

Resolvent method: exact solutions of half-space transport problems by elementary means

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Abstract. A new version of the resolvent method for solving half-space transport problems is presented. We solve the integral transfer equation with the Λ -operator. The approach is based on the use of a non-linear equation relating the surface Green function $G_0(\tau)$ for the semi-infinite medium and the Green function $G_\infty(\tau)$ for the infinite medium. This $G_\infty(\tau)$ is assumed known. The equation is first used to derive, literally in one line, the famous $\sqrt{\epsilon}$ law. Then, by a short series of elementary transformations, we obtain the Fock-Chandrasekhar explicit expression for the H -function. Assuming that the kernel function of the Λ -operator is a Laplace-type integral, we derive the usual explicit expression for $G_0(\tau)$ in the form of a similar integral. The integrand involves only the functions appearing in the explicit expression for $G_0(\tau)$, and the H -function. And again, the derivation is elementary, i.e., without any use of the complex plane, and surprisingly short; the non-linear H -equation is completely by-passed. Alternative equations satisfied by the H -function are presented, among them a linear integral equation with a non-singular kernel. The standard non-linear H -equation is used to find the asymptotic behavior of the solution of the homogeneous transfer equation in deep layers of the semi-infinite atmosphere. In conclusion, a brief comment on the development of the resolvent method is given.

Key words: radiative transfer – methods: analytical

1. Introduction

There are a dozen of books (Chandrasekhar 1950; Kourganoff 1952; Sobolev 1956; Busbridge 1960; Ivanov 1969; Sobolev 1972; van de Hulst 1980 and others) and hundreds of papers on transport theory in which the advantages of the resolvent method, invariance principles, non-linear H -equations and related techniques for solving half-space problems are advocated.

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And nevertheless, the need of concise readable modern presentation of the fundamentals of this approach is acute. The present paper is aimed at meeting this need.

The version of the resolvent method which we develop here is substantially different from what is usually given in the literature. Essentially, it is a radical simplification, mainly in presentation rather than in ideology, of the approach we proposed a quarter of a century ago (Ivanov 1969, Chap. 5).

In classical approaches to half-space problems such as the Wiener-Hopf technique (Hopf 1934; Kourganoff 1952; Morse & Feshbach 1953; Busbridge 1960), the Case method and the like (Case & Zweifel 1967; Frisch & Frisch 1982; Cercignani 1988), the main instrument of analysis is the complex plane, with analytic continuation, factorization technique and Riemann-Hilbert problem inevitably playing the central role. We show that all this is unnecessary. If the full-space transport problem is solved, which is of course easy (although needs contour integration), the explicit half-space solutions can be readily obtained by the resolvent method without any further appeal to the complex plane. It seems that the easiest and shortest path from full-space to half-space solutions of linear transport problems is not via the complex plane, but along the real axis; yet, with excursions from the exhausted province of linearity to the wild land of non-linearity.

2. Basic equations: the Green functions

Consider the integral equation for the source function S

$$S(\tau) = \lambda \int_0^\infty K(\tau - \tau') S(\tau') d\tau' + Q(\tau), \quad (1)$$

or, in short operator notation,

$$S = \lambda \Lambda S + Q. \quad (2)$$

We assume that the kernel function K is even and normalized to unity:

$$K(-\tau) = K(\tau), \quad \int_{-\infty}^\infty K(\tau) d\tau = 1. \quad (3)$$

Hence the kernel of Eq. (1) is symmetric. The parameter $\epsilon \equiv 1 - \lambda \in [0, 1]$ is the photon destruction probability per scattering.

If we assume in addition that K is non-negative (which is indeed not necessary), then the kernel function is a probability density, and probabilistic interpretations given below apply. In radiative transfer problems the condition $K(\tau) \geq 0$ is usually met.

The explicit expression for $K(\tau)$ depends on the physics of the photon scattering process (monochromatic or with complete frequency redistribution, isotropic or anisotropic etc.), but for our purposes it is irrelevant. At this stage we do *not* assume that $K(\tau)$ for $\tau \geq 0$ is a Laplace-type integral, i.e., may be represented as a superposition of the exponentials $k(s)e^{-s\tau}$ with various s . Although Eq. (1) appears in many branches of physics, we are interested in radiative transfer, and hence use the terminology and notation of this field.

The solution of Eq. (1) for arbitrary Q is essentially the search for its Green function $G(\tau, \tau_1)$ satisfying

$$G = \lambda \Lambda G + \delta(\tau - \tau_1), \quad (4)$$

since

$$S(\tau) = \int_0^\infty G(\tau, \tau_1) Q(\tau_1) d\tau_1. \quad (5)$$

In (4), δ is the Dirac delta-function. In radiative transfer problems, the Green function $G(\tau, \tau_1)$ is the source function at depth τ in the half-space with a thin plane primary source at τ_1 . The non-singular part of the Green function is known as the resolvent Γ of the kernel K . Since K is even, the Green function G is symmetric:

$$G(\tau, \tau_1) = G(\tau_1, \tau). \quad (6)$$

Let now $G_0(\tau)$ be the surface Green function, i.e., G -function for $\tau_1 = 0$. In view of (6),

$$G_0(\tau) \equiv G(\tau, 0) = G(0, \tau). \quad (7)$$

The non-singular part of G_0 is known as the resolvent function Φ :

$$G_0(\tau) = \delta(\tau) + \Phi(\tau). \quad (8)$$

Traditionally, in radiative transfer one uses Γ and Φ rather than G and G_0 . However, we prefer G and G_0 which makes formulas more compact and clear.

The heart of the resolvent method is the expression for the G -function in terms of G_0 , namely

$$G(\tau, \tau_1) = \int_0^{\tau} G_0(\tau - t) G_0(\tau_1 - t) dt, \quad \tau \equiv \min(\tau, \tau_1). \quad (9)$$

Somewhat unexpectedly, G_0 appears here non-linearly. As we shall see shortly, it is this non-linearity which is at the root of the $\sqrt{\epsilon}$ law, non-linearity of the H -equation etc. In essence, Eq. (9) is the equivalent of a factorization. It reveals the key role of the surface Green function $G_0(\tau)$ in all half-space problems.

Physically, Eq. (9) is nearly obvious. Transfer of a particle (in our case, a photon) occurs along a trajectory. Let t be the depth at which the trajectory approaches the surface most (the highest point of the trajectory, if the boundary is at the top). Evidently $t \leq \min(\tau, \tau_1)$. At this point t scattering occurs, since otherwise the photon would continue to approach the boundary. This scattering event divides the trajectory leading from τ_1 to τ in two parts. The probability of transition from τ_1 to t along the first part of the trajectory is evidently $G(0, \tau_1 - t)$, since presence of the layers above t is not felt, and otherwise the trajectory is arbitrary. After scattering at t , random walk continues, again without intersecting this level. Hence we may assume that a new photon is injected at the boundary of a half-space, and it has to arrive at $\tau - t$. The corresponding transition probability is obviously $G(\tau - t, 0)$. Since there is no memory in the random walk, the total transition probability $\tau_1 \rightarrow \tau$ along all the trajectories with the highest point at t is the product of the transition probabilities before scattering at t , i.e., $G(0, \tau_1 - t)$, and after it, $G(\tau - t, 0)$. Taking into account symmetry of G , Eq. (7), and summing contributions from the trajectories with all t , $t \leq \min(\tau, \tau_1)$, we arrive at (9). For the proof of Eq. (9) valid for an arbitrary symmetric $K(\tau)$ (not necessarily non-negative), see Appendix A.

Now we derive a non-linear equation satisfied by the surface Green function G_0 . This equation, along with Eq. (9), is the basis of all our analysis. Essentially, it is a limiting form of Eq. (9).

First we introduce the infinite medium Green function $G_\infty(\tau)$ satisfying

$$G_\infty = \lambda \Lambda_\infty G_\infty + \delta(\tau), \quad (10)$$

where Λ_∞ is the operator of the same form as Λ , but with integration in τ' over $]-\infty, \infty[$, and not $[0, \infty[$ as for Λ . Evidently,

$$G_\infty(\tau) = \lim_{T \rightarrow \infty} G(\tau + T, T), \quad (11)$$

$$G_\infty(-\tau) = G_\infty(\tau). \quad (12)$$

Now we turn to (9). We substitute $\tau + T$, T and $T - t$ for τ , τ_1 and t , respectively, and then go to the limit $T \rightarrow \infty$. The result is

$$G_\infty(\tau) = \int_0^\infty G_0(t) G_0(\tau + t) dt, \quad \tau \geq 0. \quad (13)$$

This is the desired equation for G_0 . Its probabilistic interpretation is similar to that of Eq. (9). Strangely enough, this non-linear equation can be easily solved, nearly by inspection, provided some natural assumptions on $G_0(\tau)$ are met. This is the subject of Sect. 5.

Integrating (10) over all τ , we get

$$\int_{-\infty}^\infty G_\infty(\tau) d\tau = \epsilon^{-1}. \quad (14)$$

This is, of course, trivial. A useful non-trivial result is, however, obtained if we integrate (13) over all $\tau \geq 0$:

$$\left(\int_0^\infty G_0(\tau) d\tau \right)^2 = \epsilon^{-1}, \quad (15)$$

whence

$$\int_0^\infty G_0(\tau) d\tau = \epsilon^{-1/2}. \quad (16)$$

The physical significance of this so-called $\sqrt{\epsilon}$ law is that the mean number of scatterings of a particle which starts its random walk at the boundary of the half-space is exactly the square root of the mean number of its scatterings in full space.

We note that the sign ambiguity in obtaining Eq. (16) from Eq. (15) is resolved by observing that, according to (4) and (7), for $\epsilon = 1$ the integral on the left in Eq. (16) equals +1.

3. Laplace transforms of the Green functions

Let us now obtain the Laplace-transformed version of Eq. (9). It is widely used.

Applying to (9) the Laplace transformation with respect to τ_1 , we get

$$\bar{G}(\tau, s_1) = \bar{G}_0(s_1) \int_0^\tau G_0(\tau - t) e^{-s_1 t} dt, \quad (17)$$

where $\bar{f}$ is the Laplace transform of f , and s_1 is the transformation parameter. Equation (4) Laplace-transformed with respect to τ_1 is

$$\bar{G}(s_1) = \lambda \Lambda \bar{G}(s_1) + e^{-s_1 \tau}. \quad (18)$$

As usual, in this short operator form of the equation we have suppressed the space variable τ in $\bar{G}$, so that $\bar{G}(s_1)$ in (18) stands for $\bar{G}(\tau, s_1)$. We emphasize that the Λ -operator does *not* act on the variable s_1 , and we can integrate and differentiate $\bar{G}(\tau, s_1)$ with respect to s_1 under the Λ -operator symbol. Equation (18) shows that physically, $\bar{G}(\tau, s_1)$ is the source function in the half-space with exponentially distributed primary sources $e^{-s_1 \tau}$.

Laplace-transforming Eq. (17) with respect to τ , with the transformation parameter denoted by s , we obtain the famous result going back to Hopf (1934),

$$\bar{\bar{G}}(s, s_1) = \frac{\bar{G}_0(s) \bar{G}_0(s_1)}{s + s_1}. \quad (19)$$

Here, $\bar{\bar{G}}(s, s_1)$ is the double Laplace transform of $G(\tau, \tau_1)$ with respect to both space variables. The symmetry in s, s_1 is caused by the symmetry of the Green function in τ and τ_1 , while non-linearity in $\bar{G}_0$ comes from non-linearity in G_0 in Eq. (9). Probabilistically, the root of this non-linearity is the Markovian nature of the photon random walk. The most important feature of Eq. (19) is, of course, variable separation. The Laplace transform of the surface Green function appearing here,

$$\bar{G}_0(s) \equiv \int_0^\infty G_0(\tau) e^{-s\tau} d\tau, \quad (20)$$

plays a crucial role in our further developments. In the version of the resolvent method which we present here, one needs $\bar{G}_0(s)$ only for real s . The relations between G_0 and $\bar{G}_0$ are non-trivial, as we shall see later [cf. Eqs. (38) – (39) below].

In radiative transfer, the function $\bar{G}_0(1/s)$ is known as the H -function:

$$H(s) \equiv \bar{G}_0(s^{-1}). \quad (21)$$

According to (20) and (16),

$$H(0) = \bar{G}_0(\infty) = 1, \quad H(\infty) = \bar{G}_0(0) = \epsilon^{-1/2}, \quad (22)$$

which is the alternative form of the $\sqrt{\epsilon}$ law. Equation (19) is essentially the well-known expression, in terms of the H -functions, of the intensity of radiation diffusely reflected from semi-infinite atmosphere illuminated by monochromatic unidirectional radiation.

From Eq. (17) we have, taking into account the contribution to the integral from the δ -term in G_0 ,

$$\bar{G}(0, s_1) = \bar{G}_0(s_1). \quad (23)$$

Hence, the H -function may also be looked at as the surface value of the source function in a medium with exponentially distributed sources.

4. The H -function: explicit expression

As a first step in our search for the surface Green function $G_0(\tau)$, we obtain in this section an explicit expression for its Laplace transform $\bar{G}_0(s)$ for real s . In the language of radiative transfer, it is the same thing as obtaining explicitly the H -function, cf. Eq. (21).

[At this point the reader may directly go to Sect. 5, just assuming then that $\bar{G}_0(s)$ is known.]

We have four basic equations. Two of them are the defining equations for the Green functions $G(\tau, \tau_1)$ and $G_\infty(\tau)$ for, respectively, half-space, Eq. (4), and full space, Eq. (10). The other two equations are less trivial. These are the non-linear expression of $G(\tau, \tau_1)$ in terms of the surface Green function $G_0(\tau)$, Eq. (9), and the non-linear relation (13) between G_∞ and G_0 . Only a few steps of elementary rearrangements of the first three of these equations are needed to show that

$$\log \bar{G}_0(s) = -\frac{s}{2\pi} \int_{-\infty}^{\infty} \log(1 - \lambda \tilde{K}(\omega)) \frac{d\omega}{s^2 + \omega^2}, \quad (24)$$

where $\tilde{K}(\omega)$ is the Fourier transform of the kernel function

$$\tilde{K}(\omega) = \int_{-\infty}^{\infty} K(\tau) e^{i\omega\tau} d\tau. \quad (25)$$

Since $K(\tau)$ is even, $\tilde{K}(\omega)$ for real ω is real.

To get a hint on how this can be proved, we compare (24) with the well-known explicit expression for the full-space Green function G_∞ :

$$\begin{aligned} G_\infty(\tau) &\equiv \delta(\tau) + \Phi_\infty(\tau) \\ &= \delta(\tau) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda \tilde{K}(\omega)}{1 - \lambda \tilde{K}(\omega)} e^{-i\tau\omega} d\omega. \end{aligned} \quad (26)$$

It is easily obtained by Fourier transform from the full-space convolution equation (10). This comparison reveals that

$$\lambda \frac{\partial \log \bar{G}_0(s)}{\partial \lambda} = \bar{\Phi}_\infty(s), \quad (27)$$

where $\bar{\Phi}_\infty(s)$ is the Laplace transform of the non-singular part of $G_\infty(\tau)$.

Therefore to obtain (24), it suffices to prove (27). This is a non-linear relation between the Laplace transforms of full- and half-space Green functions. Somewhat unexpectedly, it involves a derivative with respect to the parameter λ . This differentiation with respect to λ is the clue to the proof of (24) by elementary means.

Now it is clear how to proceed. From the defining equation for the half-space surface Green function

$$G_0 = \lambda \Lambda G_0 + \delta(\tau) \quad (28)$$

we have

$$\frac{\partial G_0}{\partial \lambda} = \lambda \Lambda \frac{\partial G_0}{\partial \lambda} + \Lambda G_0. \quad (29)$$

According to (28), the term ΛG_0 in the RHS of Eq. (29) may be rewritten as $\lambda^{-1}(G_0(\tau) - \delta(\tau))$. Therefore, relation (29) may be considered as the equation for $\partial G_0/\partial \lambda$ of our usual form (2), with

$$Q(\tau) = \lambda^{-1}(G_0(\tau) - \delta(\tau)). \quad (30)$$

Introducing this Q into (5) and using symmetry of $G(\tau, \tau_1)$, we obtain

$$\lambda \frac{\partial G_0(\tau)}{\partial \lambda} = \int_0^\infty G(\tau_1, \tau) [G_0(\tau_1) - \delta(\tau_1)] d\tau_1. \quad (31)$$

In the integrand in the RHS of this equation there is a product of *three* functions G_0 , cf. Eq. (9). Now we take the Laplace transform of Eq. (31) and substitute for $\bar{G}(\tau_1, s)$ in the RHS its expression from (17). Interchanging the order of integration in τ_1 and in t in the double integral which thus appears, we get an integral in τ_1 of the product of two functions G_0 , in which Eq. (13) enables one to recognize $\bar{\Phi}_\infty(t)$. Hence this manipulation yields

$$\lambda \frac{\partial \bar{G}_0(s)}{\partial \lambda} = \bar{G}_0(s) \int_0^\infty \bar{\Phi}_\infty(t) e^{-st} dt, \quad (32)$$

which is the same as (27). Thus Eq. (24) is proved.

Knowledge of $\bar{G}_0(s)$ for real s is not yet the complete solution of the basic equation (1). The surface Green function $G_0(\tau)$ is still to be found. However, we made a large step ahead. As it was already mentioned, Eq. (19) essentially expresses, in terms of $\bar{G}_0(s)$, the intensity of radiation emergent from the atmosphere with exponentially distributed primary sources. By explicitly obtaining $\bar{G}_0(s)$ we thus have found this intensity.

Two comments seem in order. First, we note a very wide domain of applicability of Eq. (24). It applies to Eq. (1) with an arbitrary symmetric kernel. The kernel function $K(\tau)$ may

have either exponential or algebraic decrease at infinity. It is not necessary for $K(\tau)$ to be non-negative or to be representable as a superposition of exponentials. The second point which we would like to make is that in the derivation of Eq. (24) neither Eq. (19) nor the non-linear H -equation was used. Indeed, this latter equation has not yet appeared in our analysis, and it could not, since it applies only to the kernels which are Laplace-type integrals (for $\tau \geq 0$).

5. Resolvent function for Laplace-type kernels

To reach our final goal of obtaining half-space Green function, from this point on we restrict our analysis to even kernel functions $K(\tau)$ which, for $\tau \geq 0$, are the Laplace-type integrals, i.e. to $K(\tau)$ of the form

$$K(\tau) = \int_0^\infty k(s) e^{-|\tau|s} ds, \quad (33)$$

with possibly (but not necessarily) $k(s) \equiv 0$ for some part of $\mathbb{R}^+$. The function $k(s)$ may also be of the form $k(s) = (s_0/2) \delta(s - s_0)$ or contain a contribution of such δ -terms. The kernel functions (33) may be both exponentially and algebraically decreasing at infinity.

For the kernels (33), the non-singular part of the infinite-medium Green function G_∞ is also a superposition of exponentials:

$$G_\infty(\tau) = \delta(\tau) + \int_0^\infty g(s) e^{-|\tau|s} ds. \quad (34)$$

This may be shown by deforming the path of integration in calculating the inverse Fourier transform in (26). Here, $g(s)$ is a function explicitly expressed in terms of $k(s)$. It may contain δ -functions (even when there are no δ -functions in $k(s)$). These δ -terms in $g(s)$ produce isolated exponentials in G_∞ . Essentially, these are the pole terms.

The function $g(s)$ will be assumed known. Its explicit expression does not play any role in our further analysis. For the sake of completeness, it is given in Appendix B. In Appendix C it is used to obtain, by elementary means, a representation of the H -function alternative to (24) which applies, of course, only to Laplace-type kernels (33).

Our aim is to find the half-space surface Green function $G_0(\tau)$. It is natural to assume that

$$G_0(\tau) = \delta(\tau) + \int_0^\infty g(s) X(s) e^{-\tau s} ds, \quad \tau \geq 0, \quad (35)$$

where $X(s)$ is an unknown function. The problem is to find this $X(s)$.

Equation (13) relating G_0 to G_∞ enables one to solve this problem surprisingly easily. We introduce into (13) G_∞ and G_0 from (34) and (35), respectively, and perform integration with respect to t in the RHS. The resulting equation is the equality of two Laplace transforms. We conclude that the integrands of the transforms are equal, which leads to

$$X(s) \left(1 + \int_0^\infty \frac{X(s')}{s + s'} g(s') ds' \right) = 1. \quad (36)$$

But, by definition, the Laplace transform of (35) is $\bar{G}_0(s)$, and hence

$$\bar{G}_0(s) = 1 + \int_0^\infty \frac{X(s')}{s+s'} g(s') ds'. \quad (37)$$

Comparison of the last two equations immediately reveals that $X(s) = 1/\bar{G}_0(s)$, so that

$$G_0(\tau) = \delta(\tau) + \int_0^\infty g_0(s) e^{-\tau s} ds, \quad \tau \geq 0, \quad (38)$$

with

$$g_0(s) = \frac{g(s)}{\bar{G}_0(s)}. \quad (39)$$

This is the end of the story. The surface Green function G_0 and hence, according to Eq. (9), the Green function G as well, are found explicitly.

6. H -equations

And where is the celebrated Ambartsumian–Chandrasekhar H -equation? Usually it is taken for granted that it is an intrinsic part of the resolvent method, its soul, so to say. Our version of this method completely by-passes its use. But it is, of course, nearby.

In standard (Sobolev's) exposition of the resolvent method (Sobolev 1956, 1972), the non-linear H -equation indeed plays the central role. We yielded this role to the non-linear equation (13) relating G_0 and G_∞ . This gave us the possibility to use full-space solutions as the basis for the construction, by elementary means, of their half-space counterparts, and to avoid any factorization. In the usual formulation of the resolvent method, it is the H -equation which serves as a means of factorization. But this equation is useful in many other respects as well, and that is why we include its brief discussion.

The standard non-linear H -equation applies to Laplace-type kernels (33). It is derived as follows. Multiplying Eq. (18) by $k(s_1)$ and integrating with respect to s_1 over $\mathbb{R}^+$, we get Eq. (2) with $Q(\tau) = K(\tau)$, the solution of which is evidently $\lambda^{-1} \Phi(\tau)$. Therefore,

$$G_0(\tau) = \delta(\tau) + \lambda \int_0^\infty \bar{G}(\tau, s_1) k(s_1) ds_1. \quad (40)$$

In view of (19), the Laplace transformed version of this relation is

$$\bar{G}_0(s) = 1 + \bar{G}_0(s) \lambda \int_0^\infty \frac{\bar{G}_0(s_1)}{s+s_1} k(s_1) ds_1, \quad (41)$$

which is the usual H -equation, written, however, as the equation for $\bar{G}_0(s)$, and not for $H(s) \equiv \bar{G}_0(1/s)$. One can easily transform Eq. (41) into

$$\bar{G}_0(s) \left(\epsilon^{1/2} + s \lambda \int_0^\infty \frac{\bar{G}_0(s_1)}{s+s_1} k(s_1) \frac{ds_1}{s_1} \right) = 1 \quad (42)$$

by using the fact that

$$\lambda \int_0^\infty \bar{G}_0(s) k(s) \frac{ds}{s} = 1 - \epsilon^{1/2}. \quad (43)$$

Equation (43) immediately follows from Eq. (41) with $s = 0$ if use is made of (22).

In the conservative case ($\epsilon = 0$) Eq. (42) assumes the form

$$s \bar{G}_0(s) \int_0^\infty \frac{\bar{G}_0(s_1)}{s+s_1} k(s_1) \frac{ds_1}{s_1} = 1, \quad \epsilon = 0. \quad (44)$$

Equation (44) is a very useful instrument in studying the asymptotic behavior of $\bar{G}(s)$ for small s and thereby the behavior of $G_0(\tau)$ for large τ . We briefly touch these matters in the next section.

We note that, along with the non-linear Eqs. (41) and (42), the H -function obeys also a linear integral equation. It is derived in Appendix D, essentially by combining Eqs. (36) and (42).

7. Homogeneous equation

For simplicity, we consider only the conservative case of the homogeneous equation with the Λ -operator,

$$S_h = \Lambda S_h, \quad S_h(0) = 1. \quad (45)$$

Differentiating this equation with respect to τ and using the well-known expression for the derivative of the Λ -operator (Appendix A, Eq. (1)), we have

$$S'_h = \Lambda S'_h + K, \quad (46)$$

whence $S'_h = \Phi$, so that

$$S_h(\tau) = \int_0^\tau G_0(t) dt, \quad \epsilon = 0, \quad (47)$$

or, in view of (38),

$$S_h(\tau) = 1 + \int_0^\infty g_0(s) (1 - e^{-\tau s}) \frac{ds}{s}. \quad (48)$$

A question of substantial interest is the asymptotic behavior of $S_h(\tau)$ for $\tau \rightarrow \infty$. Although, of course, it may be found from Eq. (48), it is not the shortest way since this would require knowledge of $g_0(s)$ for small s .

Let us consider the functional form of the leading term of $S_h(\tau)$ for $\tau \rightarrow \infty$. The case of a finite second moment of the kernel function

$$\sigma^2 \equiv \int_{-\infty}^\infty \tau^2 K(\tau) d\tau \quad (49)$$

is trivial. By expanding $S_h(\tau')$ under the Λ -operator in (45) in the Taylor series around $\tau' = \tau$, for large τ we obtain the conservative diffusion equation $S''_h = 0$, whence we conclude that

$$S_h(\tau) \sim \tau, \quad \tau \rightarrow \infty. \quad (50)$$

The case of $\sigma = \infty$ is much more interesting, and here the H -equation demonstrates its power. Let us suppose that the functional form of the large- τ behavior of the Laplace-type kernel is

$$K(\tau) \sim \tau^{-2\gamma-1}(\log \tau)^{-2\beta}, \quad \tau \rightarrow \infty, \quad (51)$$

with $\gamma \in]0, 1[$ and $\beta \in \mathbb{R}$. Then from (33) it follows that $k(s) \sim s^{2\gamma}(\log 1/s)^{-2\beta}$ for small s . Using this, from the alternative form (44) of the conservative H -equation we easily find that $\bar{G}_0(s) \sim s^{-\gamma}(\log 1/s)^\beta$. But from (47) and (20) we have

$$s^{-1}\bar{G}_0(s) = \int_0^\infty S_h(\tau) e^{-s\tau} d\tau, \quad \epsilon = 0, \quad (52)$$

whence we can finally infer that for large τ the solution of the homogeneous equation (44) diverges as

$$S_h(\tau) \sim \tau^\gamma(\log \tau)^\beta, \quad \tau \rightarrow \infty. \quad (53)$$

The constant factors appearing in all these asymptotic expressions can be found explicitly in the course of the calculation just outlined. They depend on γ and may be expressed in terms of the Euler gamma-function $\Gamma(\gamma)$.

As far as we know, this is the shortest way of obtaining the asymptotic behavior of $S_h(\tau)$ for kernels with tails of the form (51). Such kernels are typical for problems of transfer of radiation in spectral lines with complete frequency redistribution. For further information see, e.g., Ivanov 1969, 1991; Mihalas 1978; Frisch 1988.

The evident conclusion is that trying to avoid the H -equation in all cases would be unwise. The H -equation is indeed a very powerful analytical (and numerical) tool — but not the best one for obtaining explicit exact solutions of half-space problems.

8. Discussion

There is practically no hope to find something really new in a field so old and so carefully studied by so many people from so many branches of physics as the linear half-space transport problems. The author is not sure whether in the whole paper there is at least something that one cannot find somewhere in the literature. But what is definitely true is that in radiative transfer the various elements have never been put together in such a combination.

Some historical comments may be useful. The resolvent method is usually associated with the name of Sobolev who, in the fifties, has found Eq. (9) and clearly demonstrated the fundamental role of the function $\Phi(\tau)$. This function is now known in radiative transfer theory as the Sobolev resolvent function (Sobolev 1956). Although his activity was indeed very important, he was not the first to introduce this function. According to Davison (1957, Chap. 6), for the particular case of the Schwarzschild–Milne kernel $K(\tau) = E_1(|\tau|)/2$, where $E_1(x)$ is the first exponential integral, Eq. (9) was obtained by Placzek in 1945. In 1955, Elliott found Eq. (9) for an arbitrary $K(\tau)$ (Elliott 1955). The very first hint at the resolvent approach may be

traced back as far as 1929, when Ambartsumian (1929; reprinted in Ambartsumian 1960) has shown that the solution of the classical conservative Milne problem $S_h(\tau)$ satisfies $S'_h(\tau) = \Phi(\tau)$. Sobolev's notation $\Phi(\tau)$ for the resolvent function was borrowed from this early Ambartsumian paper.

The first probabilistic proof of Eq. (9) was also given by Sobolev (1956, Sect. 9 of Chap. 6). The derivation presented in Sect. 2 first appeared in the author's book (Ivanov 1969, Sect. 5.2). It implicitly uses the translational invariance of a homogeneous half-space and thus incorporates the basic idea of Ambartsumian's invariance principle. The non-linear Eq. (13) relating G_∞ and G_0 may also be found in the author's book (Sect. 5.2). Demonstration of the fact that knowledge of full-space solutions enables one to construct solutions for the half-space by elementary means was first given a quarter of a century ago (Ivanov 1969, Sects. 5.3 and 5.4). The derivation of the explicit expression (38) for $G_0(\tau)$ which we give here is a substantial simplification and clarification of our earlier one.

It seems that the derivation of the explicit expression (24) for the H -function given in Sect. 4 is really new. However, Eq. (27), with its unexpected derivative with respect to the parameter λ , which is at the core of the derivation, for the particular case of Laplace-type kernels was found earlier (see Ivanov 1969, Sect. 5.4). The physical significance of this remarkable formula is unclear. As regards the result itself, i.e., the expression (24) for the H -function, it goes back at least to Fock (1944); later it appeared in Chandrasekhar's studies (Chandrasekhar 1950, Sect. 39).

The history of the invariance principles and the non-linear H -equation which is closely related to them is now well documented (Mnatsakanian & Pickichian 1989). The possibility of using the H -equation for obtaining asymptotic solutions of line transfer problems with complete frequency redistribution is known since the beginning of the sixties. Indeed this was the way how they were first found; the explicit solutions of these problems were then not yet known. Thus, the asymptotic form $S_h(\tau) \sim \tau^{1/2}(\log \tau)^{1/4}$ for the case of line formation with complete frequency redistribution and the Doppler profile was found in the very first author's paper on radiative transfer (Ivanov 1960). Now there is a rich choice of instruments for asymptotic analysis of line transfer problems (see, e.g., Frisch 1988), but this earliest derivation is still unsurpassed in its simplicity.

Many of the results presented here may be (and indeed have been) substantially generalized. Thus, generalizations of the basic formula (9) are known for the matrix case, the case of non-symmetric kernel function and a finite (rather than semi-infinite) integration interval in the basic integral equation (1); the condition of Hölder continuity used in Appendix B may be relaxed etc. The discussion of these matters from the viewpoint of a mathematician may be found in Greenberg et al. 1987.

Although throughout the paper we used the terminology of radiative transfer, the results are general enough to be used for solving half-space problems appearing in other branches of physics such as rarefied gas dynamics, wave propagation etc.

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Appendix A: half-space Green functions: relation between G and G_0

Let D be the operator of differentiation with respect to the space variable. We first establish that

$$D\Lambda f = \Lambda Df + f(0)K, \quad (1)$$

provided $f(0)$ is finite. The most natural way of proving this relation would be by integrating the identity

$$f(\tau') \frac{\partial}{\partial \tau} K(\tau - \tau') = -f(\tau') \frac{\partial}{\partial \tau'} K(\tau - \tau') \quad (2)$$

with respect to τ' over $\mathbb{R}^+$ and then transforming the RHS by integration by parts. There is, however, a hitch: the integral on the left may diverge. This is the case, e.g., for the Milne kernel $K(\tau) = E_1(|\tau|)/2$. The problem is, of course, in the validity of interchanging integration in τ' and τ -differentiation. More accurate treatment may be found in standard texts on radiative transfer, e.g., Kourganoff 1952, Sect. 13; Sobolev 1956, Sect. 4 of Chap. 3. In the usual derivation we 1) split the domain of τ' -integration in two parts, $[0, \tau]$ and $[\tau, \infty[$; 2) use the arguments of the kernel function in each of the integrals, $\tau - \tau'$ and $\tau' - \tau$ respectively, as the new integration variables; 3) perform the τ -differentiation; the term with $f(0)$ appears; 4) in both integrals come back to the original integration variables, and finally 5) join the two integrals into one. The essence of Eq. (1) is completely lost in this derivation.

The simplest way to get (1) is to proceed directly from the definition of the derivative. We denote $f_1 \equiv \Lambda f$. Then

$$f_1(\tau + \Delta\tau) = \int_0^\infty K(\tau + \Delta\tau - \tau') f(\tau') d\tau', \quad (3)$$

whence

$$f_1(\tau + \Delta\tau) = \int_{-\Delta\tau}^\infty K(\tau - \tau') f(\tau' + \Delta\tau) d\tau'. \quad (4)$$

The origin of both terms in the RHS of (1) is now clear. Equation (4) shows that the real root of the remarkable formula (1) is the fact that K is a displacement kernel. It explains the fundamental role of the differentiation formula (1) in establishing Eq. (9) which expresses a sort of translational invariance, as we shall now show.

Let us introduce the operator of infinitesimal translation relative to the medium of the source at τ_1 and the detector at τ , with $\tau - \tau_1$ kept fixed:

$$T = \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau_1}. \quad (5)$$

Using (1) and the fact that $T\delta(\tau - \tau_1) = 0$, since evidently $TF = 0$ for an arbitrary F depending on $\tau - \tau_1$, we have from Eq. (4), for $\tau_1 > 0$,

$$TG = \lambda \Lambda TG + \Phi(\tau_1) \lambda K(\tau). \quad (6)$$

We used here also Eqs. (7) and (8) with $\tau = \tau_1 > 0$. According to Eqs. (28) and (8), the resolvent function Φ satisfies the equation

$$\Phi = \lambda \Lambda \Phi + \lambda K. \quad (7)$$

Comparison of (6) and (7) reveals that

$$TG(\tau, \tau_1) = \Phi(\tau) \Phi(\tau_1), \quad (8)$$

whence

$$G(\tau, \tau_1) = \delta(\tau - \tau_1) + \Phi(\bar{\tau} - \underline{\tau}) + \int_0^{\underline{\tau}} \Phi(t) \Phi(\bar{\tau} - \underline{\tau} + t) dt, \quad (9)$$

with $\bar{\tau} = \max(\tau, \tau_1)$, $\underline{\tau} = \min(\tau, \tau_1)$. This is identical to Eq. (9) which we prove. To make this clear, we rewrite Eq. (9) as

$$G(\tau, \tau_1) = \int_0^{\underline{\tau}} G_0(t') G_0(\bar{\tau} - \underline{\tau} + t') dt' \quad (10)$$

and note that if $\tau \neq \tau_1$, then of the two δ -functions implicitly present in the integrand of Eq. (10), $\delta(t')$ and $\delta(\bar{\tau} - \underline{\tau} + t')$, only the first contributes to the integral thus producing off-integral Φ -term in Eq. (9); if $\tau = \tau_1$, the two δ -functions in the integrand give rise to the δ -term in (9).

Appendix B: infinite-medium Green function G_∞ for Laplace-type kernels

We introduce the dispersion function $\mathcal{L}(z)$ of the kernel $K(\tau)$ by

$$\mathcal{L}(z) = 1 - \lambda \tilde{K}(iz), \quad (11)$$

where $\tilde{K}(\omega)$ is the Fourier transform of $K(\tau)$, Eq. (25). For the Laplace-type kernels of the form of (33) we have

$$\mathcal{L}(z) = 1 - \lambda \int_0^\infty \left(\frac{1}{s' - z} + \frac{1}{s' + z} \right) k(s') ds'. \quad (12)$$

Hence, in the complex plane z cut along the real axis $\mathcal{L}(z)$ is an analytic function of z provided $k(s)$ is Hölder continuous.

The limiting values of $\mathcal{L}(z)$ as z approaches the cut are given by

$$\mathcal{L}(s \pm i0) \equiv \mathcal{L}^\pm(s) = 1 - \lambda l(s) \mp i\pi\lambda k(|s|), \quad (13)$$

where s is real and $l(s)$ is the Cauchy principal value integral

$$l(s) = 2 \int_0^\infty \frac{k(s')}{s'^2 - s^2} s' ds'. \quad (14)$$

We note that since $k(0) = 0$, we have

$$\mathcal{L}^+(0) = \mathcal{L}^-(0). \quad (15)$$

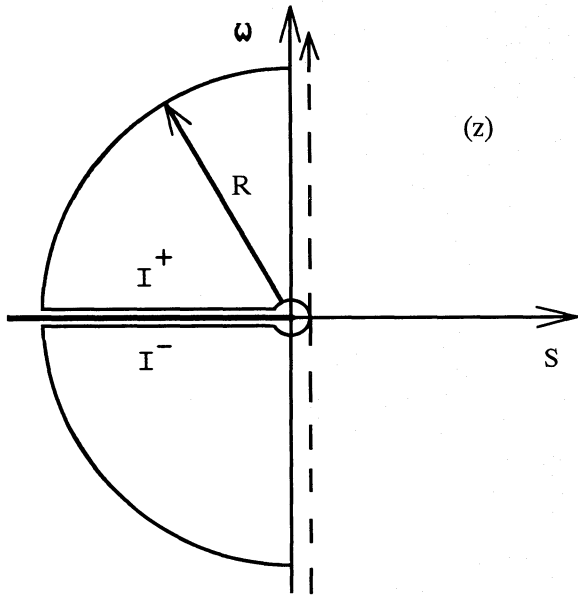


Fig. 1. Path of integration in evaluating the explicit expression for the infinite-medium Green function $G_\infty(\tau)$

Now we rewrite Eq. (26) for $\tau \geq 0$ as

$$G_\infty(\tau) = \delta(\tau) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\mathcal{L}(i\omega)} - 1 \right) e^{-i\tau\omega} d\omega. \quad (16)$$

To obtain the solution of Eq. (10) which tends to zero as $\tau \rightarrow \infty$, we deform the path of integration as shown in Fig. 1. The integral along the semi-circle of a large radius R tends to zero as $R \rightarrow \infty$ since $\mathcal{L}(z) \rightarrow 1$ for $z \rightarrow \infty$. As a result, the integral in (16) reduces to the sum of the integrals I^+ and I^- along the negative real semi-axis above and below the cut, respectively, and the integral I_0 along the small circle around the origin. We have

$$I^\pm(\tau) = \mp \frac{1}{2\pi i} \int_{-\infty}^0 \left(\frac{1}{\mathcal{L}^\pm(s)} - 1 \right) e^{\tau s} ds, \quad (17)$$

whence, using (13), one can easily find that

$$I^+(\tau) + I^-(\tau) = \lambda \int_0^\infty \frac{k(s)}{|\mathcal{L}^\pm(s)|^2} e^{-\tau s} ds, \quad (18)$$

where $|\mathcal{L}^\pm(s)|$ is the modulus of $\mathcal{L}^\pm(s)$:

$$|\mathcal{L}^\pm(s)|^2 = [1 - \lambda l(s)]^2 + [\lambda \pi k(s)]^2. \quad (19)$$

As to I_0 , in the non-conservative case it is zero, since, according to (12),

$$\mathcal{L}(0) = \epsilon. \quad (20)$$

In the conservative case ($\epsilon = 0$) it may happen that I_0 does not vanish. However, the cases for which $I_0 \in]0, \infty[$ so far had not been encountered in radiative transfer problems. On the contrary, the case of $I_0 = \infty$ is of real interest. It appears, e.g., for

conservative monochromatic scattering. If $I_0 = \infty$, then $G_\infty(\tau)$ diverges (although $G_0(\tau)$ remains finite; for further details, see Sects. 4.1, 5.3 and 5.5 in the English version of Ivanov 1969). For simplicity, we assume here that $I_0 = 0$.

The transformation which we have just described yields ($\tau \geq 0$)

$$G_\infty(\tau) = \delta(\tau) + \lambda \int_0^\infty \frac{k(s)}{|\mathcal{L}^\pm(s)|^2} e^{-\tau s} ds. \quad (21)$$

Comparison with Eq. (34) reveals that

$$g(s) = \frac{\lambda k(s)}{|\mathcal{L}^\pm(s)|^2}. \quad (22)$$

This is the final result.

We note that there may exist such an interval $[a, b]$, $a, b \in \mathbb{R}^+$, that $k(s) \equiv 0$ for $s \in [a, b]$. This does *not* necessarily imply that $g(s) \equiv 0$ for all s from this interval. Indeed, if there are such $s_0 \in [a, b]$ that

$$|\mathcal{L}^\pm(s_0)| = 0, \quad (23)$$

so that $k(s_0) = 0$ and $\lambda l(s_0) = 1$, then (22) yields 0/0. Such points produce δ -functions in $g(s)$ centered at $s = s_0$. These, in turn, give rise to isolated exponentials in (21). The equation $\lambda l(s_0) = 1$, or

$$2\lambda \int_0^\infty \frac{k(s)}{s^2 - s_0^2} s ds = 1, \quad (24)$$

is known as the characteristic equation of the problem at hand. It can be used to find all such exponentials (if there are any).

The situation which we have just discussed is encountered, e.g., for isotropic monochromatic scattering. In this case we have the Schwarzschild-Milne kernel

$$K(\tau) = \frac{1}{2} E_1(|\tau|) = \frac{1}{2} \int_1^\infty e^{-|\tau|s} \frac{ds}{s}, \quad (25)$$

so that $k(s) \equiv 0$ for $0 \leq s < 1$ and $k(s) = 1/(2s)$ for $s \geq 1$. The characteristic equation (24) assumes the form

$$\frac{\lambda}{2s_0} \log \frac{1+s_0}{1-s_0} = 1. \quad (26)$$

It has one non-negative root which gives rise to an isolated exponential term in $G_\infty(\tau)$. It is this term that determines the asymptotic behavior of $G_\infty(\tau)$ for $\tau \rightarrow \infty$, namely, $G_\infty(\tau) \sim O(e^{-s_0\tau})$.

It is useful to formulate the main result of this Appendix in operator form. Due to the multiple scattering in full space, $k(s)$ is transformed into $g(s)$, or, in operator notation,

$$g = \lambda M_\infty k. \quad (27)$$

Let L be the operator of Laplace transformation. Combining Eqs. (10), (33) and (34), one can readily show that

$$M_\infty = L^{-1}(E - \lambda \Lambda_\infty)^{-1} L, \quad (28)$$

where E is the identity operator. The equation (22) assumes the form

$$M_{\infty} k = \frac{k}{|\mathcal{L}^{\pm}|^2}. \quad (29)$$

One final remark. Equation (29) holds only for smooth enough $k(s)$, namely Hölder continuous $k(s)$. The case of a "bad" $k(s)$, with jumps, infinite discontinuities and δ -terms, has not been studied so far. The theory of analytic functions will be of little use in this situation. Numerical experiments may help to get a feeling of what really happens here, i.e., how $k(s)$ is transformed into $g(s)$. Such experiments are highly desirable.

Appendix C: H -functions for Laplace-type kernels

Along with the explicit expression (24) for the H -function which is valid for arbitrary $K(\tau)$, for Laplace-type kernels there is an alternative representation.

Let $\theta(s)$ be the argument of the complex-valued function $\mathcal{L}^-(s)$ defined by Eq. (13):

$$\theta(s) = \arctan \frac{\lambda \pi k(s)}{1 - \lambda l(s)}, \quad 0 \leq \theta(s) \leq \pi. \quad (30)$$

Elementary calculation gives

$$\frac{\partial \theta(s)}{\partial \lambda} = \frac{\pi k(s)}{|\mathcal{L}^{\pm}(s)|^2}. \quad (31)$$

Comparison with (22) yields

$$g(s) = \frac{\lambda}{\pi} \frac{\partial \theta(s)}{\partial \lambda}, \quad (32)$$

so that Eq. (34) for $\tau \geq 0$ may be rewritten also as

$$\Phi_{\infty}(\tau) \equiv G_{\infty}(\tau) - \delta(\tau) = \frac{\lambda}{\pi} \int_0^{\infty} \frac{\partial \theta(s')}{\partial \lambda} e^{-\tau s'} ds', \quad (33)$$

and hence the Laplace transform of $\Phi_{\infty}(\tau)$ assumes the form

$$\bar{\Phi}_{\infty}(s) = \frac{\lambda}{\pi} \int_0^{\infty} \frac{\partial \theta(s')}{\partial \lambda} \frac{ds'}{s + s'}. \quad (34)$$

Now, from (10) we have $G_{\infty}(\tau) = \delta(\tau)$ for $\lambda = 0$, so that $\bar{G}_{\infty}(s) = 1$ for $\lambda = 0$. Using this, from (27) and (34) we finally obtain

$$\log \bar{G}_0(s) = \frac{1}{\pi} \int_0^{\infty} \theta(s') \frac{ds'}{s + s'}. \quad (35)$$

This is the desired alternative integral representation of the H -function.

Appendix D: linear equation for the H -function

The standard non-linear H -equation

$$\bar{G}_0(s) = 1 + \bar{G}_0(s) \lambda \int_0^{\infty} \frac{\bar{G}_0(s_1)}{s + s_1} k(s_1) ds_1 \quad (36)$$

involves an integral with the weighting factor $k(s)$. This $k(s)$ is the inverse Laplace transform of the kernel function $K(\tau)$, cf. Eq. (33).

The non-linear equation (36) for a closely related function

$$X(s) \equiv \frac{1}{\bar{G}_0(s)} \quad (37)$$

at first glance resembles (36) in its general structure. The weighting function of the integral appearing in this equation is, however, not $k(s)$, but $g(s)$, i.e., the inverse Laplace transform of $\Phi_{\infty}(\tau)$, cf. Eq. (34). Rewritten in terms of $\bar{G}_0$, Eq. (36) becomes

$$\bar{G}_0(s) = 1 + \int_0^{\infty} \frac{g(s')}{\bar{G}_0(s')} \frac{ds'}{s + s'}. \quad (38)$$

Equation (38) is easily transformed into a *linear* equation for $\bar{G}_0(s)$. Indeed, from (36) we have

$$\frac{1}{\bar{G}_0(s')} = 1 - \lambda \int_0^{\infty} \frac{\bar{G}_0(s_1)}{s' + s_1} k(s_1) ds_1. \quad (39)$$

Introducing this in the integrand in (38), using (43) and performing elementary partial fraction rearrangements, one can show that

$$\bar{G}_0(s) = \sqrt{\epsilon} \bar{G}_{\infty}(s) + \lambda \int_0^{\infty} K(s, s_1) \bar{G}_0(s_1) k(s_1) \frac{ds_1}{s_1}, \quad (40)$$

where

$$K(s, s_1) = K(s_1, s) = \frac{s \bar{G}_{\infty}(s) - s_1 \bar{G}_{\infty}(s_1)}{s - s_1} \quad (41)$$

and $\bar{G}_{\infty}(s)$ is the Laplace transform of $G_{\infty}(\tau)$:

$$\bar{G}_{\infty}(s) = 1 + \int_0^{\infty} \frac{g(s')}{s + s'} ds'. \quad (42)$$

Equation (40) can be rewritten in a slightly different form

$$\bar{G}_0(s) = \bar{G}_{\infty}(s) + \lambda \int_0^{\infty} K_1(s, s_1) \bar{G}_0(s_1) k(s_1) ds_1 \quad (43)$$

with

$$K_1(s, s_1) = K_1(s_1, s) = \frac{\bar{G}_{\infty}(s) - \bar{G}_{\infty}(s_1)}{s - s_1}. \quad (44)$$

Although particular cases of Eqs. (40) and (43) or their slightly modified versions may be found in the literature (e.g. for the Schwarzschild-Milne kernel), these exotic equations are not widely known.

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