table 4d. Solution for the variable ζ_4 (inclination and node of Dione). The series is expressed in complex exponential

n° ζ_{o4}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0000591	184.579	0.00000000			22.3
2	0.0001655	89.189	−0.53763153	11.69	$-\rho_2 + \Phi_4$	62.5

we give the maximum difference observed in this comparison over two time spans. Over 3 years, we are near the precision of 5 kilometers which is required for the CASSINI mission. We have also given the precision over one century because the Earth-based observations cover approximatively this time span. As we will see in Sect. 5.3, the precision of these observations are about 0.15 second of degree, which represents 1000 kilometers on positions. So the internal precision is sufficient for our analysis. We have already noted in Paper III that the representation of the motion of Japetus is not as good as that of the other satellites.

At last, to obtain an absolute position of a satellite in the sky of the Earth, we have used the ephemerides of the Earth and of Saturn elaborated at the Bureau des Longitudes in Paris and based on VSOP82 (Bretagnon 1982). These ephemerides are referred to the J2000 system like TASS. When it was necessary, the angles of precession and nutation were computed from the formulas given in the "Connaissance des Temps" (1990).

3. The used observations

TASS is able to compute positions of the satellites Mimas, Enceladus, Dione, Rhea, Titan and Japetus, as well as the partial derivatives of these positions with respect to the parameters we want to adjust. So, in order to build equations which allow such adjustment, it is necessary to compare the position of a given satellite measured in the sky at a given date, with the corresponding position computed with TASS.

The first step is then to gather all the available observations. We have been efficiently helped by using the catalogue of observations compiled by Strugnell & Taylor (1990) which contains the majority of observations in a consistent format. We have also included in our analysis the observations made by Pascu at the U.S. Naval Observatory between 1974 and 1980 which are not published, and the observations made by Veillet & Dourneau (1992) at Mauna Kea, at Pic du Midi and at European Southern Observatories between 1979 and 1985. Note that the Voyager observations are not yet available in a usable form.

The second step is to put these observations in a form usable by TASS. Many observations are given in the system B1950. We have put them in the J2000 system with a routine made

Table 5a. Solution for the variable p_5 (mean motion of Rhea). The series is expressed in cosinus

p_{o5} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0006263	180.000	0.00000000			220.1
Δp_5	(rad)	(deg)	(rad/year)	(days)		(km)
2	0.0000650	327.198	728.17054577	3.15	$2\lambda_{o5} - 2\lambda_{o6}$	22.8

Table 5b. Solution for the variable λ_5 (mean longitude of Rhea). $\lambda_5 = 6.2213409 + 508.00931975 \times t + \delta\lambda_5 + \Delta\lambda_5$. The series is expressed in sinus

$\delta\lambda_5$ n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0004983	273.022	0.00192554	3263.07	$-\Phi_8$	262.7
2	0.0003255	189.303	0.00893124	703.51	$-\Phi_6$	171.6
3	0.0000639	164.150	0.42659824	14.73	$2\lambda_9$	33.7
4	0.0000615	210.089	0.17546762	35.81	$-\Phi_5$	32.4
5	0.0000469	252.650	0.21329912	29.46	λ_9	24.7
$\Delta\lambda_5$	(rad)	(deg)	(rad/year)	(days)		(km)
6	0.0000927	327.198	728.17054577	3.15	$2\lambda_{o5} - 2\lambda_{o6}$	48.9
7	0.0000523	253.599	364.08527289	6.30	$\lambda_{o5} - \lambda_{o6}$	27.6

Table 5c. Solution for the variable z_5 (eccentricity and pericenter of Rhea). The series is expressed in complex exponential

z_{o5} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0009713	154.007	0.00893386	703.30	ϕ_6	512.1
2	0.0001672	3.070	0.17554922	35.79	ϕ_5	88.1
Δz_5	(rad)	(deg)	(rad/year)	(days)		(km)
3	0.0003116	356.456	508.00932017	4.52	λ_{o5}	164.3
4	0.0001108	209.258	−220.16122560	10.42	$-\lambda_{o5} + 2\lambda_{o6}$	58.4

Table 5d. Solution for the variable ζ_5 (inclination and node of Rhea). The series is expressed in complex exponential

ζ_{o5} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0004207	184.578	0.00000000			221.8
2	0.0029705	150.509	−0.17546762	35.81	Φ_5	1566.2
3	0.0001788	355.495	−0.00893124	703.51	Φ_6	94.2

by Rapaport (private communication). This routine is based on the expressions derived by (Aoki et al. 1983). This transformation is possible directly if the coordinates are α and δ. If only differential coordinates are given, the transformation has been done in three steps : first, we compute with the representation of Dourneau (1987), the absolute position of the reference object only (Saturn or a satellite) in order to obtain α_{B1950} and δ_{B1950} of objects; then, we transform them in the J2000 system; finally, with α_{J2000} and δ_{J2000} we compute the coordinates in the same form as the initial one. At the precision level of the observations, the measurement is not affected by this transformation. Other observations are given in the equinox and the ecliptic of the date. We have kept them in this form and we have done the appropriate rotation on the computed positions.

The third step is to transform the catalogue in order to have as much as possible intersatellite coordinates. This method eliminates the effects of errors on the computed position of Saturn and on its observed position, as well as the effects of systematic errors in the absolute positions of the satellites. When we have N satellites observed at the same date (generally they correspond to a photographic plate), we have made the difference

Table 6a. Solution for the variable p_6 (mean motion of Titan). The series is expressed in cosinus

n°	amplitude	phase	frequency	period	identification	amplitude
p_{o6}	(rad)	(deg)	(rad/year)	(years)		(km)
1	0.0001348	180.000	0.00000000			109.8
Δp_6	(rad)	(deg)	(rad/year)	(days)		(km)
2	0.0000251	253.599	364.08527289	6.30	$\lambda_{o5} - \lambda_{o6}$	20.5

Table 6b. Solution for the variable λ_6 (mean longitude of Titan). $\lambda_6 = 4.9367922 + 143.92404785 \times t + \delta\lambda_6 + \Delta\lambda_6$. The series is expressed in sinus

n°	amplitude	phase	frequency	period	identification	amplitude
$\delta\lambda_6$	(rad)	(deg)	(rad/year)	(years)		(km)
1	0.0014892	256.852	0.00192554	3263.07	$-\Phi_8$	1819.7
2	0.0006278	9.406	0.00893124	703.51	$-\Phi_6$	767.1
3	0.0002065	164.777	0.42659824	14.73	$2\lambda_9$	252.3
4	0.0001840	253.966	0.21329912	29.46	λ_9	224.8
5	0.0000321	138.716	0.00686799	914.85	$19\mu/\phi_6 - \phi_8/ - \Phi_6 + \Phi_8$	39.2
6	0.0000291	236.833	0.63989736	9.82	$3\lambda_9$	35.6
7	0.0000278	119.512	0.01786773	351.65	$2\phi_6$	34.0

Table 6c. Solution for the variable z_6 (eccentricity and pericenter of Titan). The series is expressed in complex exponential

n°	amplitude	phase	frequency	period	identification	amplitude
z_{o6}	(rad)	(deg)	(rad/year)	(years)		(km)
1	0.0289265	153.988	0.00893386	703.30	ϕ_6	35346.6
2	0.0001921	34.663	−0.00893386	703.30	$-\phi_6$	234.8
3	0.0000745	199.650	0.41766438	15.04	$-\phi_6 + 2\lambda_9$	91.0
4	0.0000243	257.661	0.00700832	896.53	$\phi_6 + \Phi_8$	29.7
5	0.0000239	229.451	0.01085941	578.59	$\phi_6 - \Phi_8$	29.2
6	0.0000172	196.134	0.00197469	3181.86	ϕ_8	21.0
Δz_6	(rad)	(deg)	(rad/year)	(days)		(km)
7	0.0000669	282.857	143.92404729	15.95	λ_{o6}	81.7

Table 6d. Solution for the variable ζ_6 (inclination and node of Titan). The series is expressed in complex exponential

n°	amplitude	phase	frequency	period	identification	amplitude
ζ_{o6}	(rad)	(deg)	(rad/year)	(years)		(km)
1	0.0056024	184.578	0.00000000			6845.8
2	0.0027899	355.503	−0.00893124	703.51	Φ_6	3409.2
3	0.0001312	289.015	−0.00192554	3263.07	Φ_8	160.4
4	0.0001126	348.599	0.42659824	14.73	$2\lambda_9$	137.6
5	0.0000192	291.921	−0.21329912	29.46	$-\lambda_9$	23.4

between the absolute positions of each possible combination of pairs of satellites. So, we obtain $\frac{N(N-1)}{2}$ "observations" but only $N-1$ of them are independent. To take into account this redundance, we have included in the weight of the corresponding equations of condition the factor $\frac{2}{N}$. Of course, the case $N = 1$ corresponds to an absolute position of one satellite (α and δ), so we cannot do such a transformation. In fact, as Taylor et al. (1991) have shown, these observations are better analysed if they are considered as observations of Saturn itself. So, such observations have been omitted from our analysis. For most of them, they concern the satellites Titan and Japetus and come from the Carlsberg and Bordeaux automatic meridian circles. Note that the observations involving the angle of position p are transformed into $\cos p$ and $\sin p$. The corresponding equations of condition have the factor $\frac{1}{\sqrt{2}}$ in order to take into account the redundance. They have also the factor s_c in order to normal-

Table 8a. Solution for the variable p_8 (mean motion of Japetus). The series is expressed in cosinus

n°	amplitude	phase	frequency	period	identification	amplitude
p_{o8}	(rad)	(deg)	(rad/year)	(years)		(km)
1	0.0004932	180.000	0.00000000			1171.1
Δp_8	(rad)	(deg)	(rad/year)	(days)		(km)
2	0.0014226	93.339	114.99552496	19.96	$\lambda_{o6} - \lambda_{o8}$	3377.9
3	0.0000128	168.918	114.99745050	19.96	$\lambda_{o6} - \lambda_{o8} - \Phi_8$	30.5
4	0.0000117	17.760	114.99359941	19.96	$\lambda_{o6} - \lambda_{o8} + \Phi_8$	27.9
5	0.0001039	206.479	57.43044643	39.96	$2\lambda_{o8} - 2\lambda_{o9}$	246.7
6	0.0000132	206.473	57.43044643	39.96	$2\lambda_{o8} - 2\lambda_{o9} + \Omega_9$	31.3
7	0.0000482	278.307	86.06897732	26.66	$\lambda_{o6} - 2\lambda_{o8} + \phi_8$	114.4
8	0.0000379	222.187	258.91063838	8.86	$2\lambda_{o6} - \lambda_{o8} - \phi_6$	90.0
9	0.0000348	166.938	479.08079784	4.79	$\lambda_{o5} - \lambda_{o8}$	82.6
10	0.0000211	353.466	28.92654764	79.34	$\lambda_{o8} - \phi_8$	50.1
11	0.0000118	215.851	28.91958847	79.36	$\lambda_{o8} - \phi_6$	28.0
12	0.0000204	136.858	57.21714732	40.11	$2\lambda_{o8} - 3\lambda_{o9} + \varpi_9$	48.3
13	0.0000187	284.764	809.58234803	2.83	$\lambda_{o4} - \lambda_{o8}$	44.4
14	0.0000164	178.797	172.85449516	13.28	$\lambda_{o6} + \lambda_{o8} - \Phi_8$	38.9
15	0.0000153	283.216	172.85256962	13.28	$\lambda_{o6} + \lambda_{o8}$	36.3
16	0.0000156	186.678	229.99104991	9.98	$2\lambda_{o6} - 2\lambda_{o8}$	37.0
17	0.0000123	9.883	57.85704466	39.67	$2\lambda_{o8} - \Omega_9$	29.2
18	0.0000118	189.889	57.85704466	39.67	$2\lambda_{o8} - 2\Omega_9$	27.9
19	0.0000113	110.661	1186.73540673	1.93	$\lambda_{o3} - \lambda_{o8}$	26.8
20	0.0000089	213.013	28.50389879	80.51	$\lambda_{o8} - 2\lambda_{o9} + \phi_8$	21.1

ize the units (s_c is the computed angle of separation). It would had been simpler if we had transformed the two observed coordinates p and s into $s\cos p$ and $s\sin p$, but it is not possible because p and s are not given at the same date since they are visual observations.

At last, we have performed the correction of refraction in the differential coordinates for the data recorded before 1966 except those from the U.S. Naval Observatory. As it was said in Strugnell & Taylor (1990), in all other data sets this correction is supposed to be already made by the observers.

In conclusion, we can say that the set of observations used in the present work is almost the same as the set used by Harper & Taylor (1994) in their similar work using their representation of the motion of the satellites. Furthermore, we have used the same rejection level : observations which have residuals greater than 1″ are not retained. That is why in Sect. 5. we may compare both results. Note that the very few and very precise observations made by Aksnes et al. (1984) are included in the present analysis while they are not in that of Harper & Taylor. On the contrary, observations listed in Table 1 of (Harper & Taylor 1994) are not considered in our analysis.

Remark. About the 14 observations made by Aksnes et al., we have to indicate here an error in the catalogue of Strugnell & Taylor. These observations are very accurate because they correspond to mutual events (see Table 10). We obtain for these 14 observations a root-mean-square residual equal to 0″.015. To reach this value it suffices to interchange in the catalogue the "reference object" and the "observed object". This correction agrees with the definitions given in the paper of Aksnes et al. from which these observations have been taken to be put in the catalogue. This inversion was detected by Rapaport (private communication) when computing the root-mean-square residuals of these observations with the representation of Harper & Taylor (1993). He obtains 0″.030. But in a recent paper Harper & Taylor (1994) find 0″.05 in $\Delta\alpha\cos\delta$ and 0″.14 in $\Delta\delta$. So, it seems that Harper & Taylor have not yet made the correction.

4. Method to estimate the parameters

We are then able to compare the observations to TASS. From this comparison we want to give an estimation of the parameters on which TASS depends. We have done this estimation by the least square procedure. This method is well-known and extensively used. This fact is due to its simplicity and because it is easy to compute. It is not our purpose to present here all the theory of the least square method, but we want to emphasize some points. More precisely, we want to recall the hypothesis of the method and then to discuss whether the choice of weights verifies them. This discussion would lead to a better understanding of the results.

4.1. The least square procedure

Most of the formulas of this sub-section have been written with the help of the theory presented by Eichhorn (1993). The reader can find in this paper more details and more references.

Table 8b. Solution for the variable λ_8 (mean longitude of Japetus). $\lambda_8 = 0.1661250 + 28.92852233 \times t + \delta\lambda_8 + \Delta\lambda_8$. The series is expressed in sinus

n° $\delta\lambda_8$	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.1928387	75.444	0.00192554	3263.07	$-\Phi_8$	686837.3
2	0.0011977	166.491	0.00385109	1631.54	$-2\Phi_8$	4265.9
3	0.0011258	252.911	0.21329912	29.46	λ_9	4009.8
4	0.0007466	195.470	0.00893124	703.51	$-\Phi_6$	2659.3
5	0.0003004	239.655	0.42852378	14.66	$-\Phi_8 + 2\lambda_9$	1070.0
6	0.0002400	167.668	0.42659824	14.73	$2\lambda_9$	855.0
7	0.0001785	223.291	0.00691978	908.00	$19\mu/\phi_6 - \phi_8/ - \Phi_6 + \Phi_8$	635.9
8	0.0000739	315.313	0.43044933	14.60	$-2\Phi_8 + 2\lambda_9$	263.3
9	0.0000528	356.302	0.21137358	29.73	$\Phi_8 + \lambda_9$	188.2
10	0.0000408	216.372	0.01083538	579.88	unkown	145.2
11	0.0000403	62.796	0.20646793	30.43	$\lambda_9 - 19\mu$	143.5
12	0.0000381	312.253	0.64182290	9.79	$-\Phi_8 + 3\lambda_9$	135.8
13	0.0000362	35.395	0.01282203	490.03	unkown	128.9
14	0.0000359	139.576	0.21522466	29.19	$-\Phi_8 + \lambda_9$	127.7
15	0.0000349	238.299	0.63989736	9.82	$3\lambda_9$	124.3
16	0.0000238	185.596	0.10318689	60.89	287μ	84.7
17	0.0000215	190.457	0.02002818	313.72	unkown	76.5
18	0.0000202	306.111	0.01786773	351.65	$2\phi_6$	72.1
19	0.0000160	142.421	0.42264886	14.87	$-2\phi_8 + 2\lambda_9$	57.1
20	0.0000131	336.536	0.31639186	19.86	880μ	46.8
21	0.0000099	26.889	0.64374845	9.76	$-2\Phi_8 + 3\lambda_9$	35.2
22	0.0000075	234.190	0.22013031	28.54	$\lambda_9 + 19\mu$	26.6
$\Delta\lambda_8$	(rad)	(deg)	(rad/year)	(days)		(km)
23	0.0007283	273.339	114.99552496	19.96	$\lambda_{o6} - \lambda_{o8}$	2593.8
24	0.0001227	206.479	57.43044643	39.96	$2\lambda_{o8} - 2\lambda_{o9}$	437.0
25	0.0000144	206.473	57.43044643	39.96	$2\lambda_{o8} - 2\lambda_{o9} + \Omega_9$	51.4
26	0.0000073	102.053	57.43237198	39.96	$2\lambda_{o8} - 2\lambda_{o9} - \Phi_8 + \Omega_9$	26.0
27	0.0001170	166.266	0.71654053	3202.80	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_8 - \Phi_8$	416.6
28	0.0001078	270.966	0.71461498	3211.43	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_8$	384.1
29	0.0000888	180.525	0.72048991	3185.24	$-\lambda_{o6} + 5\lambda_{o8} - \Phi_8$	316.3
30	0.0000808	25.895	0.70958135	3234.21	$-\lambda_{o6} + 5\lambda_{o8} - \phi_6 - \phi_8 - \Phi_8$	287.6
31	0.0000729	130.724	0.70765581	3243.01	$-\lambda_{o6} + 5\lambda_{o8} - \phi_6 - \phi_8$	259.5
32	0.0000727	77.569	0.72241545	3176.75	$-\lambda_{o6} + 5\lambda_{o8} - 2\Phi_8$	259.0
33	0.0000353	282.813	0.71856436	3193.78	$-\lambda_{o6} + 5\lambda_{o8}$	125.8
34	0.0000302	64.357	0.71846607	3194.21	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_8 - 2\Phi_8$	107.5
35	0.0000233	333.142	0.72434100	3168.31	$-\lambda_{o6} + 5\lambda_{o8} - 3\Phi_8$	82.9
36	0.0000219	114.321	0.70370643	3261.21	$-\lambda_{o6} + 5\lambda_{o8} - \phi_6 - 3\phi_8$	77.9
37	0.0000167	278.826	0.71150690	3225.45	$-\lambda_{o6} + 5\lambda_{o8} - \phi_6 - \phi_8 - 2\Phi_8$	59.6
38	0.0000148	327.720	0.69674725	3293.78	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_6 - 2\phi_8$	52.5
39	0.0000140	242.920	0.70262218	3266.24	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_6 - \Phi_8$	49.8
40	0.0000118	347.311	0.70069664	3275.22	$-\lambda_{o6} + 5\lambda_{o8} - 2\phi_6$	42.0
41	0.0000088	67.941	0.71851522	3193.99	$-\lambda_{o6} + 5\lambda_{o8} - \phi_8 - \Phi_8$	31.4
42	0.0000085	172.701	0.71658967	3202.58	$-\lambda_{o6} + 5\lambda_{o8} - \phi_8$	30.3
43	0.0000062	188.530	0.72942115	3146.24	$-\lambda_{o6} + 5\lambda_{o8} - \Phi_6 - \Phi_8$	22.1

Let us consider N_D equations of conditions from which we want to compute the most probable values of n_p parameters (gathered in p) :

$$\underset{(N_D, n_p)}{A} \quad \underset{(n_p, 1)}{p} \quad = \quad \underset{(N_D, 1)}{b} \tag{3}$$

This equality is matricial. The dimension of each matrix is given under it. In the case of the present work we have :

$$N_D \approx 50\,000 \qquad \text{and} \qquad n_p = 55$$

A is given by the theory, and the matrix b is estimated by b_o as follows :

$$b = b_o + \delta b \tag{4}$$

b_o is issued from observations and contains the classical observed-minus-computed residuals. So, each component of δb

Table 8b. (continued)

n° $\Delta\lambda_8$	amplitude (rad)	phase (deg)	frequency (rad/year)	period (days)	identification	amplitude (km)
44	0.0000060	50.154	0.70573027	3251.86	$-\lambda_{o6} + 5\lambda_{o8} - \phi_6 - \phi_8 + \Phi_8$	21.3
45	0.0000264	268.343	143.92207259	15.95	$\lambda_{o6} - \phi_8$	93.9
46	0.0000240	136.858	57.21714732	40.11	$2\lambda_{o8} - 3\lambda_{o9} + \varpi_9$	85.6
47	0.0000222	42.195	258.91063838	8.86	$2\lambda_{o6} - \lambda_{o8} - \phi_6$	79.2
48	0.0000218	346.938	479.08079784	4.79	$\lambda_{o5} - \lambda_{o8}$	77.7
49	0.0000204	211.225	28.50389879	80.51	$\lambda_{o8} - 2\lambda_{o9} + \phi_8$	72.6
50	0.0000177	186.678	229.99104991	9.98	$2\lambda_{o6} - 2\lambda_{o8}$	63.0
51	0.0000138	189.889	57.85704466	39.67	$2\lambda_{o8} - 2\Omega_9$	49.2
52	0.0000134	9.883	57.85704466	39.67	$2\lambda_{o8} - \Omega_9$	47.7
53	0.0000068	265.464	57.85897020	39.66	$2\lambda_{o8} - \Phi_8 - \Omega_9$	24.1
54	0.0000118	104.764	809.58234803	2.83	$\lambda_{o4} - \lambda_{o8}$	42.2
55	0.0000103	178.797	172.85449516	13.28	$\lambda_{o6} + \lambda_{o8} - \Phi_8$	36.7
56	0.0000095	283.216	172.85256962	13.28	$\lambda_{o6} + \lambda_{o8}$	33.8
57	0.0000073	290.661	1186.73540673	1.93	$\lambda_{o3} - \lambda_{o8}$	25.9
58	0.0000060	99.874	86.06897732	26.66	$\lambda_{o6} - 2\lambda_{o8} + \phi_8$	21.4

is a noise, that is the difference between the true value and the observed value.

The components of b_o are correlated and have varied precisions, so, we have to consider the $N_D \times N_D$ covariance matrix of the observations V_{obs}. Note that in the theory of the least square, V_{obs} is supposed to be known.

Then, the condition is : the density of probability of δb must be maximum. We deduce an estimation of p :

$$p_o = \left(A^t V_{obs}^{-1} A\right)^{-1} A^t V_{obs}^{-1} b_o \tag{5}$$

and the corresponding covariance matrix is :

$$V(p_0) = \left(A^t V_{obs}^{-1} A\right)^{-1} \tag{6}$$

In practical cases, the main problem is to estimate the matrix V_{obs}. First, it is too large to be well used in (5) and (6). Second, most of its elements are not really known. A drastic way is to suppose that all observations have the same precision ε and are not correlated. In this case $V_{obs} = \varepsilon^2 I$, where I is the unit $N_D \times N_D$ matrix; then Eqs. (5) and (6) become :

$$p_o = \left(A^t A\right)^{-1} A^t b_o \tag{7}$$

$$V(p_0) = \varepsilon^2 \left(A^t A\right)^{-1} \tag{8}$$

and ε may be estimated by :

$$\varepsilon_o^2 = \frac{1}{N_D - n_p} \sum_{k=1}^{N_D} r_k^2 \quad \text{with} \quad (r_k) = A p_o - b_o \tag{9}$$

Another simple case but more realistic, is to suppose that the observations are independent but with varied precisions. So, we have :

$$V_{obs} = \begin{pmatrix} \varepsilon_1^2 & 0 & \dots & 0 \\ 0 & \varepsilon_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_{N_D}^2 \end{pmatrix}$$

We can still use the formulas (7) and (8) (with $\varepsilon = 1$), but before that, we must divide each line k of A and of b_o by ε_k. The number $\frac{1}{\varepsilon_k^2}$ is then called the weight of the k^{th} observation.

In the present work, as in most of the similar works, we have considered the last case : we have supposed that the correlation between the observations is negligible.

4.2. The choice of the weights

The used formulas need us to know each ε_k (which represents the precision of the k^{th} observation). But usually observers do not give the value of ε_k explicitly. So, we have to choose a way to estimate it.

We have divided the whole set of observations into different data sets. Each set is defined by its author, the used instrument and the observed satellite. For such a set we determine, with a formula like (9), the root-mean-square of the residuals $\varepsilon_{i,j}$ (i is the reference used by Strugnell & Taylor to indicate the observer and the instrument, and j represents the satellite). We assume that the precision of an observation made by an observer with a particular instrument does not depend on the observed satellite (except Mimas). So, the equations of condition of a given reference i are divided either by $\varepsilon_{i,1}$ if Mimas is observed, either by $\min_{j\neq 1} \varepsilon_{i,j}$. To avoid that this minimum corresponds to a satellite insufficiently observed, we impose that the observations of this satellite represent at least 20% of the observations in that reference.

Harper & Taylor (1994) have shown that the precision depends on the type of the observation (p, s, $\Delta\alpha\cos\delta$, $\Delta\delta$, ...). Then, we have used different weights for the different types of observations. For example, in the sets containing position angles and separations, we have computed a weight for the measures of separation and another for the measures of position angle.

Table 8c. Solution for the variable z_8 (eccentricity and pericenter of Japetus). The series is expressed in complex exponential

z_{o8} n°	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.0010161	292.601	0.00000000			3619.0
2	0.0293565	192.436	0.00197469	3181.86	ϕ_8	104559.5
3	0.0009954	334.126	0.00893386	703.30	ϕ_6	3545.2
4	0.0007357	345.828	−0.00197469	3181.86	$-\phi_8$	2620.4
5	0.0006699	272.328	−0.00390023	1610.98	$-\phi_8 + \Phi_8$	2386.0
6	0.0004152	271.809	0.00390023	1610.98	$\phi_8 - \Phi_8$	1478.8
7	0.0003789	157.294	0.42462355	14.80	$-\phi_8 + 2\lambda_9$	1349.6
8	0.0001992	256.999	0.00700832	896.53	$\phi_6 + \Phi_8$	709.4
9	0.0001900	201.542	−0.00582578	1078.51	$-\phi_8 + 2\Phi_8$	676.7
10	0.0001235	49.578	0.01085941	578.59	$\phi_6 - \Phi_8$	439.8
11	0.0001012	214.593	−0.00893386	703.30	$-\phi_6$	360.4
12	0.0000693	319.489	−0.01085941	578.59	$-\phi_6 + \Phi_8$	246.8
13	0.0000489	230.042	0.63792267	9.85	$-\phi_8 + 3\lambda_9$	174.2
14	0.0000208	67.258	−0.01278495	491.45	$-\phi_6 + 2\Phi_8$	74.2
15	0.0000204	262.710	0.21132443	29.73	$-\phi_8 + \lambda_9$	72.6
16	0.0000180	13.416	0.41766438	15.04	$-\phi_6 + 2\lambda_9$	64.1
17	0.0000160	357.207	0.21329912	29.46	λ_9	56.8
18	0.0000134	266.386	0.21527381	29.19	$\phi_8 + \lambda_9$	47.8
19	0.0000129	302.968	−0.21132443	29.73	$\phi_8 - \lambda_9$	45.9
20	0.0000113	262.106	0.42264886	14.87	$-2\phi_8 + 2\lambda_9$	40.3
21	0.0000107	0.201	0.42852378	14.66	$-\Phi_8 + 2\lambda_9$	37.9
22	0.0000066	66.486	0.43049847	14.60	$\phi_8 - \Phi_8 + 2\lambda_9$	23.5
Δz_8	(rad)	(deg)	(rad/year)	(days)		(km)
23	0.0005938	282.857	143.92404729	15.95	λ_{o6}	2114.8
24	0.0002739	96.179	−86.06700263	26.66	$-\lambda_{o6} + 2\lambda_{o8}$	975.4
25	0.0002533	9.518	28.92852233	79.33	λ_{o8}	902.1
26	0.0001049	343.040	−28.50192410	80.52	$-\lambda_{o8} + 2\lambda_{o9}$	373.7
27	0.0000133	343.046	−28.50192410	80.52	$-\lambda_{o8} + 2\lambda_{o9} - \Omega_9$	47.3
28	0.0000067	87.465	−28.50384965	80.51	$-\lambda_{o8} + 2\lambda_{o9} + \Phi_8 - \Omega_9$	23.8
29	0.0000206	52.660	−28.28862499	81.13	$-\lambda_{o8} + 3\lambda_{o9} - \varpi_9$	73.3
30	0.0000172	51.705	287.83916071	7.97	$2\lambda_{o6} - \phi_6$	61.3
31	0.0000165	356.456	508.00932017	4.52	λ_{o5}	58.6
32	0.0000130	269.645	−57.14045499	40.16	$-\lambda_{o6} + 3\lambda_{o8} - \phi_8$	46.1
33	0.0000125	175.103	0.71851522	3193.99	$-\lambda_{o6} + 5\lambda_{o8} - \phi_8 - \Phi_8$	44.4
34	0.0000114	279.526	0.71658967	3202.58	$-\lambda_{o6} + 5\lambda_{o8} - \phi_8$	40.7
35	0.0000123	179.635	−28.92852233	79.33	$-\lambda_{o8} + \Omega_9$	43.7
36	0.0000118	359.629	−28.92852233	79.33	$-\lambda_{o8} + 2\Omega_9$	41.9
37	0.0000062	284.055	−28.93044787	79.33	$-\lambda_{o8} + \Phi_8 + \Omega_9$	22.0
38	0.0000115	35.997	86.35896876	26.57	$3\lambda_{o8} - 2\lambda_{o9}$	41.1
39	0.0000105	10.722	−143.92597283	15.95	$-\lambda_{o6} + \Phi_8$	37.4
40	0.0000098	266.302	−143.92404729	15.95	$-\lambda_{o6}$	34.9
41	0.0000105	2.840	−201.06252758	11.41	$-2\lambda_{o6} + 3\lambda_{o8}$	37.3
42	0.0000103	102.714	−114.99355026	19.96	$-\lambda_{o6} + \lambda_{o8} + \phi_8$	36.8
43	0.0000090	114.282	838.51087036	2.74	λ_{o4}	32.2
44	0.0000080	182.984	57.85506997	39.67	$2\lambda_{o8} - \phi_8$	28.6

Most often, as it was explained in Sect. 3, we have inter-satellite coordinates and thus, such an observation involves two satellites. To define which satellite is concerned, we use the ordered following list: Rhea, Titan, Dione, Tethys, Enceladus, Mimas and Japetus. For example, an observation involving the pair Titan-Japetus is considered for the computation of the weight as an observation of Japetus.

Another remark concerns the computation of the root-mean-square residuals. According to (9), we have to know the values of the parameters p_o. Because p_o is the result of the least square procedure, these values are unknown the first time we want to compute the weight. In fact, the weights are defined by implicit equations. We resolve them by iterations. Starting with $p_o = 0$, convergence occurs in two iterations. We have verified that the

Table 8c. (continued)

n° Δz_8	amplitude (rad)	phase (deg)	frequency (rad/year)	period (days)	identification	amplitude (km)
45	0.0000071	326.988	−229.98211605	9.98	$-2\lambda_{o6}+2\lambda_{o8}+\phi_6$	25.2
46	0.0000063	22.580	−450.15227551	5.10	$-\lambda_{o5}+2\lambda_{o8}$	22.3
47	0.0000057	300.179	1215.66392906	1.89	λ_{o3}	20.2

Table 8d. Solution for the variable ζ_8 (inclination and node of Japetus). The series is expressed in complex exponential

n° ζ_{o8}	amplitude (rad)	phase (deg)	frequency (rad/year)	period (years)	identification	amplitude (km)
1	0.1320165	184.580	0.00000000			470205.8
2	0.0679455	289.223	−0.00192554	3263.07	Φ_8	242002.8
3	0.0006892	80.102	0.00192554	3263.07	$-\Phi_8$	2454.9
4	0.0002731	176.129	−0.00893124	703.51	Φ_6	972.6
5	0.0002641	348.659	0.42659824	14.73	$2\lambda_9$	940.7
6	0.0001817	64.344	0.42852378	14.66	$-\Phi_8+2\lambda_9$	647.1
7	0.0000457	209.876	−0.00385109	1631.54	$2\Phi_8$	162.8
8	0.0000449	291.806	−0.21329912	29.46	$-\lambda_9$	160.1
9	0.0000337	61.433	0.63989736	9.82	$3\lambda_9$	120.2
10	0.0000301	247.712	0.21329912	29.46	λ_9	107.2
11	0.0000287	64.768	0.00390023	1610.98	$\phi_8-\Phi_8$	102.3
12	0.0000283	216.316	−0.21522466	29.19	$\Phi_8-\lambda_9$	100.9
13	0.0000283	182.190	0.21137358	29.73	$\Phi_8+\lambda_9$	100.8
14	0.0000235	137.055	0.64182290	9.79	$-\Phi_8+3\lambda_9$	83.7
15	0.0000192	95.511	0.00587493	1069.49	$2\phi_8-\Phi_8$	68.2
16	0.0000139	355.208	−0.00683119	919.78	-19μ	49.5
17	0.0000136	90.131	−0.01085678	578.73	$\Phi_6+\Phi_8$	48.5
18	0.0000112	121.717	0.01786773	351.65	$2\phi_6$	39.7
19	0.0000099	173.180	0.21522466	29.19	$-\Phi_8+\lambda_9$	35.3
20	0.0000099	204.432	0.00893124	703.51	$-\Phi_6$	35.1
21	0.0000066	337.999	0.01090856	575.99	$\phi_6+\phi_8$	23.4
22	0.0000059	128.157	−0.00592407	1060.62	$-3\phi_8$	21.1
$\Delta\zeta_8$	(rad)	(deg)	(rad/year)	(days)		(km)
23	0.0000299	91.241	−114.99552496	19.96	$-\lambda_{o6}+\lambda_{o8}$	106.4
24	0.0000160	195.660	−114.99745050	19.96	$-\lambda_{o6}+\lambda_{o8}+\Phi_8$	56.9
25	0.0000201	287.796	172.85256962	13.28	$\lambda_{o6}+\lambda_{o8}$	71.6
26	0.0000108	183.376	172.85449516	13.28	$\lambda_{o6}+\lambda_{o8}-\Phi_8$	38.5

process converges to the same weight if we give, at the first step, equal weight for the whole set.

4.3. The probable errors and precisions

We have seen that the estimation of the weight of each equation of condition results from a choice. Although our procedure is close to that of Dourneau (1987) or Harper & Taylor (1993) in their similar works, this choice is not the only one possible and may be criticized. Furthermore, we have neglected the correlations between the observations (the matrix V_{obs} is supposed to be diagonal). So, we think that the choice of the matrix V_{obs} does not really respect the hypothesis of the least square theory. It seems that there is no influence on the determination of the parameters but there are sometimes some difficulties to consider their probable errors as good indicators of the precision of the parameters.

For example the Figs. 2 in Harper & Taylor (1993) illustrate this purpose. For each parameter they have determined, they have plotted its value and its probable error bar. They have also plotted for comparison the similar results obtained by other authors. In some cases, error bars are far from one another (usually one considers three times the probable error). So, assuming zero mean Gaussian measurement errors, the probability to find the parameter in the intersection of error bars is very small. Of course, for most of the cases we can explain this fact. For example, the difference between their determinations and those of Taylor & Shen (1988) is certainly explained by the fact that Taylor & Shen use only the observations made after 1966. But,

Table 9. Maximum difference in kilometers, over the indicated time-span, between positions computed with TASS and those issued from a numerical integration. The physical models are the same in both case: Saturn's oblateness, intersatellites and solar perturbations. The initial conditions used in the numerical integration are given by TASS

body	over 100 years	over 3 years
Mimas	27	10
Encelade	60	14
Tethys	23	10
Dione	30	5
Rhea	10	6
Titan	13	4
Japetus	173	37

in both cases, if the hypothesis of the least square theory were respected, error bars should be closer.

So, we conclude that the least square procedure as it is used here, leads to a good estimator of the parameters but most often overestimates their precision. This precision has to be estimated in another way. This fact was also observed by Campbell & Anderson (1989). Following them we have measured the sensitivity of the determination with respect to the method used. For that, the adjustment has been done with the weight described in Sect. 4.2. A second adjustment has been done by squaring the weight normally used. Then, our estimation of the standard error is computed so that the corresponding error bar contains both determinations. This standard error is denoted in Tables 10 by $\pm$ and it is given in addition to the probable error σ.

This way for estimating the precision is not really objective but we think that it is more realistic. Another way would be to find a good evaluation of the matrix V_{obs}. It would be necessary to analyse again all the observations to obtain errors on measures. But that seems to be very difficult to do.

5. Results

We present, in this section, the results of the comparison between TASS and the Earth-based observations. There are three kinds of results :

First, we present the values obtained for all the parameters of TASS (i.e. p_o of the preceding section). The part of p_o concerning the initial conditions cannot be easily compared to previous works because the variables and the reference frame we use are different. On the contrary, because of their physical meaning, the part of p_o concerning the masses and the oblateness coefficients may be compared with previous works. We emphasize the mass of Saturn which gives the scale of the dynamical system.

Second, we give the residuals of the observations (r_k in (9)).

Third, we discuss the new solution obtained. Because of the value of p_o, the solution is slightly different from the one presented in Papers II and III. We will comment about the main differences.

The results of two adjustments are presented : TASS1.5 and TASS1.6. In TASS1.5, all the parameters are fitted except the oblateness coefficient J_6 which is fixed at the value determined by Campbell & Anderson (1989). In TASS1.6, all the parameters are fitted except the masses of Saturn and Titan, the coefficients J_2 and J_4, and the position of the equatorial plane. These fixed parameters are also taken from Campbell & Anderson (1989). These values come from spacecraft determinations (Pioneer and Voyager) and are well determined. Thus, TASS1.6 is up to now the best solution of our equations (Tables 1 to 8). It is ready to produce ephemerides of the satellites. TASS1.5 allows to see how TASS is able to determine all the parameters. It is also an indicator of the sensitivity of the parameters in the adjustment.

5.1. Physical parameters of the Saturnian system

The determination of the gravity field of the Saturnian system, presented in Tables 10, agrees very well with almost all other determinations.

Our determination of the masses of Mimas, Tethys, Dione does not differ from those obtained by Harper & Taylor (1993) and our probable errors are slightly smaller.

The mass of Enceladus is not well determined: TASS1.5 and TASS1.6 do not agree for this parameter and the standard error (opposed to the probable errors, see Sect. 4.3) is large. Note that both determinations by Harper & Taylor and by Dourneau are not really independent, because the representations of the motions they use are very similar. Then, we have not enough determinations of this mass to decide which of them is the better one. If we suppose the radius of Enceladus equal to 250 kilometers (from the Explanatory Supplements 1994), the value from TASS1.6 leads to a density of 0.6 g/cm^3, from TASS1.5 to 1 g/cm^3 and from Harper & Taylor and Douneau to 1.9 g/cm^3. The three other inner satellites have the approximate folowing densities: Mimas, 1.2 g/cm^3 (radius $\approx$ 200 km); Tethys, 1. g/cm^3 (radius $\approx$ 530 km); Dione, 1.5 g/cm^3 (radius $\approx$ 560 km).

Our values for the mass of Rhea are in good agreement with the determination by Campbell & Anderson. Both determinations have the same precision.

The mass of Titan has a great influence on the motion of Hyperion, but the work which consists in including Hyperion in the present analysis is still in progress. So the present determination comes mainly from the motions of Rhea and Japetus only.

Although the precision of our determination of the mass of Japetus is not very good, it agrees well with the determination by Campbell & Anderson.

Concerning the oblateness coefficients J_2 and J_4, our determinations agree very well with all other determinations. Note that our precisions (probable error and standard error) and those of Campbell & Anderson are very similar. Despite the fact that these two coefficients are correlated (99%), we have obtained a good determination of J_4 and the probable error is significantly smaller than that quoted by Harper & Taylor.

Table 10. Determination of some parameters of the Saturnian system. TASS 1.6 is a fit in which the masses of Saturn and Titan, the coefficients J_2 and J_4, the position of the equatorial plane, are not adjusted but are fixed to the values determined by Campbell & Anderson (1989). TASS 1.5 is a fit in which only J_6 is not adjusted but is fixed to the value determined by Campbell & Anderson. We also give the determinations (when it is possible) by Harper & Taylor (1993), Dourneau (1987), Campbell & Anderson and Simpson & Tyler (1983) (line "Pioneer & Voyager"), Null et al (1981, line "Pioneer"), Sinclair & Taylor (1985), Elliot at al (1993, line "Hubble") and the value recommended by IAU. **a** Determination of the masses of Mimas (1), Enceladus (2), Tethys (3), Dione (4), Rhea (5), Titan (6) and Japetus (8), in units of Saturn's mass

parameters $\times 10^6$	m_1	m_2	m_3	m_4	m_5	m_6	m_8
TASS 1.6	0.0634	0.069	1.060	1.963	4.32	–	3.1
TASS 1.5	0.0640	0.107	1.068	1.950	3.68	235.36	5.1
σ	8	21	13	21	38	18	5
$\pm$	25	88	42	121	64	62	2.3
IAU						241.	
Harper & Taylor	0.0646	0.213	1.076	1.916			
σ	11	46	18	36			
Dourneau	0.0648	0.206	1.088	1.954			3.7
σ	21	55	31	58			1.7
Pioneer & Voyager			1.186		4.059	236.638	2.79
σ			105		53	8	8
$\pm$			263		105	26	26
Pioneer					4.0	238.8	3.4
σ					1.	3.	1.3

Table 10b. Determination of the oblateness coefficients of Saturn J_2, J_4 and J_6, of the inclination of the equatorial plane i_a and of the longitude of its node Ω_a referred to the J2000 system, and of the Saturn's mass by $\frac{M_\odot}{M_S}$. The coefficients J_2 and J_4 from Dourneau (1987) were calculated by Harper & Taylor (1993) from the nodes rates determined by Dourneau. In the same way we have calculated the Saturn's mass from the semi major axes determinated by Harper & Taylor and Dourneau (see Sect 5.2)

parameters	$10^6 J_2$	$10^6 J_4$	$10^6 J_6$	i_a	Ω_a	$\frac{M_\odot}{M_S}$
TASS 1.6	–	–	95.	–	–	–
TASS 1.5	16285.	−959.	–.	28°0665	169°5339	3497.2
σ	5.	17.	11.	14	29	.1
$\pm$	12.	39.	70.	06	48	.2
IAU	16270.	−980.				3499.4
Sinclair & Taylor	16508.					3498.2
σ	900.					3
Harper & Taylor	16298.	−1076.	–	28°0588	169°5357	(3491.)
σ	38.	274.		16	35	3.
Dourneau	(16326.)	(−841.)	–	28°0752	169°5082	(3496.)
σ	54.	401.		35	89	4.
Pioneer & Voyager	16298.	−915.	103.	28°0512	169°5291	3498.790
σ	5.	26.	41.			3
$\pm$	10.	40.	50.	10	25	20
Pioneer	16299.	−917.	–			
σ	18.	37.				
Hubble				28°0541	169°5252	
σ				51	100	

About J_6, we cannot say that we have determined its value because, for computing it, J_2 and J_4 were fixed to the values given by Campbell & Anderson. Nevertheless, we can say that Earth-based observations analysed with TASS confirm the value from spacecraft observations.

Our determination of the position of the equatorial plane is in good agreement with those based on the motion of the satellites (Harper & Taylor and Dourneau) and with that based on a stellar occultation observed by the space telescope Hubble. But a significative difference subsists with the precise determination obtained from the Voyager missions (Simpson & Tyler 1983). This difference occurs mainly in i_a.

Table 11. Root-mean-square residuals ($\times 100$ in seconds of degree) for several major data sets and compared to those obtained by Harper & Taylor (1993, column "HT"). The last line of this table represents the mean value of the other lines, weighted by the corresponding root-mean-square residual. The column "weight" gives the influence of the corresponding data set : the value is the sum of the weight of each observation in that data set. So, the data sets presented in this table represent 72% of the whole set of observations used in the analysis

				HT	TASS	TASS (by satellite)						
Ref	Observer	Nb. obs.	weight (%)			1	2	3	4	5	6	8
1	USNO (1877-1887)	1080	1.6	40	36	35	33	30	31	34	36	40
3	USNO (1911)	3720	9.6	27	24	32	28	21	18	21	26	34
4	USNO (1929)	2390	8.2	18	15	17	15	14	14	15	15	22
5	USNO (1954)	2520	6.2	22	20	24	22	20	19	19	18	
6	Struve (1933)	2790	5.4	30	24	24	24	22	24	24	33	35
9	Struve (1898)	1780	7.4	14	12	15	9	11	12	12	23	20
10	Alden & O Connell (1928)	590	3.2	12	12		13	9	6	6	11	24
18	Kisseleva et al (1977)	360	1.5	16	17	38	27	17	15	6	7	17
26	Sinclair (1974,1977)	860	1.8	12	13			14	11	12	12	12
31	Pascu (1982)	2330	12.7	13	13	19	14	10	10	8	8	15
33	Tolbin (1985)	720	2.1	14	12	14	15	12	11	6	7	19
42	Aksnes & al (1984)	14	0.7	3.0	1.5							
46	Dourneau et al (1989)	950	1.4	28	27			31	29	27	26	24
47	Veillet & Dourneau (1992)	300	1.7	18	11	15	10	10	9	8	9	
48	Veillet & Dourneau (1992)	1400	8.4	13	12	18	13	12	10	9	10	14
	mean value			17	15	18	14	13	13	12	13	19

5.2. Mass of Saturn and the semi major axes

The mass of Saturn is well determined from the spacecraft measures (Campbell & Anderson 1989). From Earth-based observations of the satellites, it is difficult to obtain such a precise determination (Table 10b). The best comparable determination is that of Sinclair & Taylor (1985). They have adjusted the mass of Saturn (with other parameters) from a numerical integration of Titan, Hyperion and Japetus fitted on observations of these satellites made after 1966. Despite this relatively good value, they think that there is a systematic scale error in the observations of the satellites of Saturn. Taylor & Shen (1988) have confirmed this scale error. They have compared the observed semi major axes with the values derived from the observed mean motions using Kepler's third law. Vienne (1991) emphasized the possible confusion between the "observed semi major axis" and the mean radius of the orbit. He was not able to find which correction must be applied to find the observed value. This is because the semi major axis and the mean motion are fitted independently and then one or both values may be affected by a default in the theory. This is not the case with TASS : the mean motion is adjusted (more exactly the mean mean motion) and the semi major axis is a computed value. In TASS1.6 we have used the mass of Saturn determined by Campbell & Anderson, and the residuals are presented in Table 11. We will comment these residuals in the next section, but we can see that our residuals are at least as good as those of Harper & Taylor (1993). So, we think that there is no evidence for systematic scale errors. Furthermore, with the value of the mass of Saturn determined by TASS1.5, the residuals are not significantly different: these residuals are not very sensitive to any reasonable change on the mass of Saturn. Note that the precision quoted in the table is certainly overestimated.

If we adopt the value of Campbell & Anderson, we can say that our value is relatively good but overestimated. A better value can be obtained if we keep only the post 1966 observations in the fit. We find : $\frac{M_\odot}{M_S}$ = 3498.3 which is close to the Sinclair-Taylor value. With observations before 1966 only, we find 3493.6. So it seems that the oldest observations lead to overestimate the mass of Saturn.

5.3. Residuals of the observations

The residuals of the observations are presented in Table 11 in the form of root-mean-square residuals of several major data sets. We have compared these results with those from Harper & Taylor (1993).

In general, the improvement of precision is small and occurs mainly for the oldest observations (except those of Veillet & Dourneau, reference 47). Furthermore, it seems that the distribution of the residuals does not depend on the theory. So, we can say that the residuals presented in the table represent the precision of each set of observations.

A complete discussion of the comparative quality of the data sets is done in Harper & Taylor (1994). We only emphasize here the sets of Pascu, of Veillet & Dourneau and of Struve which contain numerous observations of good quality.

The observations of Aksnes et al. (1984) are very precise. They correspond to mutual events, with precision limited only by the timing of each event. This precision ($0\farcs02$ as seen from the Earth $\approx$ 130 km) corresponds to a fraction of the diameter of the involved satellites. Unfortunately, up to now there are very few observations of this kind. We see here the importance of the next campaign for observing mutual events in 1995-1996 (Arlot & Thuillot 1993).

The results given for each satellite show that Enceladus, Tethys, Dione, Rhea and Titan have the best residuals. Residuals for Mimas and Japetus are not as good. In fact, Harper & Taylor (1994) have shown that these satellites are much less observed than the others.

5.4. Remarks about the new orbits obtained

Tables 1 to 8 present the adopted solution TASS1.6. The presentation is limited here to perturbations greater than 20 km but the fortran program which computes the positions uses the complete series. This program including the series is available from the authors. It is not our purpose to describe here the whole solution because this was already done in Papers II and III. We only want to emphasize two points.

The first one concerns the mass of Enceladus. The value found in TASS1.6 is smaller than that in Dourneau or in Harper & Taylor (about three times our value). The main effect is seen in the mean longitude of Dione where both terms are smaller than those found in Paper III. Note that in all cases, these terms are rather small (to compare to the precision of the observations $\approx$ 1000 km). It is why the mass of Enceladus is rather badly determined.

The second one concerns the eccentricity of Tethys. There are in the mean longitude of Mimas, some very long period terms whose arguments contain $2\Phi_1 + \phi_3$ (terms n° 7 and 10 in Table 1b). These terms were more numerous in Paper III because the eccentricity of Tethys was 5 times greater than that found in the present work (Table 3c). Indeed, because the argument $2\Phi_1 + \phi_3$ contains the pericenter of Tethys, the amplitude of such terms depends on its eccentricity. The influence of these very long period terms does not seem to be as important as mentioned in Vienne et al. (1992), where it was explained that these terms could be confused with a secular acceleration. Nevertheless, the standard errors on the eccentricity of Tethys are rather large (about half its value) and then the question about the eccentricity of Tethys is still open. Furthermore, the corresponding small divisors exist in any case even if the eccentricity of Tethys is very small. The study of this secular resonance has to be done. Perhaps it would be an explanation of the nearly circular orbit of Tethys.

6. Conclusion

TASS with its version TASS1.6 is now ready to produce ephemerides. In comparison with previous theories, the improvement in precision seems small because the residuals reach the precision of the observations. The interest of TASS is mainly elsewhere.

The internal precision of the nominal solution is far better. So, it will be a good tool to account with the future observations of high precision (mutual events in 1995-1996, spacecraft observations).

TASS is a representation of the motion of the satellites which is coherent with all the dynamical parameters of the Saturnian system, specially with the masses, the oblateness coefficients, the mean mean motions and the semi major axes.

We have now to work again to improve TASS into a version TASS.2. The versions TASS1. are all issued from the nominal solution presented in Paper III. They differ only because they depend on the partial derivatives. In the present work, we have found large variations for some of the parameters (e.g.: eccentricity of Tethys, node of Dione, eccentricity and pericenter of Japetus,...) and this fact degrades the precision of the representation. The first interest in computing a new version of TASS is to update the nominal solution. We recall that the nominal solution of TASS1. (Paper III) is based on the representation by Dourneau (1987) which is very different from ours. We can then expect smaller variations of the parameters. The second interest is to improve some points quoted in Paper III in order to improve the nominal solution (specially for Japetus, see Table 9).

With the aim of the CASSINI mission as well as for a better knowledge of the dynamical system of the Saturn' satellites, it is very important to collect high quality observations and also to reduce some older ones such as those from the Voyager missions or those from new observations by CCD cameras (Colas 1991). The mutual events which occur in 1995-1996 are especially of great interest.

Acknowledgements. We would like to thank Dr. M. Rapaport for many helpful discussions. We are very grateful to Dr. Dan Pascu for his permission to use his unpublished observations of the satellites of Saturn.

References

Aksnes K., Franklin F., Millis R. et al.: 1984, AJ 89, 280.

Arlot J.E., Thuilot W.: 1993, Icarus 105,

Aoki S., Sôma M., Kinoshita H., Inoue K.: 1983, A&A 128, 263

Bretagnon P: 1982, A&A 114, 278

Campbell, J.K., Anderson, J.D.: 1989, AJ 97, 1485.

Colas, F.: 1991, 'Nouvelles observations CCD astrométriques pour l'étude dynamique des satellites des planètes', *Thèse, Observatoire de Paris.*

Dourneau, G.: 1987, 'Observations et étude du mouvement des huit premiers satellites de Saturne', *Thèse, Bordeaux.*

Duriez, L., Vienne, A.: 1991, A&A 243, 263

Eichhorn H.: 1993, Celest. Mech. 56, 337.

Elliot, J.L., Bosh, A.S., Cooke, M.L. et al.: 1993, AJ 106, 2544.

Harper D., Taylor, D.B.: 1993, A&A 268, 326

Harper D., Taylor, D.B.: 1994, A&A 284, 619

Null G.W., Lau E.E., Biller E.D., Anderson J.D.: 1981, AJ 86, 456.

Simon, J.L., Bretagnon, P.: 1984, A&A 138, 169

Simpson, R.A., Tyler, G.L.: 1983, AJ 88, 1531.

Sinclair, A.T., Taylor, D.B.: 1985, A&A 147, 241

Strugnell P.R., Taylor, D.B.: 1990, A&AS 83, 289

Taylor, D.B., Shen K.X.: 1988, A&A 200, 269

Taylor, D.B., Morrison L.V., Rapaport M.: 1991, A&A 249, 569

Veillet C., Dourneau G.: 1992, A&AS 94, 291

Vienne, A.: 1991, 'Théorie Analytique des Satellites de Saturne', *Thèse, Lille.*

Vienne, A., Duriez, L.: 1991, A&A 246, 619

Vienne, A., Duriez, L.: 1992, A&A 257, 351

Vienne, A., Sarlat, J.M., Duriez, L.: 1991, 'About the secular acceleration of Mimas', in 'Chaos, Resonance and Collective Dynamical phenomene in the solar system' I.A.U Symposium 152, *ed. Ferraz-Mello*