H.W.M. OLBERS AS A FORERUNNER OF THE SPACECRAFT ASTRODYNAMICS

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ABSTRACT. - In this paper the Olbers' work, in the field of astrodynamics, is considered. It is pointed out that he was a forerunner of the ballistic missile trajectories theory. Obviously there are many shortcomings in his work but it is very interesting to note his pioneer's studies.

1. - Introduction

The astronomer Olbers is well known, in the field of cosmology, on account of the problem of the darkness of the night sky, or Olbers' paradox, which received noticeable attention specially after the formulation of the steady-state theory in 1948. He was undoubtedly a sharp and versatile scientist (1758-1840), many astronomical discoveries are present in his "curriculum operis": the asteroids Pallas and Vesta, six comets. Speaking about last work on comets, he gathered theoretical models and calculation methods within a treatise with the title «Ueber die leichteste und bequemste Methode die Bahn eines Cometen zu berechnen » i.e. « On the simplest and easiest method for calculating the comet orbit » (Figure 1). Olbers published also a paper, in the year 1820, in which he

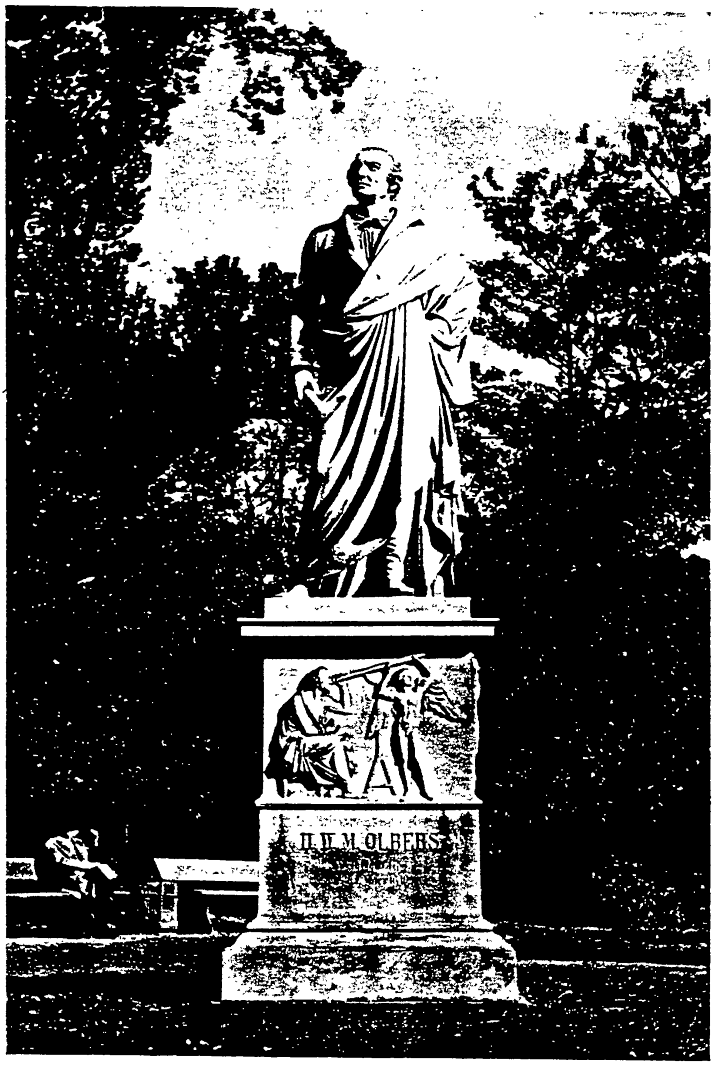

Figure 1. Olbers' monument in Bremen.

approached the "general ballistic missile problem". In simpler words, he studied the launch of a body in order to strike the Moon (Ref.1).

2. - The Olbers' problem

The analytical development of the problem lies on the basis of newtonian mechanics paradigm. With respect to the pratical realization, Olbers pondered on rocket use. We must think that, at start of XIX century, there were many advancements in the use of rockets for military purposes. English army used, following the Colonel W. Congreve ideas, about 13,000 rockets (during the Napoleonic wars), some of that launched as far as about 5 km (Ref. 2). Olbers was very surprised by the efficacy of this new weapon and therefore he was oriented towards this type of engine for its missile. Unfortunately he did not develop any analytical approach for calculating the trajectory of a possible spacecraft endowed with continuous thrust.

Olbers' study, here considered, proved the impossibility of using a gun in order to throw a projectile from Earth's surface to Moon's surface, he had the full consciousness of the limits of his time technology.

3. - Analytical development of the problem

Let us take up, from the Olbers' text (Ref. 1), the theoretical model, based on simple and fundamental concepts of classical mechanics, using the same analytical notation. Let T and L (Figure 2) two mass-points, Earth and Moon respectively, and a the average distance between them. Furthermore:

r = Earth's radius, M_T = Earth's mass,

G = gravitation constant, M_L = Moon's mass.

If we consider a point E between T and L, with coordinate x at time t, taking in E the unit-mass in motion towards the Moon, Olbers' writes:

$$\frac{dv}{dt} = - \frac{G M_T}{x^2} + \frac{G M_L}{(a - x)^2}, \qquad (1)$$

in which v is the istantaneous velocity at point E of the unit-mass. Taking account the gravity acceleration at the ground, i.e. $M = G M_T / r^2$, it follows that

$$\frac{G M_T}{x^2} = \frac{r^2 M}{x^2}.$$

Introducing also the gravity acceleration, owing to the Moon, at distance r from its center, i. e. $C = G M_L / r^2$ it follows that

$$\frac{G M_L}{(a - x)^2} = \frac{C r^2}{(a - x)^2}.$$

With these notations Eq. (1) becomes

$$\frac{dv}{dt} = - \frac{r^2}{x^2} M + \frac{C r^2}{(a - x)^2}$$

and then $\left(\text{because } v = \frac{dx}{dt} \right)$

$$v\, dv = - \frac{r^2}{x^2} M\, d x + \frac{C r^2}{(a-x)^2} dx. \qquad (2)$$

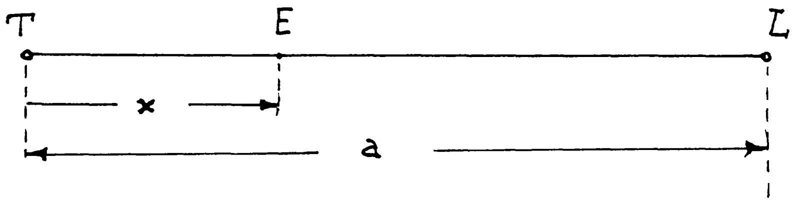

Figure 2. - Geometry for Equation (2).

Equation (2) is readily integrable with the initial condition: when $x = r$, $v = c$ = launching initial velocity. We obtain the solution

$$v^2 = c^2 - 2\,r\,M\left(1 - \frac{r}{x}\right) + \frac{2r^2 C\,(x-r)}{(a-r)\,(a-x)} \qquad (3)$$

With Eq. (3) Olbers gives the variation law $v = v\,(x)$ together with the parameters c,r,M,C,a. In order to get the minimum value of v he writes Eq. (3) so:

$$v^2 = c^2 - K\,(x,r,M,C,a) \qquad (4)$$

and then he calculates the maximum of $K\,(x)$.

Putting $\frac{dK}{dx} = o$, i. e.

$$\frac{d}{dx}\left[2rM\left(1 - \frac{r}{x}\right) - \frac{2\,r^2\,C\,(x-r)}{(a-r)\,(a-x)}\right] = o$$

he easily obtains the value of x corresponding to the maximum of $K\,(x)$. This value x_o is

$$x_o = \frac{a\sqrt{M}}{\sqrt{M} + \sqrt{C}},$$

whereas $K_o = K(x_o)$ is the maximum value of $K(x)$ connected with $v(x_o)$, the minimum value of $v(x)$. Following Eq. (4), the velocity $v(x)$ decreases as x increases until to x_o; then it increases as far as L. Putting $v(x_o) = o$ we have

$$c = \sqrt{K_o}, \tag{5}$$

in which

$$K_o = 2\,r\,M\left(1 - \frac{r}{x_o}\right) - \frac{2\,r^2\,C\left(x_o - r\right)}{(a - x_o)\,(a - r)}$$

Consequentially we rewrite Eq. (5) so:

$$c = \sqrt{2r\,M}\quad k = \sqrt{2}\sqrt{GM_T/r}\quad k, \tag{6}$$

in which

$$k = \sqrt{1 - \frac{r}{x_o}\,\frac{M_L}{M_T}\,\frac{x_o - r}{(a - x_o)\,(a - r)}}$$

The expression $\sqrt{2}\sqrt{\frac{GM_T}{r}}$ is the parabolic escape velocity from Earth's surface.

The astronomical data, at Olbers' time, give us:
$a = 54\,r$ instead to-day value $a = 60\,r$,

$$\frac{M_L}{M_T} = \frac{C}{M} = \frac{1}{66} \text{ instead to-day value } \frac{M_L}{M_T} = \frac{1}{81.3}.$$

Therefore Olbers obtains x_o = 48.08 x Earth's radius and k = 0.988. Using to-day astronomical data we calculate now x_o = 54 x Earth's radius.

At first he notes that the initial velocity for the projectile is

$$c = 11.2 \text{ km s}^{-1} \times 0.988 = 11.066 \text{ km s}^{-1},$$

by far greater than the initial velocity of a gun projectile, at his time, i. e. 0.48 km s^{-1}. The propellent force should be increased greatly and then it should be necessary a "technological jump" with respect to the projectile propellent material. The second Olbers' note deals with the fact of giving the velocity c = 11.066 km s^{-1} suddenly, i. e. at the starting instant. He rightly says that, owing to air resistance, the impressed initial velocity shall decrease drastically in such a manner that the projectile shall arrive, sooner or later, back to the ground. There is a very eloquent passage in Olbers'text:

"...Auch die glühenden Steine und halbverglaszten Massen, die unsere Vulkane auswerfen, erheben sich nur zu einer verältniszmäszig sehr geringen Höhe wenn gleich ihre anfängliche Geschwindigkeit weit gröszer ist als die einer Canonenkugel: und so wird man, besonders wenn man auf den schon erwähnten Wiederstand der Luft sieht, leicht zu geben, dasz sich nie schwere Theile von unserer Erde bis zu irgend einem anderen Weltkörper schleudern laszen...".

"...Also the incandescent stones and semi-vitrified bodies,thrown by our volcanoes, shall rise as far as not so great height, in spite of their initial velocity, much more greater that that of gun projectile. Therefore it should be very difficult, owing to air resistance, to throw a heavy piece of our Earth as far as the satellite's or planet's surface...".

The most important conclusion of Olbers' study is the theoretical demonstration of the impossibility of launching, with a gun, a projectile on Moon's surface (taking into account the technology status of 1820). There are some hints dealing with a possible rocket application, but they are not sufficiently developed.

4. - Criticism on Olbers' analytical treatment

After this synthetic outline of Olbers' study, we may pass to a more detailed consideration of the various analytical steps which are involved. Our purpose is to verify the correctness, from the point of view of the analysis, of the examined treatment.

In Figure 2 we see that mass-poins T and L are fixed in mutual position. Theoretically speaking this is the Euler's problem of planar motion of a mass-point in the gravitational field of two fixed centers of attraction. A classical example of a "mathematically integrable" dynamical system, discussed in various text of dynamics (Ref. 3,4,5); however this problem is a pure mathematical one. The restricted three-body problem, specially for our application, is more important in this context. In fact if one of the three masses is very small, i.e. the projectile or the spacecraft with respect to the primary masses Earth and Moon, it has no appreciable effect on the motion of other two, the possible motions of the small mass are considerably expanded. The primary bodies shall revolve around their common center of mass in circular orbits, then the mutual distance shall be constant. Taking the x-axis fixed, as Olbers made, with Earth and Moon in a fixed inertial system, called the sidereal system, is physically not acceptable. If we consider a synodic system,with x-axis passing through T and L rotating with the constant angular velocity n (mean motion of Earth-Moon system which is equal to $2.66\ 10^{-6}$ rad s^{-1}), we may study the relative motion of the small mass with respect to primaries T and L, fixed on the rotating x-axis. This is the realistic model for the correct approach to our problem, or to calculate a spacecraft motion in the Earth-Moon system.

With simpler words, to describe events taking place on a train you might well decide to travel with the train rather than to stay in the station.

On the contrary the Olbers' model is quite incorrect by the theoretical point of view. Eq. (1), in item 3, must be replaced by the following vector differential equation in the synodic system:

$$\frac{d\bar{v}}{dt} = \bar{F}_{GT} + \bar{F}_{GL} + \bar{F}_{C} + \bar{F}_{COR} , \qquad (7)$$

in which

$\bar{v}$ = vector velocity of the small mass,

$\bar{F}_{GT}$ = gravitational force (per unit mass) exerted by the Earth,

$\bar{F}_{GL}$ = gravitational force (per unit mass) exerted by the Moon,

$\bar{F}_{C}$ = centrifugal force (per unit mass),

$\bar{F}_{COR}$ = Coriolis force (per unit mass).

Going back to the restricted three-body problem, which is exhaustively studied in any good textbook of Celestial Mechanics (Ref. 6,7), it is demonstrated that there exist five relative equilibrium position (with respect to P_1 and P_2), called also libration points. Three of these points are located on the line (x - axis) connecting the primaries and two points form equilateral triangles with the primaries. Figure 3 shows the x-axis and y-axis of the synodic system, P_1 and P_2 as primary bodies and O as center of mass.

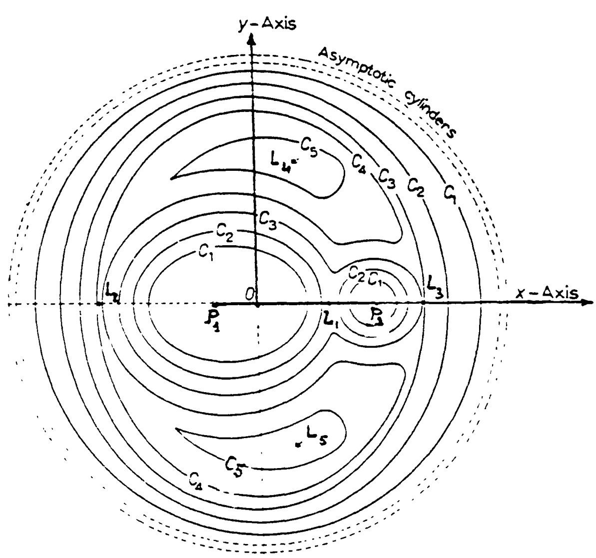

Figure 3. The zero-velocity curves in the restricted three-body problem. The constant distance between P_1 and P_2 is normalized to unity.

We can see also a family of curves with C parameter (C_1, C_2 ...); these are the zero-velocity curves. In each point of these curves the small body is at rest with respect to primaries. We see also the five libration points: L_1, L_2, L_3, L_4, L_5.

We are interested, for our problem, to the point L_1; its abscissa x (pratically the distance corresponding to x_o, since $\frac{M_L}{M_T}$ is small) is easily found with the following equation

$$n^2 x = \frac{GM_T}{x^2} - \frac{GM_L}{(a-x)^2} . \tag{8}$$

Comparing Eqs. (7) and (8) we note that the centrifugal force is $n^2 x$, whereas $\frac{d\bar{v}}{dt}$ and $\bar{F}_{COR}$ are both zero since we stay at equilibrium point L_1. Working on Eq. (8), taking into account also the third Kepler's law, we get the distance x_1 of L_1 from P_1, which is equal to

$$x_1 = 1 - \rho ,$$

$$\text{being } \rho = \left(\frac{M_L}{3 M_T}\right)^{\frac{1}{3}} - \frac{1}{3}\left(\frac{M_L}{3 M_T}\right)^{\frac{2}{3}} - \frac{1}{9}\left(\frac{M_L}{3 M_T}\right)^{\frac{3}{3}} \ldots$$

Taking Olbers' astronomical data we obtain

$$\rho \cong 0.16 \qquad \text{and } x_1 = 0.84.$$

Therefore the equilibrium point, following the Olbers' model, is at distance from T equal to 48,08 times Earth's radius whereas, with the correct model provided by the restricted three-body problem, it is at distance from T equal to 45.36 time Earth's radius. We see that the error is equal to 5.6%. The error is small, owing to the smallness of $\frac{M_L}{M_T}$ ratio, but the conceptual correction is important.

5. - Concluding remarks

We studied the Olbers' problem to launch a projectile, with a gun, on lunar surface. We have found that his treatment, partially rigorous, provides reasonable results. Particularly his sharp intuitions in order to apply rockets technology are precious.

Let us consider, in simpler way, the nowadays solution of Olbers' problem. Figure 4 shows schematically how we can approach this general ballistic missile problem. We note at first the circular parking orbit 1, around the

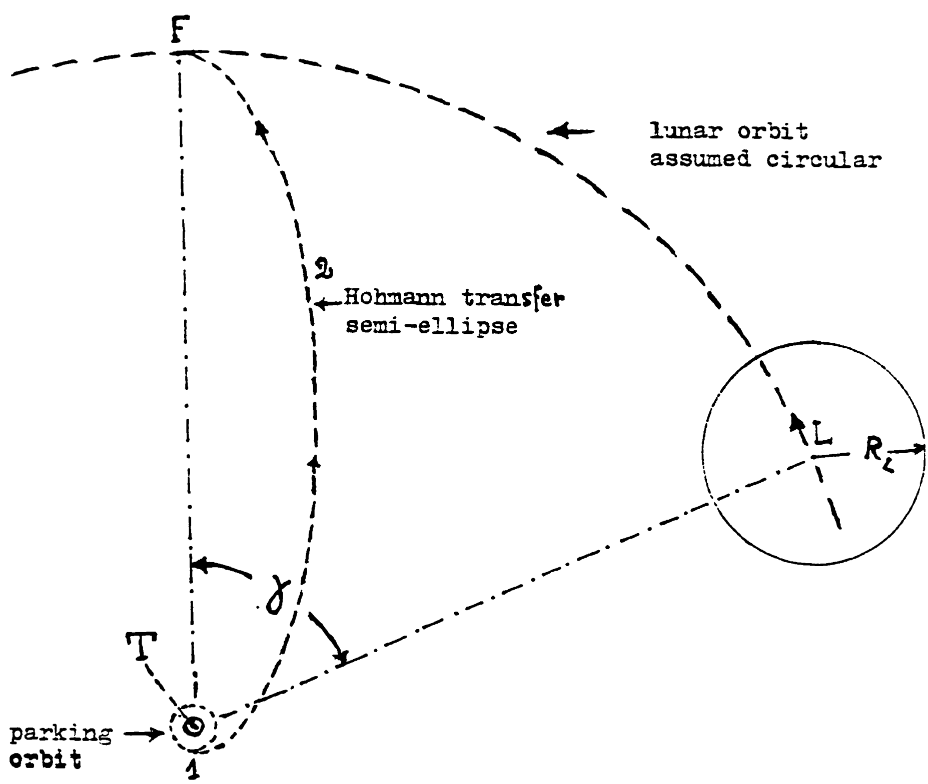

Figure 4. Geometric transfer to the lunar surface. The simplest method.

Earth, with radius equal to 12,756 km, whereas lunar orbit (considered also circular) has radius equal to TF = 384,400 km (average distance between Earth and Moon). The trajectory 2 is a semi-ellipse denoted as Hohmann elliptical transfer orbit, which is tangent to both circular orbits, i. e. the parking orbit and that of the Moon.

If the projectile is to encounter the Moon at the time it crosses the Moon's orbit, then Earth and Moon must have the correct position, with respect to point F, at departure. The angle between radius vectors at the departure TL and at arrival TF is called γ . This angle γ depends on the time-of-flight from point A and F for Hohmann transfer of the projectile. This time is easily calculated; at first we obtain the semi-major axis a_T for the Hohmann ellipse and the relevant eccentricity e_T.

We have:

$$a_T = \frac{12{,}756 + 384{,}400}{2} = 198{,}578 \text{ km}$$

$$e_T = 0{,}9355;$$

owing to the third Kepler's law

$$(384{,}400)^3 \frac{4\,\pi^2}{(28 \text{ days})^2} = G\,M_T = (198{,}578)^3 \frac{4\,\pi^2}{(z \text{ days})^2}$$

and solwing for z we obtain z = 10.4. The transfer time is then $\frac{z}{2}$ days = 5.2 days. Finally we get γ so:

$$\gamma = 5.2 \text{ days} \frac{360°}{28 \text{ days}} = 67°.$$

Indeed the transfer time should be smaller than the foregoing one since near the arrival there is the action of our natural satellite. In Figure 4 we can see the Lyttleton's circle, with radius R_L , which is the trace, on

projectile orbit plane of Moon's sphere of influence. Within this sphere of influence, suggested by Laplace, the force of attraction of the Moon is greater than that provided by the Earth.
Lyttleton radius R_L is simply (Ref.8)

$$R_L = \sqrt{\frac{M_L}{3\,M_T}}\;T\,F.$$

As final observation we calculate the projectile velocity v_i at starting point of Hohmann ellipse. With the application of the conservation of total mechanical energy theorem we get

$$v_i = \sqrt{\frac{G\,M_T}{a_T}\;\frac{1+e_T}{1-e_T}} \cong 7.8 \text{ km s}^{-1},$$

about 70% of the escape parabolic velocity from Earth. Olbers' value is obviously greater since the departure is from Earth's surface.

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