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CONTRIBUTIONS TO THE EARTH'S OBLIQUITY RATE, PRECESSION, AND NUTATION

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ABSTRACT

The precession and nutation of the Earth's equator arise from solar, lunar, and planetary torques on the oblate Earth. The mean lunar orbit plane is nearly coincident with the ecliptic plane. A small tilt out of the ecliptic is caused by planetary perturbations and the Earth's gravitational harmonic J_2. These planetary perturbations on the lunar orbit result in torques on the oblate Earth which contribute to precession, obliquity rate, and nutation while the J_2 perturbations contribute to precession and nutation. Small additional contributions to the secular rates arise from tidal effects and planetary torques on the Earth's bulge. The total correction to the obliquity rate is $-0.024''$/century, it is an observable motion in space (the much larger conventional obliquity rate is wholly from the motion of the ecliptic, not the equator), and it is not present in the IAU-adopted expressions for the orientation of the Earth's equator. The J_2 effects have generally been allowed for in past nutation theories and some precession theories. For the planetary effect, the contributions to the 18.6 yr nutation are -0.03 mas (milliarcseconds) for the in-phase $\Delta\psi$ plus out-of-phase contributions of 0.14 mas in $\Delta\psi$ and -0.03 mas in $\Delta\epsilon$. The latter terms demonstrate that out-of-phase contributions can arise by means other than dissipation. The sum of the contributions to the precession rate is considered and the inferred value of the moment of inertia combination $(C-A)/C$, which is used to scale the coefficients in the nutation series, is evaluated. Using an updated value for the precession rate, the rigid body $(C-A)/C=0.003\,273\,763\,4$ which, in combination with a satellite-derived J_2, gives a normalized polar moment of inertia $C/MR^2=0.330\,700\,7$. The planetary contributions to the precession and obliquity rates are not constant for long times causing accelerations in both quantities. Acceleration in precession also arises from tides and changing J_2. Contributions from the improved theory, masses, ecliptic motion, and measured values of the precession rate and obliquity are combined to give expressions (polynomials in time) for precession, obliquity, and Greenwich Mean Sidereal Time.

1. INTRODUCTION

Torques on the oblate Earth due to the gravitational attraction of the Sun and Moon cause the Earth's equator to precess and nutate. The precession is retrograde and its rate is $50''$/yr, roughly 1/3 of it due to the Sun and 2/3 from the Moon. The rate depends on the lunar and solar masses and distances, the orbital eccentricities and inclinations, and the obliquity angle between the Earth's equator and ecliptic planes.

Recent decades have seen impressive advances in the accuracies of techniques measuring positions of artificial satellites, the Moon, and radio sources. Accurate theories for the motion of the Earth's equator in space are needed. This paper examines several theoretical contributions to precession, obliquity change, and nutation.

The orbit of the Earth–Moon system about the Sun defines the ecliptic plane. The lunar orbit is inclined 5° to the ecliptic plane and the strong solar torques drive the precession of the lunar orbit plane along the ecliptic with an 18.6 yr period. But several influences cause a slight tilt of the mean plane of orbital precession with respect to the ecliptic. The Earth's oblateness contributes a small torque which attempts to precess the lunar orbit along the equator. The net result of these two torques is a lunar orbit precession along a plane tilted $8''$ with respect to the ecliptic and this plane intersects the ecliptic at the dynamical equinox, the intersection of the ecliptic and equator planes. This small influence of the Earth's oblateness on the lunar orbit in turn causes a small change in the precession of the Earth's equator.

The orbit planes of the planets have small inclinations with respect to the ecliptic plane. As a consequence of the planetary attractions, the ecliptic plane moves. The Moon's mean plane of orbital precession follows the moving ecliptic closely, but not perfectly. This motion causes a $1.4''$ tilt of the plane of orbital precession to the ecliptic. There are also direct planetary torques on the lunar orbit which contribute a smaller displacement. These two influences on the lunar orbit result in torques on the oblate Earth which modify its orientation. In addition, the planets directly torque the Earth. The torques from these three planetary influences are not aligned with respect to the dynamical equinox. Consequently, they contribute to both the precession of the equator and the obliquity rate. While the precession rate must be a measured quantity, the obliquity rate is not a free parameter of the dynamics. These planetary influences are not included in the IAU-adopted theory of precession and obliquity change (Lieske *et al.* 1977). Neither have all of the consequences of the planetary tilts on the lunar orbit been included in recent nutation theories.

The above sources of precession and obliquity rate also cause accelerations. Acceleration corrections also arise from tidal effects and the Earth's changing J_2.

The above outlined corrections to the motion of the

Earth's equator are developed in the following sections. To these corrections are added precession corrections developed by Kinoshita & Souchay (1990) due to the Earth's J_4 and second-order corrections due to nutation. From the revised theory are developed new polynomial expressions for the motion of the Earth's equator and revised values of the Earth's fractional moment of inertia, $(C-A)/C$, and the normalized polar moment, C/MR^2.

2. FUNDAMENTALS

This section sets up the fundamental equations for calculating the motion of the Earth's equator (or pole) in space. As the computations of the subsequent two sections are limited to small effects, it is reasonable to introduce simplifications. The Earth will be treated as a rigid body without oceans and without the influence of a liquid core. Small differences in the directions of the axes of angular momentum, instantaneous spin, and figure (equivalent to celestial ephemeris pole for a rigid body) are ignored. The equations will be written for the angular momentum axis, but strictly speaking it is the motion of the figure axis of the rigid Earth which is desired. Also ignored are second-order effects due to the change of the Earth's orientation, e.g., precession and nutation modifying the computation of precession and nutation.

The oblate, rigid Earth is torqued by an external body. The attracting body has a geocentric distance r and a product of the gravitational constant and mass Gm. The Earth has moments of inertia A, A, C with $A<C$, and mass M. The z axis is aligned with the Earth's principal axis corresponding to the maximum moment C, and the x axis points toward the intersection of the ecliptic and equator planes, the dynamical equinox. The potential energy of the external body in the gravity field of the oblate Earth is

$$V=Gm[M/r-(C-A)(3\sin^2\delta-1)/2r^3], \tag{1}$$

where the declination of the attracting body is δ and the right ascension is α. Equivalently, the vector r has components (x,y,z). The torque $\underline{T}$ on the Earth is

$$\underline{T}=-\underline{r}\times\underline{\nabla}V, \tag{2}$$

$$\underline{T}=\frac{3Gm(C-A)\sin\delta\cos\delta}{r^3}\begin{pmatrix}\sin\alpha\\ -\cos\alpha\\ 0\end{pmatrix},$$

$$\underline{T}=\frac{3Gm(C-A)}{r^5}\begin{pmatrix}yz\\ -xz\\ 0\end{pmatrix}. \tag{3}$$

The rate of change of the vector angular momentum $\underline{L}$ is governed by $d\underline{L}/dt=\underline{T}$. Given the orbit of the external attracting body, the resulting precession and nutation of the Earth can be calculated.

The analytical theories for the Sun, planets, and Moon are referred to the ecliptic plane. Consequently, the conversion from geocentric ecliptic coordinates (X,Y,Z) to equatorial coordinates (x,y,z) requires a rotation about the x axis by the obliquity ϵ

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}X\\ Y\cos\epsilon-Z\sin\epsilon\\ Z\cos\epsilon+Y\sin\epsilon\end{pmatrix}. \tag{4}$$

In the torque vector the products of equatorial coordinate components become

$$\begin{pmatrix}yz\\ -xz\\ 0\end{pmatrix}=\begin{pmatrix}(1/2)(Y^2-Z^2)\sin 2\epsilon+YZ\cos 2\epsilon\\ -XZ\cos\epsilon-XY\sin\epsilon\\ 0\end{pmatrix}. \tag{5}$$

The ecliptic coordinates (X,Y,Z) of the attracting body can be written in terms of the geocentric distance r and the geocentric ecliptic longitude λ and latitude β

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}=r\begin{pmatrix}\cos\beta\cos\lambda\\ \cos\beta\sin\lambda\\ \sin\beta\end{pmatrix}. \tag{6}$$

Because the Earth's path about the Sun is well approximated by an elliptical orbit in the ecliptic plane, the solar torque may be computed with good accuracy with little effort. Averaged over an integral number of revolutions the average x component of torque is

$$T_x=3Gm(C-A)\sin\epsilon\cos\epsilon/2a^3(1-e^2)^{3/2},$$

where a is the semimajor axis, e the orbital eccentricity, and ϵ the obliquity. The x component of the torque gives rise to a retrograde precession along the ecliptic with rate $d\psi/dt=T_x/C\omega_z\sin\epsilon$, where ω_z is the major component of the Earth's angular velocity and $C\omega_z$ approximates the total angular momentum of the Earth's spin.

$$d\psi/dt=3Gm(C-A)\cos\epsilon/2a^3(1-e^2)^{3/2}C\omega_z. \tag{7}$$

G/a^3 may be replaced with the square of the mean motion divided by the sum of the masses (Sun+Earth+Moon) using Kepler's third law. The analogous precession from the Moon includes an inclination factor of $1-1.5\sin^2 i$. The other two torque components have zero average, but of course the first two components have time variations which contribute periodic nutation terms.

The elliptical approximation above works well for the solar-induced precession of the Earth's equator along the ecliptic, but it is a coarser approximation for the lunar effect because the lunar orbit is strongly perturbed by the Sun. These difficulties in the major precession and nutation effects have been dealt with by Kinoshita & Souchay (1990). Their computation for the solar precession is only larger by 2×10^{-6} so Eq. (7) is a very good approximation for the Sun. The lunar orbit is highly perturbed and the equivalent equation for the lunar-induced precession, including the inclination factor, is less precise. The computation of many small corrections in this paper can use the elliptical approximation.

3. EFFECTS DUE TO THE TILTED LUNAR MEAN PLANE

The lunar orbit precesses along a plane which is tilted slightly with respect to the ecliptic plane. The Earth's J_2 causes an 8″ tilt and planetary effects cause a 1.5″ tilt. As a consequence of these small sizes, expansions will be used.

The lunar latitude arises from the 5.15° inclination of the orbit to the ecliptic i, and smaller perturbations $\Delta\beta$ so that

$$\sin \beta \approx \sin i \sin F + \Delta\beta, \tag{8}$$

where F is the mean argument of latitude. Similarly the lunar longitude λ may be written in terms of its mean longitude L, mean anomaly l, eccentricity e, and smaller contributions ΔL

$$\lambda = L + 2e \sin l + \Delta L. \tag{9}$$

The perturbing terms most important for precession and nutation are selected from Chapront-Touzé & Chapront (1988, 1991). For J_2 perturbations

$$\begin{aligned}\Delta\beta &= -8.045'' \sin L + 0.326'' \sin(L-2F),\\ \Delta L &= 7.063'' \sin \Omega + 0.361'' \sin(L+F),\end{aligned} \tag{10}$$

where Ω is the lunar node ($\Omega = L - F$). The important planetary-induced terms are

$$\begin{aligned}\Delta\beta &= 1.510'' \sin(L+96.68°),\\ \Delta L &= -0.289'' \sin(\Omega+95.13°)\\ &\quad -0.062'' \sin(L+F+95.13°).\end{aligned} \tag{11}$$

The J_2 and planetary effects also cause radial perturbations, but compared with the longitude and latitude perturbations they are relatively ineffective in modifying precession and nutation.

For the purposes of expansions, the above perturbations in ecliptic longitude and latitude will be represented symbolically as

$$\begin{aligned}\Delta\beta &= B \sin(L+\phi) + B' \sin(L-2F),\\ \Delta L &= E \sin(\Omega+\varphi) + E' \sin(L+F+\varphi).\end{aligned} \tag{12}$$

The ecliptic plane is rotating about a line which is displaced from the dynamical equinox by $\varphi - 90° = 5.13°$. The phase ϕ is different from φ because the first term in latitude combines both the direct effect of the planets with the indirect effect of the ecliptic motion. After introducing these perturbations into the differential equations of the previous section and carrying out the expansions through first degree in e and $\sin i$ there are contributions to both rate and periodic terms in ψ and ϵ. The rate terms are

$$\begin{aligned}\frac{d\psi}{dt} &= \frac{3Gm(C-A)\cos 2\epsilon}{4a^3 C\omega_z \sin\epsilon}\\ &\quad\times[2B\cos\phi + (E'-E)\sin i \cos\varphi],\\ \frac{d\epsilon}{dt} &= \frac{-3Gm(C-A)\cos\epsilon}{4a^3 C\omega_z}\\ &\quad\times[2B\sin\phi + (E'-E)\sin i \sin\varphi].\end{aligned} \tag{13}$$

A contribution to the obliquity rate requires phase shifts. The planetary effects contribute a -0.254 mas/yr (mas = milliarcsecond) correction to the obliquity rate while J_2 perturbations contribute nothing. To the precession, J_2 perturbations contribute -2.630 mas/yr while planetary effects contribute -0.056 mas/yr for a total of -2.686 mas/yr.

The largest of the nutation corrections has the 18.6 yr period of the lunar node (rate $\dot{\Omega}$). While the rigid-body nutation caused by the main lunar theory only contains in-phase terms (sines of the arguments for $\Delta\psi$ and cosines for $\Delta\epsilon$), the phase shifts with the planetary effects also induce out-of-phase terms (cosines for $\Delta\psi$ and sines for $\Delta\epsilon$)

$$\begin{aligned}\sin\epsilon\Delta\psi &= \frac{3Gm(C-A)\sin\epsilon\cos\epsilon}{4a^3 C\omega_z\dot{\Omega}}\{[-7\sin i\, B\cos\phi\\ &\quad +2E'\cos\varphi + 6\sin i\, B']\sin\Omega\\ &\quad +[5\sin i\, B\sin\phi - 2E'\sin\varphi]\cos\Omega\}\\ \Delta\epsilon &= \frac{3Gm(C-A)\sin\epsilon}{4a^3 C\omega_z\dot{\Omega}}\{[\sin i\, B\cos\phi\\ &\quad -2E'\cos\varphi]\cos\Omega + [\sin i\, B\sin\phi\\ &\quad -2E'\sin\varphi]\sin\Omega\}.\end{aligned} \tag{14}$$

Nutation terms at half of the nodal period must also be considered. The contributions to the nutation terms with argument twice the lunar node are

$$\begin{aligned}\sin\epsilon\Delta\psi &= \frac{3Gm(C-A)\cos 2\epsilon}{8a^3 C\omega_z\dot{\Omega}}[(\sin i\, E\cos\varphi\\ &\quad -2B')\sin 2\Omega + (\sin i\, E\sin\varphi)\cos 2\Omega]\\ \Delta\epsilon &= \frac{3Gm(C-A)\cos\epsilon}{8a^3 C\omega_z\dot{\Omega}}[-(\sin i\, E\cos\varphi\\ &\quad -2B')\cos 2\Omega + (\sin i\, E\sin\varphi)\sin 2\Omega].\end{aligned} \tag{15}$$

Finally, there are small corrections to terms with argument $2L$ (rate $2\dot{L}$)

$$\begin{aligned}\sin\epsilon\Delta\psi &= \frac{3Gm(C-A)\cos 2\epsilon}{4a^3 C\omega_z\dot{L}}[-B\cos\phi\sin 2L\\ &\quad -B\sin\phi\cos 2L],\\ \Delta\epsilon &= \frac{3Gm(C-A)\cos\epsilon}{4a^3 C\omega_z\dot{L}}[B\cos\phi\cos 2L\\ &\quad -B\sin\phi\sin 2L]\end{aligned} \tag{16}$$

Using the numerical values of the coefficients and phases for the J_2 and planetary effects from Eqs. (10) and (11), the above contributions to the nutation have been calculated. They are presented in Table 1 (units mas). The major contribution is to the 18.6 yr $\Delta\psi$ term with a lesser contribution to the 18.6 yr $\Delta\epsilon$ term. Both of these contributions increase the magnitude of the conventional 18.6 yr terms. The contributions to the 9.3 yr nutation are small. The two contributions in Eq. (15) from the J_2 effect nearly cancel and the values in the Table 1 are effectively zero. The higher frequency of the $2L(=2F+2\Omega)$ term prevents those half-month nutation corrections from being large.

Out-of-phase terms in the nutation theory will arise from dissipative processes in the oceans (Wahr & Sasao 1981; Zhu

TABLE 1. Nutation terms due to J_2 and planetary tilt effects of lunar orbit. Lunar mean longitude is $L=\Omega+F$.

	$\Delta\psi$		$\Delta\epsilon$	
Argument	sin mas	cos mas	sin mas	cos mas
J_2 Tilt				
Ω	-1.4782	0.0000	0.0000	0.1557
2Ω	0.0049	0.0000	0.0000	-0.0026
$2L$	0.0151	0.0000	0.0000	-0.0081
Planetary				
Ω	-0.0301	0.1366	-0.0277	0.0029
2Ω	-0.0005	0.0060	0.0032	0.0003
$2L$	0.0003	-0.0028	-0.0015	-0.0002

et al. 1990) and interior of the Earth (Wahr & Bergen 1986; Dehant 1988, 1990). The out-of-phase terms in Table 1 arise from the phase shifts in the planetary effects which in turn arise because the orbit planes of the planets other than Earth have no special alignment with the ecliptic plane or dynamical equinox. There are still smaller corrections with arguments of $2L+\Omega$, $2L-\Omega$, $2\varpi+\Omega$, $2\varpi-\Omega$, and 3Ω which are not given.

Woolard (1953) was aware that out-of-phase terms in nutation theory could arise from planetary perturbations on the lunar orbit. The out-of-phase 18.6 yr term for nutation in longitude occurs in his Table 24 (it is marked with a ? and a footnote, but matches the value in Table 1 of this paper), but the obliquity term was too small for his cutoff limit. Woolard also calculated the obliquity rate contribution, showing -0.256 mas/yr in his Table 24 at the year 1900. In the text (p. 127) he also comments that the planetary-induced lunar terms contribute to precession and to the acceleration of obliquity. Kinoshita (1975, 1977) considered the obliquity rate contribution to be due to an error in Woolard's equations of motion. This assertion will be discussed further in Sec. 6. Kinoshita's M_1 correction to precession is -2.68 mas/yr and it appears to correspond to the sum of the J_2 and in-phase planetary effects computed in this paper. In Kinoshita & Souchay (1990) a more elaborate "second-order" correction to precession replaces the earlier M_1 correction. It contains a -2.60 mas/yr correction to precession due to the J_2 effects, but the -0.056 mas/yr planetary effect is missing. Presumably, their nutations contain the corresponding contribution from J_2, but not from the planetary tilt.

4. RATES DUE TO DIRECT TORQUES OF PLANETS ON EARTH

The torques from the Sun and Moon dominate the precession of the Earth's equator. There are small additional torques from the planets which contribute to the precession. The inclination of the planetary orbits to the ecliptic will also cause a small obliquity rate. A calculation of the precession contribution was given by Kinoshita & Souchay (1990), but not the obliquity rate. A brief derivation of both rates is given below. Note that the effect of these direct planetary torques on the Earth is distinct from the tilt effect due to the direct planetary perturbations on the lunar orbit.

In order to compute the geocentric coordinates of the attracting planet, it is necessary to difference the heliocentric coordinates of the planet and the Earth. Primes will be used for the planet's variables, no primes for the Earth. The effects are small; so to keep the derivation from becoming unwieldy, two approximations will be introduced. The heliocentric orbits will be taken as circles and the planetary inclinations will only be carried to first degree ($\cos i'\approx 1$). Then the planet's geocentric ecliptic coordinates are

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a'\cos(u'+\Omega')-a\cos(u+\Omega') \\ a'\sin(u'+\Omega')-a\sin(u+\Omega') \\ a'\sin i'\sin u' \end{pmatrix}, \tag{17}$$

where a and a' denote the semimajor axes, i' the inclination to the ecliptic, Ω' the node on the ecliptic, and u and u' the arguments of latitude measured from that same node for both the Earth and attracting planet. The geocentric distance r is given by

$$r^2=a^2+a'^2-2aa'\cos(u-u'). \tag{18}$$

The ecliptic coordinates are rotated into equatorial coordinates following Eq. (4) and the products of coordinates of Eq. (5) are formed for substitution into Eq. (3) for the torque. There result expressions involving products of sines and cosines of u and u'. In order to isolate the secular rates from the periodic terms Gauss' method of averaging over u and u' is used. Denoting the average with $\langle\rangle$, an example is $\langle(YZ/r^5)\rangle=\int\int(YZ/r^5)du\,du'/4\pi^2$ with both integrals evaluated from 0 to 2π. To winnow out terms which will disappear during the double integration a mathematical device is useful. The transformations $(\sin u, \sin u')\rightarrow(-\sin u, -\sin u')$, $(\cos u, \cos u')\rightarrow(-\cos u, -\cos u')$, and both taken together leave unchanged the distance r which appears in the denominators of the integrals. Any component of the numerators Y^2-Z^2, YZ, XZ, or XY which reverses sign under any of the three transformations will average to zero. Also, since only $u-u'$ appears in the denominator, changing variables of integration to $u-u'$ and $u+u'$ makes it clear that additional components average to zero. Finally one gets

$$\begin{aligned} &\langle(Y^2-Z^2)/r^5\rangle=(1/2)\langle 1/r^3\rangle, \\ &\langle YZ/r^5\rangle=(a'/2)\sin i'\cos\Omega'(a'\langle 1/r^5\rangle \\ &\qquad -a\langle\cos(u-u')/r^5\rangle), \\ &\langle XZ/r^5\rangle=-(a'/2)\sin i'\sin\Omega'(a'\langle 1/r^5\rangle \\ &\qquad -a\langle\cos(u-u')/r^5\rangle), \\ &\langle XY/r^5\rangle=0. \end{aligned} \tag{19}$$

The three different averages on the right-hand sides above are only functions of $u-u'$ and they may be evaluated in terms of complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$, respectively,

TABLE 2. Precession and obliquity rates from direct planetary torques on the Earth's bulge.

Planet	ψ rate mas/yr	ϵ rate mas/yr
Mercury	0.003 651	−0.000 090
Venus	0.187 273	−0.017 372
Mars	0.005 393	0.000 255
Jupiter	0.116 665	0.002 782
Saturn	0.005 177	0.000 217
Uranus	0.000 100	0.000 001
Neptune	0.000 029	0.000 001
Total	0.318 287	−0.014 207

$$\langle 1/r^3\rangle = 2E(k)/[\pi(a+a')^3(1-k^2)],$$

$$\langle 1/r^5\rangle = 2[-K(k)+2E(k)(2-k^2)/(1-k^2)]/[3\pi(a+a')^5(1-k^2)], \tag{20}$$

$$\langle \cos(u-u')/r^5\rangle = 2[-K(k)(2-k^2)+2E(k)(1-k^2+k^4)/(1-k^2)]/[3\pi(a+a')^5k^2(1-k^2)].$$

The modulus k is the geometric mean of the two semimajor axes divided by the arithmetic mean or

$$k^2=4aa'/(a+a')^2. \tag{21}$$

The rates induced by the direct planetary torques are

$$d\psi/dt=[Gm'(C-A)/\pi(a+a')(a-a')^2C\omega_Z]\times[3\cos\epsilon\, E(k)-G\cos 2\epsilon \sin i' \cos\Omega'/\sin\epsilon],$$

$$d\epsilon/dt=[Gm'(C-A)/\pi(a+a')(a-a')^2C\omega_Z]\times[-G\cos\epsilon\sin i'\sin\Omega'], \tag{22}$$

$$G=(a^2+7a'^2)E(k)/2(a^2-a'^2)-(a-a')K(k)/2(a+a'),$$

where m' is the attracting planet's mass. In the precession rate the larger term involves $\cos\epsilon E(k)$. It contributes to precession as though the attracting body were in the ecliptic plane and it has the same $\cos\epsilon$ dependence as the dominant precession due to the Sun and Moon. For both rates the combinations $\sin i' \sin\Omega'$ and $\sin i' \cos\Omega'$, which are two of the coordinates of the planet's orbit pole direction, allow for small contributions due to the tilt of the planet's orbit plane with respect to the ecliptic.

The numerical results for the precession and obliquity rate contributions from the direct planetary torques on the Earth are given in Table 2. The 0.3183 mas/yr precession rate results from 0.3269 mas/yr due to the $\cos\epsilon E(k)$ term and −0.0086 mas/yr from the planetary inclinations. The comparison of precession rate with Kinoshita & Souchay's (1990) computations for Venus through Saturn shows differences of 3% for Venus and 1% for Jupiter. In Table 2 the largest values of the modulus k occur for Venus and Mars, 0.987 and 0.978, respectively. The obliquity rate contribution of −0.014 mas/yr combines with the larger contribution of −0.254 mas/yr from planetary effects through the lunar orbit (Sec. 3) to give −0.268 mas/yr. Tidal torques contribute an additional 0.024 mas/yr to obliquity rate; that derivation is interconnected with nonlinear contributions and will be deferred (Sec. 7) until after the summarizing of the rates. The total obliquity rate with respect to space is −0.244 mas/yr. This correction to the obliquity rate is not included in the expressions accompanying the IAU-adopted precession theory.

5. TOTAL PRECESSION AND OBLIQUITY RATES

This section summarizes the various contributions to precession and obliquity rates, gives the total values, and discusses the implications. The precession and nutation of the Earth's pole in space depend on the dynamical flattening $(C-A)/C$. Since the precession rate was measured with a smaller relative error than the nutation coefficients, the rate of precession was chosen as a primary 1976 IAU constant and recent nutation series have been computed from the derived value of $(C-A)/C$ (or proportional quantities for the Sun and Moon called k_S and k_M).

Knowledge of the precession rate and obliquity has improved since the adoption of the 1976 IAU constants. The value of $(C-A)/C$ appropriate to the IAU constants, but with the theoretical modifications of this paper and updated ecliptic motion, is 0.003 273 978 26. The featured computations will use improved values of the precession rate, obliquity, masses, mean motions, and ecliptic motion. A −3 mas/yr correction to the IAU-adopted value of the precession constant has been indicated by several lines of evidence: lunar laser ranging (Williams *et al.* 1991, 1993), very long baseline interferometry (Herring *et al.* 1991; Herring 1991; McCarthy & Luzum 1991; Steppe *et al.* 1993), the two combined (Charlot *et al.* 1991), and systematic proper motions in star catalogues (Miyamoto & Soma 1993). Several recent fits have given corrections near −3.2 to −3.3 mas/yr and a general precession rate of 5028.77″/century has been chosen for this paper. The change from the IAU general precession rate is −3.266 mas/yr and the change in the luni-solar precession rate is −3.219 mas/yr (the two do not match because the ecliptic motion is different from the IAU paper). For the obliquity at J2000 (ϵ_0), the value of 84 381.409″=23° 26′21.409″ is based on analyses of lunar and planetary observations. This obliquity and the mass ratios Earth/Moon =81.300 59 and Sun/(Earth+Moon)=328 900.560 are from the recent ephemeris DE 245 (Newhall *et al.* 1993). See Standish (1982) for the technique of extracting the obliquity from an ephemeris. The corresponding $(C-A)/C$ is 0.003 273 763 40, k_S=3475.19739″/century, and k_M=7546.737 00″/century (or 7567.305 75″/century with the $1/F_2^3$ factor).

The various contributors to precession and obliquity rates are summarized in Table 3. Taken from Kinoshita & Souchay (1990) are the first-order equations for the computation of the lunar- and solar-induced precession (the values were computed from the equations), the value of the second-order

TABLE 3. Contributions to precession and obliquity rates. $(C-A)/C = 0.003\,273\,763\,4$ and obliquity 23°26′21.409″ at J2000.

Contribution	Prec. rate arcsec/yr	ϵ rate arcsec/yr
Sun first order	15.948 870	···
Moon first order	34.457 698	···
Second order	−0.000 468	···
J_4	0.000 026	···
Tilt effects	−0.002 686	−0.000 254
Direct planetary	0.000 318	−0.000 014
Tidal	···	0.000 024
Geodesic precession	−0.019 194	···
Total space motion	50.384 565	−0.000 244
Ecliptic motion	−0.096 865	−0.468 096
General motion	50.287 700	−0.468 340

lunar plus solar effects (excluding the J_2 orbit effects), and the small value for the precession induced by the Earth's J_4 gravitational harmonic. The contribution to precession and obliquity rate due to the lunar orbit tilt comes from Sec. 3 of this paper and the planetary contribution due to direct torques on the Earth's oblateness comes from Table 2 in Sec. 4. The relativistic precession, variously called the geodesic, geodetic, and de Sitter–Folker precession, is computed from the following equation based on that in Barker & O'Connell (1970, 1975):

$$P_g = 3(na/c)^2 n/2(1-e^2), \tag{23}$$

where c is the speed of light and n and a are the mean motion and semimajor axis of the orbit of the Earth–Moon system about the Sun. The convention of measuring the precession constant in a left-handed sense (retrograde) results in a negative sign for the geodesic precession in Table 2. The tidal influence on obliquity rate is taken from Sec. 7. The sum of all of the above contributions gives the precession and obliquity rate with respect to space for the stated value of $(C-A)/C$. Conventionally the precession along the fixed ecliptic with respect to space is referred to as the "luni-solar precession" (which includes contributions from the planets as well). Clearly, it would be inappropriate to refer to the companion −0.244 mas/yr obliquity rate as luni-solar obliquity rate since most of it ultimately comes from planetary influences.

To get the precession and obliquity rate for the moving equator with respect to the moving ecliptic plane it is necessary to subtract off the motion of the ecliptic plane. This is done in the last two lines of Table 3. The values for the ecliptic motion have been improved upon since the IAU theory (Lieske *et al.* 1977). Improved ecliptic motion and its influence on the precession expressions has been considered by Bretagnon (1982), Bretagnon & Chapront (1981), Laskar (1986), and Simon *et al.* (1994) and the improved motion from Simon *et al.*, including the correction for mass changes, has been used in Table 3. There is a problem with the nomenclature of the past. What have been called "planetary precession" and obliquity rate (Woolard uses "precession in obliquity") come from the motion of the ecliptic. We now have two planetary contributions to each of precession and obliquity rate which are motions in space, not ecliptic motion. It is conventional to refer to the (mean-of-date) motion of the dynamical equinox along the moving ecliptic plane as "general precession in longitude." Consequently, the final line has been labeled general motion and by extension the obliquity rate might be called general obliquity rate.

Both very long baseline interferometry (VLBI) and lunar laser ranging (LLR) are capable of measuring the motion of the equator with respect to space rather than the moving ecliptic. Thus both measure the luni-solar precession rate, not the general precession rate (the IAU primary constant), and have the potential to measure the −0.244 mas/yr obliquity rate with respect to space. There is weak evidence for the latter in the VLBI results (Herring *et al.* 1991; Steppe *et al.* 1993). Better precession and obliquity rate measurements can be anticipated as the VLBI and LLR data spans lengthen and separation of rates and 18.6 yr nutation becomes stronger.

The nutation theory of Kinoshita & Souchay (1990) is a significant improvement on previous theories and it is important to understand the corresponding values of $(C-A)/C$, k_M, and k_S. In part due to misprints, three different values of $(C-A)/C$ and two of k_M have been published and it is important to resolve the discrepancy. From the 1976 IAU constants and Kinoshita & Souchay's numerical values and equations are calculated $(C-A)/C = 0.003\,273\,967\,71$, $k_S = 3475.414\,26''$/century, $k_M = 7547.199\,69''$/century (or 7567.769 70″/century with the $1/F_2^3$ factor). The set of their values which has internal consistency is $(C-A)/C = 0.003\,273\,967$ (Souchay & Kinoshita 1991), $k_S = 3475.4135''$/century, and $k_M = 7547.1981''$/century (7567.7681″/century with factor). The relative difference is 2.2×10^{-7}, but is small enough to only influence the nutation at the few microarcsecond level. In Kinoshita & Souchay the (Earth+Moon)/Sun mass ratio is incorrectly labeled Earth/Sun ratio in two places and correctly labeled in a third, but any discrepancy is too small to explain the difference. To adjust their theory to this paper's updated precession rate and $(C-A)/C$ with all of the corrections, multiply their nutation series by the factor 0.999 937 82.

It is instructive to consider several contributions to the above factor and the proportionate $(C-A)/C$. The largest is the correction to the IAU precession rate causing a relative change of -6.48×10^{-5}. Updating the mass ratios and obliquity causes -8×10^{-7}. The change in the ecliptic motion causes 9×10^{-7}. Theoretical differences due mainly to the planetary tilt induced precession (absent in Kinoshita & Souchay) and a somewhat different geodesic precession account for 2.2×10^{-6}. By comparison, the relative uncertainty due to the present precession rate determination is about 7×10^{-6}.

The ratio of $J_2 = (C-A)/MR^2$, where R is the Earth's equatorial radius, and $(C-A)/C$ gives C/MR^2 the normalized polar moment of inertia. Combining J_2 from the GEM-T2 solution of Marsh *et al.* (1990), including a suitable addition for the model's permanent tide, with the precession derived $(C-A)/C$ from above yields a rigid body $C/MR^2 = 0.330\,700\,7$. With the mean moment $I = (C+2A)/3$, then $I/MR^2 = 0.329\,978\,9$.

6. THE PHASE OF THE TILT TERMS

There are two reasons to consider the seemingly prosaic subject of the phase of the tilt terms used in Sec. 3. (1) It has been stated (Kinoshita 1975, 1977) that these terms do not give rise to an obliquity rate and the resolution of the difference between that claim, Woolard (1953), and this paper hinges on the origin of the phase. (2) Time variations of the phase will give rise to higher derivatives of the precession and obliquity.

The J_2-induced tilt terms in Sec. 3 have zero phase so long as L and Ω are referred to the moving equinox. They do not give rise to an obliquity rate and do not need to be considered further in this section. The planetary-induced tilt terms in the lunar orbit arise in two ways. The direct terms arise from the forces of the planets on the lunar orbit. The indirect terms arise from the force of the Sun on the lunar orbit, coupled with the motion of the ecliptic plane due to the forces of the planets changing the heliocentric orbit. The tilt from the direct effect is an order of magnitude smaller than the indirect effect. The two components have been combined in Sec. 3. The indirect contribution will dominate the following discussion.

We wish to express the secular motion of the lunar orbit plane acted upon by the Sun. The coordinates of the pole of the variable lunar orbit plane are $(P_V, -Q_V, \cos i_V)$ where $P_V = \sin i_V \sin \Omega_V$ and $Q_V = \sin i_V \cos \Omega_V$. The analogous time-varying variables for the ecliptic pole are P' and Q'. Using an inertial frame aligned with the ecliptic and equinox at the initial time, e.g., at J2000 $P' = Q' = 0$, a good approximation for the differential equations for the secular motion is (see chapters 12 and 16 of Brouwer & Clemence 1961)

$$dP_V/dt = \dot{\Omega}_0 (Q_V - Q'),$$

$$dQ_V/dt = -\dot{\Omega}_0 (P_V - P'). \tag{24}$$

In the first approximation $\dot{\Omega}_0$ is a quantity which is proportional to the mass ratio Sun/(Earth+Moon), the lunar mean motion, and the cube of the ratio of the lunar to heliocentric semimajor axes $(a/a')^3$. When the ratio of ecliptic motion to $\dot{\Omega}_0$ is small, it is 10^{-5} for the Moon, a good approximation for the solution of the differential equations is

$$P_V = P + P' - \dot{Q}'/\dot{Q}_0,$$

$$Q_V = Q + Q' + \dot{P}'/\dot{\Omega}_0. \tag{25}$$

$P = \sin i \sin \Omega_0$ and $Q = \sin i \cos \Omega_0$ represent a uniformly precessing lunar orbit plane with rate $\dot{\Omega}_0$ (retrograde 18.6 yr period) and fixed inclination i, which the additional solar perturbation terms modify. $P_V - P'$ and $Q_V - Q'$ are good approximations for the motion of the lunar orbit pole with respect to the moving ecliptic pole (P' and Q' are functions of time). In the above solution the orbit is precessing along a plane tilted slightly with respect to the moving ecliptic with the tilt given by the last terms on the right-hand side. At J2000 $\dot{Q}'/\dot{\Omega}_0 = 1.386''$ and $\dot{P}'/\dot{\Omega}_0 = -0.124''$, so the indirect term causes a 1.39″ tilt with an orientation governed by the node about which the ecliptic plane is rotating ($\Pi = 174.87°$ at J2000).

The tilt of the lunar orbit plane to the moving ecliptic can be expressed as a perturbation in the lunar latitude

$$\Delta\beta = (\dot{Q}'/\dot{\Omega}_0)\cos L_0 + (\dot{P}'/\dot{\Omega}_0)\sin L_0, \tag{26}$$

where the mean longitude L_0 is given in terms of the mean argument of latitude and node ($L_0 = F + \Omega_0$). Note that the differential Eqs. (24) and their solution (25) are written in an inertial coordinate system. While the differences $P_V - P'$ and $Q_V - Q'$ are useful for seeing that the lunar orbit nearly follows the ecliptic, the node Ω_0 is a quantity referenced to the equinox at the initial time (J2000) and its retrograde rate is with respect to inertial space. The subscript zero has been used to distinguish L_0 and Ω_0 from L and Ω which in conventional lunar theory are measured from the moving (mean of date) equinox along the moving (mean) ecliptic. The $\Delta\beta$ equation can be put in the form of an amplitude times $\sin(L_0+\text{phase})$ where the phase is given by $\tan^{-1}(\dot{Q}'/\dot{P}')$, which at J2000 is 95.13° or $270° - \Pi$. At J2000 this phase rate measured along the moving ecliptic plane is -17.36 ″/yr, twice the rate of Π measured along the fixed J2000 ecliptic. To put $\Delta\beta$ in the form of Eq. (11) with $\sin(L+\text{phase})$, with L measured from the moving (mean of date) equinox along the moving ecliptic, then for compatibility the phase must be measured from the moving equinox, along the moving ecliptic, to the node of rotation of the ecliptic on the moving ecliptic. In the notation of the IAU precession paper this phase is $270° - \Pi(T,0)$. The phase rate is then the general precession rate minus 17.36 or 32.93″/yr (this rate is wrong in Brown's lunar theory). The smaller direct contribution has its own phase which depends on the difference between the moving equinox and the planetary nodes, so in Eq. (11) the phase of the combined direct and indirect terms is slightly larger than 95.13° and the amplitude is slightly larger than 1.39″.

Woolard (1953) earlier computed the obliquity rate contribution from the planetary-induced tilt in the lunar orbit. When explaining it (pp. 127–128) he used the (direct plus indirect) $\Delta\beta$ contribution from Brown's lunar theory which is equivalent to Eq. (11). Broken into sine and cosine components of $F + \Omega = L$, only the cosine component [he gave it as $1.53'' \cos(F+\Omega)$] was displayed because the sine does not give rise to a secular obliquity rate. If the argument was measured with respect to the moving equinox, as Ω and L conventionally are in lunar theory, and if the coefficient was constant, then there would be no obliquity rate. For this reason Kinoshita (1975, 1977) argued against Woolard's obliquity rate. For the indirect contribution (there is a parallel argument for the direct contribution), Eq. (26) and its discussion show that the $\Delta\beta$ expression could have been written in terms of $\cos[L - \Pi(T,0)]$ or in the form of Eq. (26) using L_0. To write it using L requires introducing a function of $\Pi(T,0)$, and $\Pi(T,0)$ depends on the moving equinox. Woolard was only computing the linear time term (his discussion shows that he was aware of the higher powers of time). His notation was numerically suitable for the epoch [like Eq. (11) of this paper], but if taken literally it is functionally misleading because the equinox dependence is hidden in the numerical coefficient [in Eq. (11) it is in the phase]. The discussion in the appendix of Kinoshita (1977) was unaware that $\Delta\beta$

had a functional dependence on the equinox which was undisplayed in Woolard. Kinoshita's discussion depended on a finite partial derivative of $\Delta\beta$ with respect to the equinox, but the equinox dependence given above causes the partial to be zero. It is concluded here that the obliquity rate from the indirect (and direct) tilt is real and that Woolard's numerical value was reasonable.

There is an anomaly that I do not understand. The comparison by Souchay & Kinoshita (1991) of their theory with a numerical integration showed as discrepancies neither the obliquity rate term nor the out-of-phase 18.6 yr nutation terms which arise from the same source.

Since the indirect tilt terms depend on the time-varying P' and Q' and the two direct planetary effects (direct tilt and direct torque on the Earth) depend on the time-varying planetary orbits, it should be understood that all three of these contributions to the obliquity rate are not constant. Over long time scales (>10 000 yr) the obliquity exhibits quasiperiodic variations and the rate will show both signs. By contrast, the small obliquity rate due to tidal dissipation (Sec. 7) is always positive.

Both the precession and obliquity rates arising from the indirect tilt will vary with time. The time dependence of the P' and Q' derivatives in Eq. (26) can be used with Eqs. (12) and (13) to compute the precession and obliquity rates as a power series in time. For use in Sec. 8 it is convenient to express these rates in a coordinate system moving with the ecliptic and equinox. In the two-time (T,t) power series of Lieske *et al.* (1977) or Simon *et al.* (1994) differentiate P' and Q' with respect to t, set $t=0$, and use T as the time variable. The series for the indirect contribution then becomes $-0.003\,92+0.000\,703\ T$ for precession rate (units $''$ and centuries) and $-0.0233+3\times10^{-6}\ T^2$ for obliquity rate. The computation of the nonlinear contributions is similar for the direct planetary torques on the Earth and the direct planetary tilt on the lunar orbit. In Eqs. (22) the dependence of the direct torque effects on the planetary P's and Q's is explicit and power series for the P's and Q's (Laskar 1986; Simon *et al.* 1994) can be used. Venus dominates the acceleration and the result is -17×10^{-6} $''$/centuries^2 for precession and 2×10^{-6} $''$/centuries^2 for obliquity. The acceleration due to the direct tilt terms is more difficult, but is estimated to be about 40% larger than for the direct torque effect. Because the polynomials for the planets used a fixed equinox, to these figures must be added the larger accelerations which result from transforming from a fixed to a moving equinox: 411×10^{-6} $''$/centuries^2 for precession and -48×10^{-6} $''$/centuries^2 for obliquity. The total of the preceding direct and indirect tilt terms and the direct torques on Earth are listed under "planetary tilt and direct torque" in Table 4. Note that part (0.032 69 $''$/century) of the direct torque effect for precession in Eq. (22) does not depend on planetary Ps and Qs, contributes no accelerations, and is included with the entry for luni-solar precession in the table.

The coefficients of the planetary-tilt-induced nutation (Table 1) will also have secular changes. Assuming that the secular changes in the ΔL coefficients scale in proportion to those of the latitude coefficient, then the in-phase contributions are $(-0.0301+0.0050T)\sin\Omega$ to the longitude nutation and $(0.0029-0.0005T)\cos\Omega$ to the latitude nutation (units mas and centuries). The relative changes of the out-of-phase coefficients is about 10^{-3}/century and is ignorable.

TABLE 4. Time and obliquity dependence of precession and obliquity rates ($''$/century) which are needed to calculate the evolution of precession and obliquity with time.

Source	Rate in "/century	ε Dependence
	Precession	
Luni-solar, direct planetary torque	$P_0 \cos\varepsilon_0 - 0.003395\,t - 6\times10^{-6}\,t^2$	$\cos\varepsilon$
Geodesic precession	$-1.919362 + 2.7\times10^{-6}\,t$	1
Second order (M_3)	-0.03310	$6\cos^2\varepsilon - 1$
Second order	-0.01368	$3\cos^2\varepsilon - 1$
J_4 precession	$+0.00260$	$\cos\varepsilon\,(4-7\sin^2\varepsilon)$
J_2 tilt	-0.2630	$\cos 2\varepsilon / \sin\varepsilon$
Planetary tilt and direct torque	$-0.00643 + 0.001074\,t$	$\cos 2\varepsilon / \sin\varepsilon$
Tides on lunar orbit	$-0.000102\,t$	$\cos^2\varepsilon$
Tides on spin and moments	$-0.000133\,t$	$\cos^3\varepsilon$
J_2 rate	$-0.0140\,t$	$\cos\varepsilon$
	Obliquity	
Planetary tilt and direct torque	$-0.0268 - 0.000044\,t + 3\times10^{-6}\,t^2$	$\cos\varepsilon$
Tides	$+0.0024$	$\sin\varepsilon\,\cos\varepsilon$

7. TIDAL AND NONLINEAR EFFECTS

This section considers effects which cause accelerations and higher derivatives in the accumulated precession (integral of precession rate), and another contribution to obliquity rate. There are effects due to the change in the eccentricity of the orbit of the Earth–Moon system about the Sun which have been considered in previous theories, plus tidal effects in the lunar orbit and Earth rotation, and possible changes in $(C-A)/C$. Many of the results of this section can be derived from Eq. (7) and its lunar counterpart. Moving toward polynomial expressions for the precession quantities as in Lieske *et al.* (1977), the units of that paper are now adopted (arcseconds and centuries). The results of this section are summarized in Table 4. That table also gives the ϵ dependence since the change of obliquity contributes additional accelerations and higher derivatives which will be utilized while solving the differential equations for orientation in the next section.

As seen from Eq. (7), the eccentricity of the heliocentric orbit enters into the solar-induced precession of the equator and changes in the eccentricity will affect the derivatives of the precession rate. The evaluation of the eccentricity-induced acceleration (first derivative of the precession rate) in the accumulated precession in the IAU theory dates to de Sitter & Brouwer (1938). It is re-evaluated here at J2000. Using the eccentricity polynomials in Laskar (1986) or Simon *et al.* (1994), $de/dt=-42.0\times10^{-6}$/centuries. This causes the solar-induced precession rate to have a first derivative of -3362×10^{-6} $''$/centuries^2. The eccentricity of the heliocentric orbit also enters into the geodesic precession Eq. (23). The derivative of that rate (in retrograde sense) is 2.7×10^{-6} $''$/centuries^2. The lunar orbit includes perturbations by the Sun and those which depend on the heliocentric eccentricity contribute accelerations. These periodic terms in the lunar latitude and distance weakly influence the lunar-induced precession through their squares. The fractional influence on the precession rate is 1.97×10^{-6} by these radial terms and -5×10^{-8} by these latitude terms, yielding -33.3×10^{-6} "/centuries^2 or 1.0% of the solar-induced acceleration. The luni-solar acceleration is -3395×10^{-6} "/centuries^2 exclusive of the contribution of the geodesic precession. There is also a small (higher derivative) contribution due to the second derivative of e^2. The t and t^2 terms in the precession rate (without geodesic precession) due to heliocentric eccentricity changes are -3395×10^{-6} "/centuries^2 $t-6\times10^{-6}$ "/centuries^3 t^2. To convert the t coefficient to the P_1 parameter of the IAU theory divide by $\cos\epsilon_0$ to get $-0.003\,70$ "/centuries^2. The agreement with the $-0.003\,69$ "/centuries^2 used in the IAU theory is excellent, aided by the small t^2 term.

Tides are raised on the Earth by the Moon and Sun and their energy dissipation causes the Moon to recede and the Earth's rotation to slow. Lunar laser analyses indicate a secular acceleration of -26.0 "/centuries^2 and a tidal (semimajor axis) recession of 3.84 cm/yr (Williams *et al.* 1993) so $da/dt/a=1.00\times10^{-8}$/centuries. The $1/a^3$ dependence of precession in Eq. (7) implies the precession changes by -103×10^{-6} "/centuries^2. The tidal changes in the lunar orbit eccentricity e and inclination i are small (Chapront-Touzé & Chapront 1983, 1988) and lead to only 0.9 and 0.2×10^{-6} "/centuries^2, respectively, in the precession. Equation (7) depends on the obliquity ϵ, the angular momentum $C\omega_z$, and the moment difference $(C-A)$ which exhibit secular changes due to tidal effects. An angular momentum balance between the Earth's spin and the lunar orbit for long time scales gives an estimate for secular changes in these quantities. Writing the angular momentum components perpendicular and parallel to the ecliptic plane

$$\begin{aligned} H_z &= C\omega_z \cos\epsilon + \mathbf{M}m[G(\mathbf{M}+m)a(1-e^2)]^{1/2} \\ &\quad \times \cos i/(\mathbf{M}+m), \\ H_y &= C\omega_z \sin\epsilon, \end{aligned} \tag{27}$$

where m and $\mathbf{M}$ are the masses of the Moon and Earth and G the gravitational constant. Differentiating both equations for secular changes, conserving angular momentum, and combining gives

$$\begin{aligned} d\epsilon/dt &= [m/(\mathbf{M}+m)](n/\omega_z)(\mathbf{M}R^2/C)(a/R)^2 \sin\epsilon \\ &\quad \times (1-e^2)^{1/2} \cos i[da/dt/2a - e\,de/dt/(1-e^2) \\ &\quad - \tan i\; di/dt] \\ d(C\omega_z)/dt/C\omega_z &= -\cot\epsilon\; d\epsilon/dt, \end{aligned} \tag{28}$$

where n is the lunar mean motion and R the Earth's radius. Evaluating with the tidal changes in the lunar orbit (dominated by da/dt) gives $d\epsilon/dt=19.6\times10^{-4}$ "/centuries and $d(C\omega_z)/dt/C\omega_z=-2.20\times10^{-8}$/centuries. The latter causes 110×10^{-6} "/centuries^2 in the precession. Both $C-A$ and the deviation of C from the mean moment depend on the square of the Earth's rotation rate. Thus the value of $d\omega_z/dt/\omega_z=-2.19\times10^{-8}$/centuries, 0.44% less than $d(C\omega_z)/dt/C\omega_z$. The change in $C-A$ causes a precession change of -220×10^{-6} "/centuries^2. This tidal despinning of the Earth by the Moon causes changes in both lunar- and solar-induced precession. The solar tides also act to despin the Earth. The solar torque is much less well known than the lunar. It is a common approximation to assume that the ratio of solar to lunar torques is proportional to the square of the ratio of tide heights (0.46), though there is some evidence for a smaller torque ratio (Brosche & Wunsch 1990). Here the factor 1.21 is used to amend the lunar calculations for the solar contribution: the tidal obliquity rate is 24×10^{-4} "/centuries and the tidal precession change is $(-102-1.21\times110)\times10^{-6}$ "/centuries$^2=-235\times10^{-6}$ "/centuries^2. A related, but not identical, calculation of the obliquity rate by Kaula (1964), when adjusted for recent secular acceleration measurements and the solar contribution, gives an obliquity rate of 17×10^{-4} "/centuries.

There are a host of nontidal processes which change the spin rate of the Earth by exchanging angular momentum between the liquid core, solid mantle plus crust, oceans, and atmosphere, but these leave $C\omega_z$ unaffected. However, some of these processes do affect the precession through changes in $C-A$. The Earth's gravitational harmonic J_2 is proportional to $C-A$ and exhibits a small secular decrease which has been detected from the analyses of ranges to the Lageos and Starlette satellites (Yoder *et al.* 1983; Rubincam 1984; Cheng *et al.* 1989; Gegout & Cazenave 1991; Watkins & Eanes 1993; Nerem *et al.* 1993). Consequently the precession rate should also exhibit a decrease. The J_2 rates from these studies lie in the range of $(-2.5$ to $-3.6)\times10^{-9}$/centuries; they induce a sizable precession rate change in the range of $(-11.6$ to $-16.8)\times10^{-3}$ "/centuries^2. This is about 0.7% of the -2 "/centuries^2 classical acceleration induced by ecliptic motion (next section) and two orders of magnitude larger than tidally induced accelerations. Though seeming to vary on thousand year time scales, the nontidal acceleration of the Earth's spin (Stephenson & Morrison 1984, 1985) appears to be in accord with the reported J_2 rates. While the J_2 rate is clearly visible in satellite tracking data taken since 1976, that rate is imperfectly separated from the J_4 rate and there appear to be rate irregularities and questions about the separation of 18.6 yr tidal signatures which limit knowledge of the long-time average (Watkins & Eanes 1993). For Table 4 a J_2 rate of -3×10^{-9}/centuries has been adopted; this yields -0.014 "/centuries^2 in precession. This choice will give a precession acceleration valid since 1976, but future extrapolation is less certain. The J_2 rate uncertainty is the largest recognized uncertainty in the acceleration of precession. The precession is only sensitive to long-time changes in J_2 and the appropriate contribution to the precession will depend upon further monitoring of J_2 changes with artificial satellites.

8. POLYNOMIALS

The IAU precession paper (Lieske *et al.* 1977) gives polynomials in time for orienting the Earth based on the IAU-adopted general precession rate, obliquity, and masses. Equivalent matrix formulations have also been published (Lieske 1979; Fabri 1980). Improved ecliptic motion led to polynomial updates by Bretagnon & Chapront (1981),

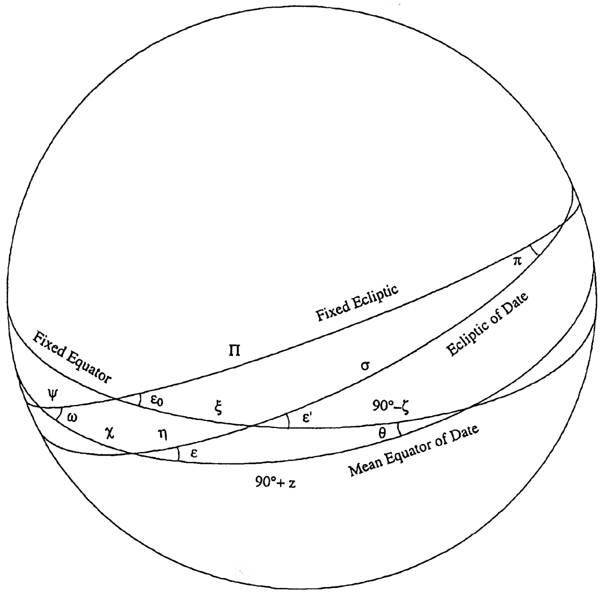

FIG. 1. Relation between the fixed equator (mean equator) and fixed ecliptic of J2000 and the moving (mean of date) equator and ecliptic. The arc from the moving equinox to the node of the moving ecliptic on the fixed ecliptic is $\Lambda_A=\Pi_A+p_A=\sigma_A+\eta_A$. The subscript A is not used with symbols in the figure.

Laskar (1986), and Simon *et al.* (1994). The latter paper includes improved precession rate, obliquity, and masses as well. In this paper theoretical contributions to precession and obliquity rates and higher derivatives have been identified and computed. In this section the theoretical improvements plus updated values for precession rate, obliquity, masses, and ecliptic motion will be used to generate revised polynomial expressions.

The notation of the IAU paper is used in this section except that the tilde has been dropped. The subscript A (for accumulated) denotes an angle. Thus ψ_A and p_A are accumulated (integrated) luni-solar and general precession rates, respectively. Rates will be indicated with derivatives. The polynomial expressions will be derived for a single time argument for use with the J2000 epoch. The fixed ecliptic and equator planes of J2000 and the moving ecliptic and equator of date constitute the basic geometry. See the IAU paper and Fig. 1 for the definition of the variables.

The basic differential equations were given in the IAU paper but they require additional terms due to the obliquity rate contributions with respect to inertial space. The obliquity and precession rates of Table 4 use a coordinate frame which is moving with the equinox. The total contribution of the obliquity rate (no ecliptic rate) from Table 4 is denoted R_ϵ and the total contribution to the precession rate multiplied by $\sin\epsilon_A$ is denoted R_ψ. These two components of the equator's rotation vector are in the plane of the moving equator. Two of the differential equations are just the projections of these two rates through the angle χ_A ("planetary precession") between the moving equinox and the intersection of the fixed ecliptic and moving equator.

$$
\begin{aligned}
d\omega_A/dt&=\cos\chi_A R_\epsilon+\sin\chi_A R_\psi,\\
\sin\omega_A d\psi_A/dt&=\cos\chi_A R_\psi-\sin\chi_A R_\epsilon.
\end{aligned}
\tag{29}
$$

The differential equation for the obliquity rate with respect to the moving ecliptic involves both the motion of the ecliptic and R_ϵ

$$
\begin{aligned}
d\epsilon_A/dt={}&\cos p_A dQ'/dt-\sin p_A dP'/dt\\
&+(1-\cos\pi_A)\cos(\Pi_A+p_A)d\pi_A/dt+R_\epsilon,
\end{aligned}
\tag{30}
$$

where $P'=\sin\pi_A\sin\Pi_A$ and $Q'=\sin\pi_A\cos\Pi_A$ describe the ecliptic pole with Π_A and π_A being the node and inclination, respectively, of the moving ecliptic on the fixed ecliptic.

In addition to P' and Q', which are input functions for the ecliptic motion, the right-hand sides of the differential equations are functions of χ_A and p_A. Two geometrical equations are needed to link these latter two variables with the others

$$
\begin{aligned}
\sin\chi_A&=\sin\pi_A\sin(\Pi_A+p_A)/\sin\omega_A,\\
\cos(\Pi_A+p_A)&=\cos\chi_A\cos(\Pi_A+\psi_A)\\
&\quad+\sin\chi_A\sin(\Pi_A+\psi_A)\cos\omega_A.
\end{aligned}
\tag{31}
$$

The three differential equations and the two geometrical equations must be solved simultaneously for ω_A, ψ_A, ϵ_A, χ_A, and p_A.

The simultaneous solution was performed with a numerical technique. The variables are represented with polynomials of time. The five equations are evaluated at equal time intervals, the polynomial coefficients are fit, and the procedure is iterated to convergence. Input quantities are the J2000 obliquity (ϵ_0) and general precession rate plus the polynomials for P' and Q'. The constant P_0 in Table 4 is determined from the initial obliquity and precession rate and the other precession rates in the table. At J2000 the rates of general and luni-solar precession are linked through the rate of the planetary precession χ_A projected on the ecliptic plane. The iterative solution for the polynomials was done in extended precision on a microcomputer.

Parameters such as the commonly used ζ_A, θ_A, and z_A are derived geometrically from the above set of variables. Numerical polynomial fits were also used. Two points are noted for generating the polynomials for ζ_A and z_A. The polynomial for ω_A is carried to one higher degree than those for ζ_A and z_A. It is necessary to include a constant in ζ_A and z_A when the zero coefficient of t in ω_A is finite. The constant is $d\omega_A/dt/\sin\epsilon_0 d\psi_A/dt$ evaluated at J2000 and has opposite signs for ζ_A and z_A.

The computer program was tested against the IAU expressions. For the IAU theory, three terms are used which have the form, but different numerical values, of those on the first two lines for precession rate in Table 4: the P_0 and t terms on the first line and a constant geodesic precession rate. With the input values used in the IAU theory, the program was able to successfully reproduce the polynomials in the IAU paper with a deviation of no more than 1 in that paper's last digit except for Π_A. As also experienced by Fabri (1980), in the IAU paper the number of digits given for Π_A exceeds those given for P' and Q' and apparently additional digits in P' and Q' were used there.

TABLE 5. Polynomial expressions for orientation of the Earth's equator (arcsec). Time t in Julian centuries from J2000 (JD 2451545.0). Greenwich mean sidereal time (s) at 0 h UT1. Time in UT1 centuries from 12 h UT1, JD 2451545.

Angle	Constant	t	t^2	t^3	t^4
P'	0.000000	4.199610	0.193971	-0.000223	-0.000001
Q'	0.000000	-46.809560	0.051043	0.000522	-0.000001
π_A	0.000000	46.997570	-0.033506	-0.000124	0.000000
Π_A	629543.967373	-867.919986	0.153382	0.000026	-0.000004
p_A	0.000000	5028.770000	1.105407	0.000076	-0.000024
ψ_A	0.000000	5038.456501	-1.078977	-0.001141	0.000133
ω_A	84381.409000	-0.024400	0.051268	-0.007727	0.000000
χ_A	0.000000	10.557700	-2.381366	-0.001208	0.000170
ε_A	84381.409000	-46.833960	-0.000174	0.002000	-0.000001
ζ_A	2.511180	2306.071060	0.299027	0.018017	-0.000005
z_A	-2.511180	2306.065079	1.092516	0.018265	-0.000029
θ_A	0.000000	2004.182023	-0.429466	-0.041822	-0.000007
ξ_A	0.000000	10.557700	0.493164	-0.000309	-0.000003
ε'_A	84381.409000	-46.809560	0.051142	0.000531	0.000000
η_A	0.000000	5038.456501	1.558353	-0.000186	-0.000027
GMST0	24110.54841	8640184.7928613	0.0927695	-0.0000003	-0.0000020

The expression for Greenwich Mean Sidereal Time (GMST) relates UT1 to the angle between the mean equinox of date and the Greenwich meridian. This paper will follow the convention that the GMST expression is a solution of the dynamical equations for rotation. The IAU-adopted polynomial expression for GMST given by Aoki *et al.* (1982) is specific to the IAU precession theory. As pointed out by Williams & Melbourne (1982) and Zhu & Mueller (1983), changing the precession expressions without changing the GMST expression would alter the determination of UT1 from observations. Consequently, an additional equation has been evaluated numerically. The fundamental parameter is the rotation rate of a rigid Earth with respect to inertial space about its symmetry axis

$$d(\mathrm{GMST}+\chi_A)/dt-\cos\,\omega_A d\psi_A/dt \tag{32}$$

with due consideration for the units. The nonlinear parts of the GMST expression at zero hour UT, GMST=GMST0 +UT1, come from

$$\int \cos\,\omega_A d\psi_A/dt\; dt-\chi_A \tag{33}$$

dividing arcseconds by 15 to convert to seconds. The coefficients of the constant and linear terms were set by imposing the condition that at J2000 there would not be a discontinuity of UT1 (the constant coefficient matches Aoki *et al.*), its rate, or the rotation rate of the Earth in space (there are small ambiguities at the level of truncated digits). In Aoki *et al.* the constant and linear coefficients were picked for continuity of UT1 determined from optical astrometric measurements of catalogue stars rather than continuity with respect to an inertial frame. Inertially referenced techniques now dominate the determination of UT1 so there is no counterpart to the catalogue equinox drift. It is conventional to derive the small nonlinear terms of GMST0 using a linear time scale for the independent time, but to evaluate the entire GMST0 expression using a UT1 time scale.

Since the IAU theory for precession appeared, there have been improvements in the computation of the motion of the ecliptic due to theoretical advances and improved planetary masses, better measured values for precession rate and obliquity, and the theoretical computations of Kinoshita & Souchay (1990) and this paper. To illustrate the resulting changes, revised expressions are presented here. The input values match those used to generate Table 3 (Sec. 5). The ecliptic motion is taken from Simon *et al.* (1994) including planetary mass corrections. The theoretical contributions of Table 4 have been used. The resulting expressions are given in Table 5. The units are arcseconds and Julian centuries measured from J2000 [t=(JD-2451545.0)/36525], except for GMST which uses seconds and centuries of UT1 measured from JD 2451545.0 UT1=12 h. UT1 on January 1, 2000. While these expressions can serve those who need the highest accuracy now, it should be anticipated that there will be future improvements: some theoretical, certainly in the measurement of the precession constant and obliquity, and hopefully in the predictive knowledge of the J_2 rate.

The polynomial expressions in Table 5 can be used for times extending out to a few millennia, but are not suitable for longer times. The polynomials are equivalent to expansions of expressions appropriate for longer times: an average precession rate and obliquity plus long-periodic, or at least quasiperiodic, terms with periods exceeding 10 000 yr (Berger 1976; Laskar *et al.* 1993). For millions of years the small tidal acceleration is inexorable and modifies the precession and obliquity behavior for ancient times (Berger *et al.* 1992). Nontidal J_2 change must be transient. Most of the Earth's oblateness is caused by its spin and fluctuations of the active Earth's J_2 from the spin-controlled equilibrium will damp down.

9. ROTATIONS

Considered in this section are sequences of rotations suitable for the various sets of precession parameters in Table 5. When combining the rotations for precession and nutation, the number of rotations can be minimized. Finally, an expedient procedure is given which is suitable for introducing the most important (linear) corrections to precession and obliquity without undertaking the more extensive and complete modifications.

Consider the sequence of rotations which can be used to orient the precessing and nutating Earth. The standard procedure is to precess from the mean equator and equinox of J2000 to the mean equator and equinox of date using the angles ζ_A, θ_A, and z_A and then to nutate to the true equator and equinox of date by rotating into the mean ecliptic of date, applying nutation in longitude to reach the true equinox of date, and then to rotate to the true equatorial plane including nutation in obliquity. The sequence of six rotations (R_i is the rotation matrix around axis i) is

$$R_1(-\epsilon_A-\Delta\epsilon)R_3(-\Delta\psi)R_1(\epsilon_A)R_3(-90°-z_A)$$
$$\times R_1(\theta_A)R_3(90°-\zeta_A). \tag{34}$$

An alternative is to precess by moving along the fixed ecliptic to the intersection with the mean equator of date, then rotate along that equator to the mean equinox of date, and then to nutate as before. The seven rotations are

$$R_1(-\epsilon_A-\Delta\epsilon)R_3(-\Delta\psi)R_1(\epsilon_A)R_3(\chi_A)R_1(-\omega_A)$$
$$\times R_3(-\psi_A)R_1(\epsilon_0). \tag{35}$$

A second alternative is to precess by moving along the fixed ecliptic to the intersection with the mean ecliptic of date, rotate back along the mean ecliptic, and then nutate

$$R_1(-\epsilon_A-\Delta\epsilon)R_3(-\Pi_A-p_A-\Delta\psi)R_1(\pi_A)$$
$$\times R_3(\Pi_A)R_1(\epsilon_0). \tag{36}$$

The third procedure requires only five rotations. If one is to nutate as well as precess, it is possible to bypass the mean equator of date and combine the precession and nutation along the moving ecliptic. Five rotations is not the minimum for combining precession and nutation. With four rotations one can move along the fixed equator to the intersection with the moving ecliptic (angle ξ_A), rotate into the moving ecliptic (ϵ_A'), combine the rotation along the ecliptic of date (η_A to the mean equinox of date) and nutation in longitude, and rotate to the true equator

$$R_1(-\epsilon_A-\Delta\epsilon)R_3(-\eta_A-\Delta\psi)R_1(\epsilon_A')R_3(\xi_A). \tag{37}$$

The angles ξ_A, ϵ_A', and η_A have not been given with conventional precession expressions in the past. They are illustrated in Fig. 1 and the expressions are given in Table 5.

Note that for the changes in precession and obliquity polynomials not due to ecliptic motion, only the last two (left) rotations in the last two sequences above would be changed; for the first two sequences the precession and obliquity changes are distributed over multiple rotations. The last two sequences make it clear that a change to the precession and obliquity rates could be added into the corresponding nutation parameters as an alternative way to introduce them (a similar conclusion was reached by Folkner *et al.* 1994 and VLBI fits to observations have often included linear terms in their "nutation" corrections). Adding $-0.0244''$/centuries t to $\Delta\epsilon$ and $-0.3219''$/centuries t to $\Delta\Psi$ (t in centuries from J2000) is an expedient way to incorporate the most important corrections without reprogramming the precession and GMST expressions. For this expedient approach the equation of equinoxes associated with $\Delta\Psi$ will automatically satisfy the concerns of Williams & Melbourne (1982) and Zhu & Mueller (1983) about precession modifications changing UT1, so that the GMST0 polynomial does not require revision. Neither does the J2000 value of the obliquity need changing for the expedient approach since the difference tends to cancel between the rotations. Other theoretical contributions, such as the nonlinear corrections, could also be added as nutation corrections. The expedient procedure does not work for geometrical revisions to ecliptic motion (purely P', Q', Π_A, and π_A but also parts of other parameters) since they appear in multiple rotations and tend to cancel, but it could be applied to the dynamical consequences of those revisions. However, it is observed that if an expedient procedure becomes too complicated then it is not expedient. It is best suited to easily inserting the linear corrections to precession and obliquity while more thoroughgoing revisions can use Table 5's polynomial expressions for precession and GMST0.

10. NUTATION CORRECTIONS, SCALING, AND COMPARISONS

The 1980 IAU nutation series (Seidelmann 1982) was a combination of the rigid-body series of Kinoshita (1977) and the elastic and structured-Earth corrections due to Wahr (1979, 1981). Since the 1980 IAU nutation working group paper there have been two revisions of rigid-body nutation (Zhu & Groten 1989; Kinoshita & Souchay 1990). The nutation theories allowing for the Earth's elasticity and core are based on the rigid-body theories and it is well to compare and understand those rigid-body theories.

Zhu & Groten utilized the earlier work of Kinoshita (1977) extending the nutation series to smaller terms, and adding both second-order terms and corrections for the Earth's J_3. It has served as the basis for nonrigid body treatments by Zhu *et al.* (1990), and the several ZMOA series of Herring (1990), Mathews *et al.* (1991), and Herring *et al.* (1991). Kinoshita & Souchay also extended the series to smaller terms, and added second-order terms, J_3 effects, and planetary terms involving planetary arguments. In addition they added small solar terms due to the offset of the Earth from the center of mass of the Earth–Moon system and revised the expression for the $(C-A)/C$ scaling of the nutation series from the precession constant. In Kinoshita (1977) and Kinoshita & Souchay the J_2 tilt effects on the scaling and 18.6 yr terms are present. The in-phase 18.6 yr nutation coefficients and the $-0.0056''$/centuries precession due to the planetary tilt effect is present in the former, but not the latter work. Kinoshita & Souchay also add $-0.014''$/centuries second-order contributions to the precession and make small revisions to the first-order contributions which are not present in the earlier works. Thus there are small differences in the scaling of Kinoshita & Souchay, Zhu & Groten, and this paper which are addressed below. Only Kinoshita & Souchay have the small center-of-mass offset corrections, three of which have amplitudes of 0.02 mas in longitude. Comparison of the J_3 contributions in the two papers shows poor agreement; the 3231 day obliquity coefficients have different signs and differ by 0.12 mas. In addition, the 6164 day coefficients disagree by a factor of two. See Souchay (1993) for further comparisons.

The coefficients of Kinoshita & Souchay's rigid-body nutation theory would have to be multiplied by 0.999 937 82 (Sec. 5) to match the precession rate and other changes of this paper. This would cause the 18.6 yr $\Delta\psi$ coefficient to increase by 1.075 mas and the 18.6 yr $\Delta\epsilon$ coefficient to change by -0.574 mas. These corrections are in addition to those of Table 1, and taken together the in-phase corrections to the 18.6 yr coefficients are 1.045 mas in $\Delta\psi$ (giving $-17.280\,76''$) and -0.571 mas in $\Delta\epsilon$ ($9.228\,00''$).

To match the constants of this paper the coefficients of Zhu & Groten's rigid-body nutation series need to be multiplied by 0.999 930 8 for the lunar terms and 0.999 929 7 for the solar terms. For the in-phase 18.6 yr coefficients this gives $-17.280\,75''$ in $\Delta\psi$ and $9.227\,92''$ in $\Delta\epsilon$. Thus after correction to a common $(C-A)/C$ and compensation for the planetary tilt effect, the 18.6 yr terms of Zhu & Groten and Kinoshita & Souchay differ by only 0.01 mas in longitude and 0.08 mas in obliquity.

To all nutation series since Woolard (1953) the out-of-phase planetary tilt contributions of Table 1 need to be added. To all nutation series since Woolard, a -0.15 mas annual term from the yearly variation of the geodesic precession needs to be added to the nutation in longitude (Voinov 1988; Gill *et al.* 1989; Fukushima 1991). For highest accuracy, nutation terms with planetary arguments, such as those of Kinoshita & Souchay, should also be included. While it causes minor changes in the resulting nutation series evaluation, the arguments of the 1980 IAU series and other series can be improved upon by using the values of Simon *et al.* (1994). The annual argument (l') differs by $5''$ at J2000, but the values of l' and L' depend very much upon which long-period terms are being carried when these arguments are fit to the time-varying heliocentric orbit. It should be compatible with the formulation used to generate the nutations with planetary arguments.

11. SUMMARY

Improvements in the accuracy of the observed motion of the Earth's equator plane and the wish to use these observations to infer the Earth's properties make improvements in the theories of precession, obliquity rate, and nutation desirable. The rate terms computed in this paper come from lunar orbit perturbations due to the planets and the Earth's J_2 plus direct planetary torques on the Earth and tidal effects.

The corrections to the obliquity rate are due to direct planetary torques on the Earth (see Sec. 4 and Table 2), torques due to planetary perturbations on the lunar orbit (Sec. 3), and tidal influences (Sec. 7, Table 4). Together these corrections are -0.244 mas/yr. This correction is a motion in space; the conventional $-0.468''$/yr obliquity rate is due solely to ecliptic motion, not to changes in the Earth's orientation. The IAU-adopted theory of precession and obliquity changes requires correction for this contribution to the obliquity rate. The largest contribution to the obliquity rate in space was earlier computed by Woolard (1953), but its reality was questioned by Kinoshita (1977). Section 6 discusses the reason for this discordant interpretation and concludes that the rate is real. The obliquity motion in space should be observable by the very long baseline interferometry and lunar laser ranging techniques.

In addition to the obliquity rate amendments, there are small contributions to the precession rate due to direct planetary torques and lunar orbit effects (Secs. 3 and 4, Table 2). The sum of the various contributions to obliquity and precession rates is given in Table 3 (Sec. 5). Based on recent measurements a general precession rate of $50.2877''$/yr at J2000 was adopted. For a rigid Earth this corresponds to the moment-of-inertia combination $(C-A)/C = 0.003\,273\,763\,4$. Combined with a satellite-determined J_2 this gives a normalized polar moment of inertia $C/MR^2=0.330\,700\,7$ and a normalized mean moment $I/MR^2=0.329\,978\,9$, with R the equatorial radius.

The contributions to obliquity and precession rates are not constant with time and the higher derivatives from these and other sources are computed in Sec. 7. Table 4 summarizes both linear and nonlinear (in time) contributions. The largest nonlinear correction arises from the Earth's J_2 rate. The theory for orienting the Earth (precession, obliquity changes, and Greenwich Mean Sidereal Time) is considered in Sec. 8 and revised polynomial expressions are presented (Table 5). In addition to the theoretical corrections of this paper, these expressions use improved values of the obliquity, precession rate, masses, and ecliptic motion.

Matrix rotations which combine precession and nutation are considered in Sec. 9. The conventional rotation scheme is not optimized for the number of rotations. A sequence of four rotations is given which incorporates both precession and nutation.

The torques, due to lunar orbit perturbations from the planets, also give rise to nutation contributions (Sec. 3 and Table 1). The largest contributions are to the 18.6 yr nutation: $-0.030 \sin\Omega + 0.137 \cos\Omega$ to $\Delta\psi$ (in mas) and $-0.028 \sin\Omega + 0.003 \cos\Omega$ to $\Delta\epsilon$. The small out-of-phase corrections arise because the perturbing planets' nodes on the ecliptic are not aligned with the dynamical equinox. Out-of-phase nutations are conventionally considered to arise from energy dissipation in the Earth and oceans, but these are exceptions.

The torques which cause precession and nutation depend on $(C-A)/C$ so that an accurate determination of the precession rate sets the scale of the nutation series. This scaling of the two most recent rigid-body theories is discussed in Sec. 10. Also discussed are the additions appropriate to each of these nutation theories.

Since the IAU expressions for precession and nutation were adopted, both theoretical improvements and refined measurements have become available. The theoretical contributions of this paper may be added to revised computations of ecliptic motion, rigid-body nutation, dissipative effects in the Earth's interior and oceans, and relativistic effects. Improved measurements of the precession constant and individual nutation terms are available. The latter have permitted refined computations of the non-rigid-body contributions to nutation. Understanding of the orientation of the Earth's equator and the fundamental influences on the orientation is advancing.

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