A DYNAMICAL PROBLEM APPLIED TO THE EARTH, SOLVED BY LEONARDO DA VINCI

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ABSTRACT. - A very interesting problem of dynamics, solved by Leonardo, is here examined. The solution is obviously of qualitative type but it is full of important suggestions and comments.The most important of them is the intuition of the I law of dynamics. A short part of the paper is devoted to the to-day solution of problem.

1. - Introduction to the problem

In the Cod. Atl. [153 V b], see Reference 1, we find this very interesting problem. Leonardo supposed the existence of a long tunnel crossing our planet, whose axis followed an Earth's diameter,between points A and B onto the terrestrial surface (Figure 3).Then he studied the motion of a body along this axis or, with Leonardo's words, "within this air canal".

At first istant the body shall fall down, on account of gravitation, towards the Earth's center.Leonardo asks: which kind of motion shall follow? This is the answer [Reference 1]:

"...certo quella tal gravità non si fermerebbe subito nel centro comune, anzi quello per l'impeto trapasserebbe insino vicino all'altra parte dell'elemento, e qui, non trovando ancora riposo, ripiglierebbe la medesima strada di prima, e ritornerebbe propinco al loco onde onde si divise, e oltre al primo moto fatto rifarebbe il secondo pel medesimo cammino con più brieve corso, e po' ritornerebbe, e così

"..certainly the gravitation action shall not cease at Earth's center.On the contrary the body, on account of its impetus, shall continue to move reaching the other extremity of the tunnel. In this position the body shall reverse its motion going back to a position near to the starting point. After this motion, another one shall follow, and so on with many others. The amplitudes however are always decreasing.

farebbe moltissime volte, a similitudine d'un peso applicato allo stremo di una corda, il quale tirato in parte e poi lasciato andare, molte volte tra qua e là si muove, sempre diminue i sua corsi, in modo che all'ultimo si ferma sotto il firmamento della corda che 'l sospende, sì che in questo modo farebbe tal peso, scorrendo molti anni tra su e giù per l'aria."

A similar case is that of a mass which is connected with a fixed suspension point by a cord. When the position of this mass is not on the vertical line passing through the sospension point, motion shall start up and down. It shall stop when, after many alternating motions with decreasing amplitudes, its position shall be on the vertical line under the sospension point. Our body, within the air canal, in similar way shall move, during many years, up and down..."

Leonardo's scientific interests were so manifold that it is impossible to deal with them here. Considering our present problem, he made original and penetrating observations in order to make the complete qualitative analysis with the correct solution. Obviously the quantitative problem solution is possible in the frame of newtonian mechanics only, and by means of the application of the differential equations theory.

Leonardo's unusually observant and intuitive mind noted the strong analogy between the motion under study and that of the simple pendulum. He was sure that the oscillations were damped. We can read in Cod. Arundel, fol.65 r [Reference 2]:

"La gravità cadente al centro del mondo non fermasi a esso centro immediate,ma co' lunghezza di tempo molte volte scorre tra su e giù intorno a tale centro. Quando il centro naturale del peso si unisce col centro del mondo,il corpo che lo include rimarrà sanza il moto. Sempre i moti del grave intorno al centro del mondo saranno fatti con tempi uguali, ancora che mai nella lor successione siano d'eguale lunghezza."

"The body falling towards Earth's center shall not stop, when it reaches the same center, but it shall oscillate up and down crossing this center. At the end of this natural motion, the body center coincides with the Earth's center. Before, during the motion, the time [period] of oscillation shall be constant, whereas the amplitudes shall be decreasing."

Two comments about these Leonardo's considerations. The first one deals with the experimented force on the body when it stays at the Earth's center. He says: this force is zero, following the pendulum analogy. The second one is simply: Leonardo has given us a splendid definition of a damped oscillatory motion. The constancy of the period is clearly expressed [Reference 3]. It is truly astounding as we ponder the Leonardo's words which so well illustrate Quattrocento's rupture with Middle Ages and its preparation for the era of modern science.

2. - <u>The Leonardo's "moto ventilante", the mechanical equilibrium condition and the impossibility of perpetual motion</u>

In this item we will study the Leonardo's concept of "moto ventilante" (in Quattrocento's italian language these two words mean "oscillatory motion") and also the concept of impetus, a development of Buridan's ideas. Leonardo claimed that, during the oscillatory motion, there is a transformation of "reflected motion" into "incident motion" and vice versa. This phenomenon is common to many types of oscillatory motion. In Reference 2 there is the description of many damped oscillatory motions. The simple pendulum, the balances, water oscillations between communicant basins and so on, are quoted as typical examples. The conceptual generalization, concerning these damped oscillatory motions, is very conspicuous.

From the point of view of the history of the science, it is important to point out that Leonardo considers the observable oscillatory motions always with irreversible decay.

Nowadays, following the standard scientific language, we say that, during the oscillation, there is a transformation of "potential energy" into "kinetic energy" and vice versa. This transformation is endowed with an irreversible loss of energy; some part of total energy transforms into heat.

The reader, at this point, will think that the author wishes to envisage a Leonardo's extension of Buridan's theory of impetus until to the modern energy-balance concept. It should be an incorrect interpretation of Leonardo's text. In fact there are many connections of Aristotle's physics with Leonardo's views. But it is undubitable that the talent for the observation was sharper than that of Aristotle.

In order to consider the question of the mechanical equilibrium and also that of the impossibility of perpetual motion, let us read this passage in Cod. Madrid I, fol. 145 r [Reference 4], which contains also a drawing of a wheel with attached many rotating little rods carrying (at the extremity) some masses. Leonardo says that, following experience:

"Qualunque peso sarà applicato alla ruota, il quale peso sia causa del moto d'essa ruota, sanza dubbio alcuno il centro di tal peso si fermerà sotto il centro del suo polo; e nessun istrumento che per umano ingegno fabbricar si possa che col suo polo si volti, potrà a tale effetto riparare. O speculatori del continuo moto, quanti vani disegni in simile ricerca avete creati! Accompagnatevi colli cercatori dell'oro!"

"...any weight, attached wheel,shall cause rotation around its axis. But sooner or later the wheel shall cease its rotation motion. At that time the center of the weight lies on vertical line passing through rotation axis. No mechanical device,made by wheels and rods, shall be able to change this phenomenon.
Oh partisans of the perpetual motion conception, how many dreams in vain following this way! Please join yourselves together with gold searchers!"

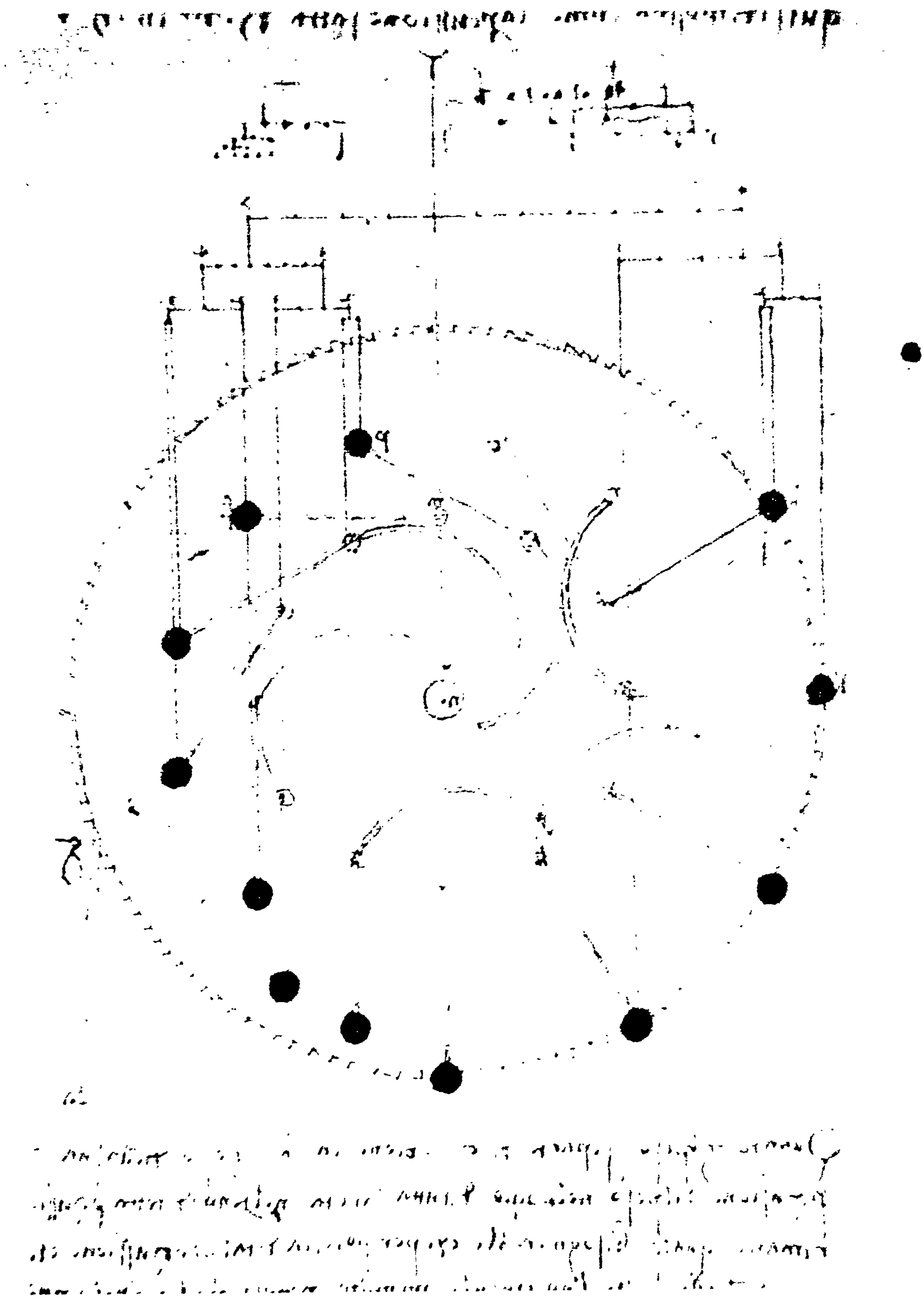

Figure 1. - Device with a wheel in order to demonstrate the impossibility of perpetual motion [Cod. Madrid I, fol. 145 r].

The Figure 1 clearly illustrates the Leonardo's thought (Reference 5).On the periphery of the wheel, at the right, there is a weight which is able to put in rotation the wheel. After some oscillations, owing to dissipative forces, the wheel shall stop in such a way that the weight position shall be the lowest one. This is the stable mechanical equilibrium condition.

The Physics nowadays states that it is impossible to obtain work without expense of energy; Leonardo reached, with his intuitive mind, this very important law. Let us go back to item 1, in the first passage "...certainly the gravitation action...". It is possible to see, at Earth's center, an anticipation of the first law of dynamics. In fact, as a confirmation, we can also read another passage in Cod. Atl. 109 V a [Reference 1]:

"Ogni moto seguiterà tanto la via del suo corso per retta linia, quanto durerà in esso la natura della violenza fatta dal suo motore."

"...every motion shall follow its rectilinear path, if the effect,made by body which impressed the motion, shall remain in it...".

In modern words we may say that, during the rectilinear motion of a body which is not subject to any force, the momentum, or the velocity, remains constant. Leonardo intuition follows certainly from Buridan's theory of impetus. This law was announced by Newton in the Principia (1686) so:

"Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by a force impressed upon it."

But it is very interesting to note that the Aristotle's dynamics remains valid during the complete Renaissance. In Figure 2, Aristotle's paradigm and Newton's paradigm are illustrated.

DYNAMICS

ARISTOTLE'S PARADIGM	NEWTON'S PARADIGM
A body moves on account of «species motrix» given to it; if «species motrix» is zero the body remains at rest. Mathematically:	A body moves on account of force $\bar{F}$ following this law:
$\bar{F} = \mu \bar{v}$	$\bar{F} = \frac{d}{dt} m \bar{v} = m \frac{d\bar{v}}{dt}$,
$\bar{F}$ = force (vector)	being m constant and
$\bar{v}$ = velocity (vector)	equall to body mass. Then
μ = resistence-to-motion factor	$\bar{F} = m \frac{d\bar{v}}{dt}$ with
$\bar{F} \neq o$, $\bar{v} \neq o$ $\forall t$	$\bar{F} \neq o$, $\bar{v} \neq o$ $\forall t$
If $\bar{F} = 0$ at $t \geq t_o$, then $\bar{v} = o$.	If $\bar{F} = o$ at $t \geq t_o$, then $\bar{v} = \bar{c}$ = constant vector

Figure 2. Schematic graph in which the comparison between Aristotle's paradigm (sublunar bodies dynamics) and Newton's paradigm (classical mechanics) is illustrated.

Studying this comparison we see that the resistance-to-motion factor μ plays its role in the law:

$$\bar{F} = \mu \, \bar{v} \qquad \begin{array}{l} \bar{F} = \text{force (vector)} \\ \bar{v} = \text{velocity (vector)}, \end{array} \tag{1}$$

whereas the corresponding law in newtonian paradigm is

$$\bar{F} = m \frac{d\bar{v}}{dt}, \qquad m = \text{mass}. \tag{2}$$

It is important to note that Aristotle's dynamics was, at first, valid for sublunar motions only. The great Kepler (1571 - 1631) extended this validity to the whole Solar System in order to found his vortical field of forces theory. By means of this theory, Kepler explained the planetary motion.

However Leonardo, except for the first law of dynamics, substantially followed the Aristotle's paradigm.

3. - The newtonian solution of the problem

Let us assume the Earth as spherical body of constant density ρ (Figure 3). The coordinate x (t) fixes the position (from Earth's center) of the body during the motion. Points A and B are the two extremities of the tunnel. Motion differential equation is the following:

$$\ddot{x} + G \frac{4}{3} \pi \frac{x^3}{x^2} \rho = 0 \tag{3}$$

in which G = gravitation constant, $\ddot{x} = \frac{d^2}{dt^2} x$, t = time.

Putting $\omega = \frac{2\pi}{T} = \sqrt{\frac{4}{3} \pi G \rho}$, Eq.(3) becomes:

$$x + \omega^2 \, x = 0, \tag{4}$$

i.e. the equation of harmonic oscillator.

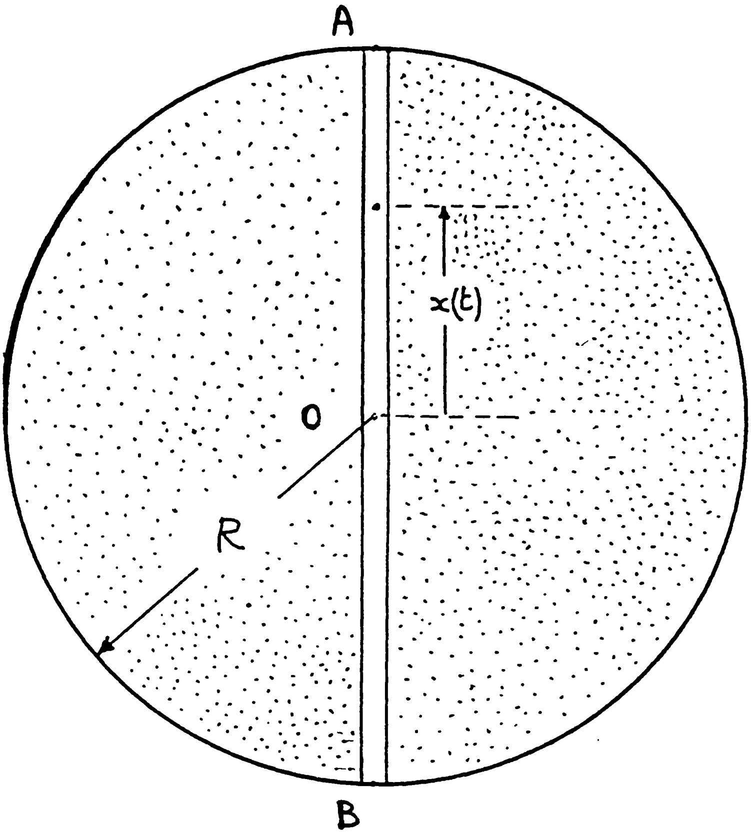

Figure 3. - Geometry for Equation (3).

Since at $t = o$, $x(o) = R$ and $\dot{x}(o) = o$ (with A starting point), we obtain

$$x = R \cos \sqrt{\frac{4}{3} \rho \pi G}\ t. \qquad (5)$$

With $G = 6.672 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$ and $\rho = 5.5 \times 10^3\ kg\ m^{-3}$ (average density of the Earth), we have T = 84.5 minutes.

Writing down Eq. (4) we have dropped the effects of Earth's own rotation. These effects are null when the choice of the tunnel axis direction is North Pole- South Pole; they are small in any other direction, owing to smallness of the ratio

$$\frac{\text{oscillation period T}}{\text{Earth own rotation period}} = \frac{84.5 \text{ min.}}{1440 \text{ min.}} = 0.059.$$

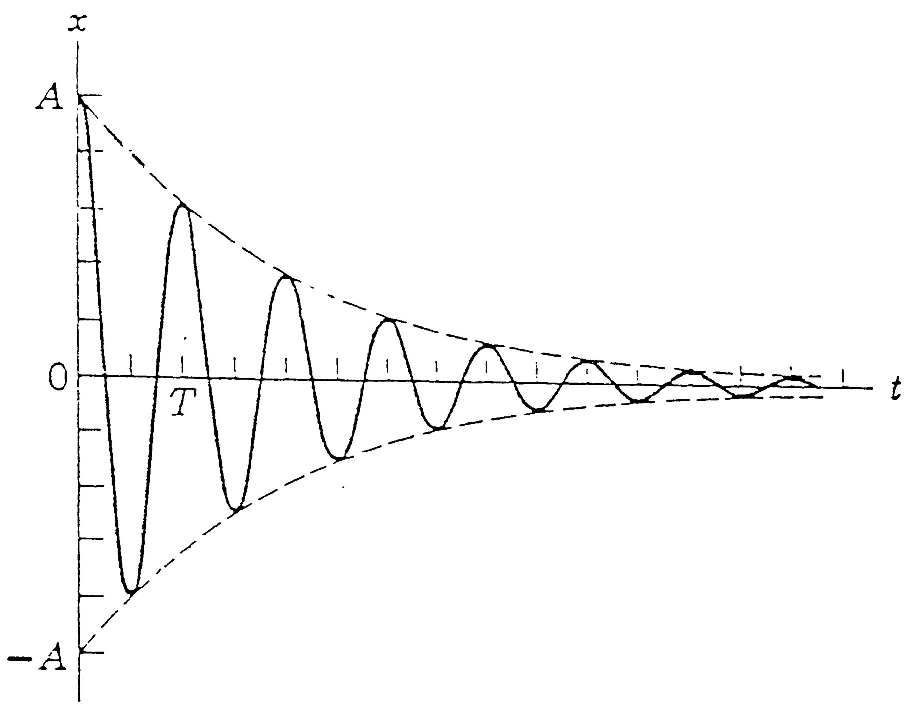

Figure 4. - A cosinusoidal damped function x (t).

In Eq. (4) we have also dropped the dissipative effects. In fact free oscillations shall be damped. The modification of Eq. (4) and its relevant solution give a function x = x (t) which is plotted in Figure 4. A third observation deals with the fact that ϱ is not constant in the real case of the Earth, but it is a monotonic decreasing function ϱ (x) (in the range 0 $\longmapsto$ R). Also in this case the motion is oscillatory and periodic (Reference 6).

Going back to Eq. (4), it is interesting to note that the solution is always a sinusoidal function, i.e. the sinusoidal form is not dependent on the amplitude. The differential equation that governs the motion of the simple pendulum is different with respect to Eq. (4), and we obtain sinusoidal solutions for small amplitudes only.

4. - Concluding remarks

Leonardo will apparently remain inexhaustible. Just as many of his unfinished works of art, unpublished notes, so he will continue to be the source of new discoveries that were concealed in his fragmentary records [References 7 and 8].

In this paper was considered a problem of dynamics which was solved, from the point of view of qualitative analysis, in clear manner. Other topics were touched, the most important of which was certainly the intuition of the first law of dynamics.

In this sense we can consider Leonardo a great scientist, approaching to the Galileo's level.

R E F E R E N C E S

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