GROWTH OF PLANETS FROM PLANETESIMALS

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The formation of terrestrial planets and the cores of Jovian planets is reviewed in the framework of the planetesimal hypothesis, wherein planets are assumed to grow via the pairwise accumulation of small solid bodies. The rate of (proto)planetary growth is determined by the size and mass of the protoplanet, the surface density of planetesimals, and the distribution of planetesimal velocities relative to the protoplanet. Planetesimal velocities are modified by mutual gravitational interactions and collisions, which convert energy present in the ordered relative motions of orbiting particles (Keplerian shear) into random motions and tend to reduce the velocities of the largest bodies in the swarm relative to those of smaller bodies, as well as by gas drag, which damps eccentricities and inclinations. The evolution of the planetesimal size distribution is determined by the gravitationally enhanced collision cross-section, which favors collisions between planetesimals with smaller velocities. Deviations from the 2-body approximation for the collision cross-section are caused by the central star's tidal influence; this limits the growth rates of protoplanets. Runaway growth of the largest planetesimal in each accretion zone appears to be a likely outcome. The subsequent accumulation of the resulting protoplanets leads to a large degree of radial mixing in the terrestrial planet region, and giant impacts are probable. Gravitational perturbations by Jupiter probably were responsible for preventing runaway accretion in the asteroid belt, but detailed models of this process need to be developed. In particular, the method of removal of most of the condensed matter (expected in a nebula of slowly varying surface density) from the asteroid region and the resulting degree of radial mixing in the asteroid belt have yet to be adequately modeled. Accumulation of Jupiter's core before the dispersal of the solar nebula may require more condensable material at 5 AU than predicted by standard minimum-mass solar-nebula models.

I. INTRODUCTION

The nearly circular and coplanar orbits of the planets argue for planetary formation in a flattened disk orbiting the Sun (Kant 1755; Laplace 1796). Astrophysical evidence suggests that such disks are the natural byproducts of the collapse of molecular cloud cores leading to star formation (Chapters by Beckwith and Sargent and by Basri and Bertout). The most highly developed theory for explaining planetary growth within such a circumstellar disk is accretion (aggregation) of solid planetesimals via binary collisions, followed in the case of Jovian type planets by accretion of gas onto solid cores (Safronov 1972; Hayashi et al. 1985).

In this chapter, we review the dynamics of the accretion process from kilometer sized planetesimals to terrestrial planets and the cores of giant planets. The formulas presented herein are valid for single stars of any mass; however, the complexities of planetary accretion in multiple star systems are not treated. We assume as our initial conditions the presence of a disk of stellar composition in orbit about the star. Moreover, our calculations begin when the bulk of the condensed material in the nebula has settled out and agglomerated into bodies at least ~1 km in size. The microphysics of the growth of the sub-centimeter grains is very different than the dynamical processes important to later stages of planetary accretion, and growth in the intermediate-size range is believed to be very rapid (Chapter by Weidenschilling and Cuzzi). Our review also neglects the accretion of the gaseous envelopes of the giant planets, except insofar as an atmosphere may enhance the planetesimal accretion cross-section of a protoplanet. This final stage of giant planet growth is reviewed in the Chapter by Podolak et al.

Strictly speaking, the planet formation process is unlikely to be as purely sequential as we have lain out. Grain growth and even the accretion of large planetesimals described in detail herein may well begin during the epoch when a protoplanetary disk is still accreting and redistributing matter (cf. Chapters by Shu and by Adams and Lin). Given that the theories of each of these epochs are still rather primitive, a sequential study of each stage is probably adequate. However, to the extent that planetary growth depends on, e.g., the initial size distribution of planetesimals, we must recognize that various processes currently being treated as separate events occur simultaneously and affect each other.

The star's gravity is the dominant force upon planetesimals, and other forces are generally included as perturbations on the planetesimal's Keplerian orbit. The dominant perturbations to a planetesimal's heliocentric trajectory are usually due to gravitational attraction of other planetesimals and protoplanets. (We use the term protoplanets to refer to exceptionally large planetesimals, not the giant gaseous protoplanets once popularized by Cameron [1962].) When a single protoplanet is the dominant perturber in a given region of the protoplanetary disk, it is convenient to treat its perturbations separately. The dominant nongravitational forces upon planetesimals are mutual inelastic collisions (which may lead to accretion and/or fragmentation), and gas drag. Whole body magnetic forces are believed to be negligible for the dynamics of bodies of kilometer size and larger. However, if electromagnetic induction heating (Herbert 1989) is sufficient to melt planetesimal interiors, then the outcome of physical collisions could be altered.

The distribution of planetesimal velocities is one of the key factors which control the rate of planetary growth. In Sec. II, we review the physical factors important to determining the equilibrium velocity distribution of a swarm of planetesimals of various sizes when accretion is neglected. The growth rate of a protoplanet in a uniform surface density disk of planetesimals with known velocity dispersion is discussed in Sec. III. In Sec. IV, we follow the simultaneous evolution of planetesimal masses and velocities and show that under a wide variety of initial conditions the largest body in any given accretion zone grows very rapidly and "runs away" from the mass distribution of other accreting bodies in its region of the solar system. The limits to such runaway accretion are quantified in Sec. V. In Sec. VI, we review models of the final stages of planetary accretion, with emphasis on the growth times of planets and the possibility of giant impacts. We conclude with a summary of the major results of planetesimal dynamics and a list of outstanding questions in Sec. VII.

II. PLANETESIMAL VELOCITIES FOR A STATIC MASS DISTRIBUTION

The simplest analytic approach for calculating the evolution of planetesimal velocities uses a "particle-in-a-box" approximation in which the evolution of the mean square planetesimal velocities are calculated via the methods of the kinetic theory of gases. Originally, the particle-in-a-box calculations were developed by Safronov (1972) using relaxation time arguments similar to those used by Chandrasekhar (1942) in stellar dynamics. More recently, these calculations have been refined by employing modern kinetic theory methods (Hornung et al. 1985; Stewart and Wetherill 1988; Barge and Pellat 1990; Ida 1990). A kinetic theory approach appears to be the only viable method for treating the initial stages of planetesimal accumulation because the number of initial planetesimals is extremely large. During the final stages of planetesimal accumulation, the number of planetesimals eventually becomes small enough that a more direct treatment of individual planetesimal orbits is feasible. With a modest number of planetesimals, the most straightforward approach would be a numerical n-body integration of the planetesimal orbits, but the exceedingly long time scales required for the final stages of planetary accumulation (e.g., between 10 and 100 Myr in the inner solar system) severely limit the usefulness of this method. An alternative approach that has enjoyed considerable success is to assume that the planetesimals follow slowly precessing elliptic orbits that are occasionally altered by rare close encounters with other planetesimals. A Monte Carlo procedure is then used to choose successive pairs of planetesimals that interact according to the two-body gravitational scattering formula. This Monte Carlo approach has been extensively developed by Wetherill (1980*b*,1985,1986,1988,1990*b*).

A. Random Velocity Distribution

In the particle-in-a-box approximation, one ignores the details of individual planetesimal orbits and uses a probability density to describe the distribution of orbital elements in the planetesimal population. Specifically, the orbital perihelia and longitudes of the ascending node are assumed to be random and the orbital eccentricities e and inclinations i are assumed to be Rayleigh distributed,

$$f(e,i) = 4\,\frac{\sigma}{m}\frac{ei}{\langle e^2\rangle\langle i^2\rangle}\exp\left[-\frac{e^2}{\langle e^2\rangle}-\frac{i^2}{\langle i^2\rangle}\right] \tag{1}$$

where m is the mass of the individual planetesimals and σ is the surface mass density of planetesimals with a particular semimajor axis; $\langle e^2\rangle$ and $\langle i^2\rangle$ are the mean square eccentricity and inclination. Although the form of Eq. (1) is difficult to justify rigorously, the randomization of orbits caused by planetesimal interactions in n-body simulations has been found to yield orbital distributions similar to Rayleigh distributions (Wetherill 1980*b*; Ida and Makino 1992*a*).

Another important property of planetesimal orbits is the fact that planetesimals with different semimajor axes orbit the star with different mean velocities. This Keplerian shear is usually introduced into a local velocity distribution by postulating a local mean velocity that reproduces the differential rotation between coplanar circular orbits. Because the orbital phase angles are averaged out in this approximation, the deviations from coplanar circular orbits are reduced to a distribution of random velocities relative to the local mean velocity of a circular orbit. The distribution of random velocities that is locally equivalent to Eq. (1) is a triaxial Gaussian distribution in cylindrical coordinates,

$$f_o(z,\mathbf{v}) = \frac{\Omega\sigma}{2\pi^2{c_r}^2{c_z}^2 m}\exp\left[-\frac{{v_r}^2+4{v_\theta}^2}{2{c_r}^2}-\frac{{v_z}^2+\Omega^2z^2}{2{c_z}^2}\right] \tag{2}$$

where $2{c_r}^2 = \langle e^2\rangle {v_K}^2$, $2{c_z}^2 = \langle i^2\rangle {v_K}^2$, v_θ is the azimuthal component of velocity relative to the local circular Keplerian velocity, $v_K = (GM_\star/r)^{1/2}$, and $\Omega = v_K/r$ is the orbit frequency. The integral of f_o over velocity space yields the local number density,

$$n = \int d^3 v f_o = \frac{\Omega\sigma}{\sqrt{2\pi}\,c_z m}\exp\frac{-\Omega^2 z^2}{2c_z^2}. \tag{3}$$

B. Velocity Evolution

Given the local random velocity distribution in Eq. (2), one can write down a kinetic equation for the time evolution of the random velocity distribution caused by mutual planetesimal interactions:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_r f-\left[\frac{GM_\star\mathbf{r}}{r^3}+(\mathbf{v}+\mathbf{v}_K)\cdot\nabla_r\mathbf{v}_K\right]\cdot\nabla_v f = \frac{\delta f}{\delta t}\Big|_{\rm coll}+\frac{\delta f}{\delta t}\Big|_{\rm grav}. \tag{4}$$

The two terms on the right-hand side of the kinetic equation denote the two kinds of planetesimal interactions: (1) physical collisions which dissipate some or all of the relative kinetic energy of the colliding bodies; and (2) gravitational scattering which conserves the relative kinetic energy but results in a rotation of the relative velocity vector. Physical collisions are usually modeled with a Boltzmann collision operator for hard spheres that has been modified to allow for inelastic collisions (see Trulsen 1971; Hornung et al. 1985). Most published calculations of planetesimal accumulation assume collisions are completely inelastic because physically plausible rebound velocities rarely exceed the mutual escape velocity except for the very smallest planetesimals in the distribution (cf. Sec. III). Violent collisions that produce a size distribution of collision fragments are more likely to occur once sizeable protoplanets are formed. In order to account for the gravitational enhancement of the collision cross-section, a correction factor must be applied to the hard sphere collision operator. An accurate determination of this enhancement factor is rather difficult in general, owing to the tidal influence of the star; a detailed discussion of its calculation is presented in Sec. III.

Gravitational scattering between planetesimals can be modeled with a Fokker-Planck operator similar to that used in stellar dynamics (cf. Binney and Tremaine 1987). The use of a Fokker-Planck operator requires two assumptions. First, the result of a close encounter must be well approximated by the 2-body Rutherford scattering formula, which ignores the gravitational influence of the star for the duration of the encounter. Between successive encounters, the star's gravitational influence is properly taken into account by the left-hand side of Eq. (4). Numerical integrations of the 3-body problem indicate that the average perturbations given by the 2-body approximation are valid within a factor of 2, so long as the random velocities are greater than 0.07 times the surface escape velocity of the two bodies at contact (Wetherill and Cox 1984). More extensive numerical investigations of the 3-body problem support the accuracy of the Fokker-Planck relaxation and energy exchange rates provided that the encounter velocities do not become too small (Ida 1990). Second, the relative velocity between planetesimals must be primarily determined by their random velocities, rather than by Keplerian shear. This assumption is valid for the early stages of planetesimal accumulation, when a very large number of planetesimals can be found within a spherical volume of radius equal to the scale height c_z/Ω of the planetesimal disk.

The essential reason why the Fokker-Planck operator cannot correctly model a gravitational encounter more distant than a scale height is that the Fokker-Planck operator implicitly assumes that gravitational encounters are local events. In particular, it assumes that the two interacting planetesimals have the same local mean velocity, so that their relative velocity is entirely determined by the random velocity distribution (Eq. 2). This assumption is violated for distant encounters, where the relative velocity between planetesimals is largely determined by the difference in their semimajor axes (i.e., by Keplerian shear). Although it is possible to write down formal expressions for the velocity evolution due to distant encounters (see Chapter by Ohtsuki et al.), an explicit analytic formula is generally not available to replace the Rutherford scattering law, except in a few limiting cases.

One such limiting case occurs when the separation in semimajor axes is large compared to both the orbital eccentricities and the Hill sphere radius h (see Sec. III.C), and is also small compared to both semimajor axes (Hénon and Petit 1986; Hasegawa and Nakazawa 1990). Weidenschilling (1989) evaluated the contribution of distant gravitational encounters in this limit and concluded that they may safely be neglected compared to close gravitational encounters except when a few protoplanets are so large that their surface escape velocities exceed the local random velocity by a factor of $\geq$100. For these reasons, the maximum encounter distance is set equal to the disk scale height in the Fokker-Planck operator.

Once the interaction terms have been specified, an approximate solution to Eq. (4) can be obtained by linearizing the kinetic equation about the Gaussian velocity distribution f_o stated in Eq. (2):

$$\left[\frac{\partial}{\partial t} + 2\Omega v_\theta \frac{\partial}{\partial v_r} - \left(\frac{\Omega v_r}{2}\right)\frac{\partial}{\partial v_\theta}\right] f_1 = \frac{\delta f_o}{\delta t}\Big|_{\rm coll} + \frac{\delta f_o}{\delta t}\Big|_{\rm grav} \tag{5}$$

where $|f_1| << |f_o|$. Note that f_o does not appear on the left-hand side of Eq. (5) because it is an integral of the free orbital motion. Linearization is an excellent approximation because the orbit frequency greatly exceeds both the collision rate and relaxation rate due to gravitational scattering. Evolution equations for the mean square random velocities are obtained by taking second-order velocity moments of the linearized kinetic equation. The rate of change of the mean square random velocity is given by

$$\frac{\partial \langle n v^2 \rangle}{\partial t} = \int d^3 v \left(\frac{\delta f_1}{\partial t}\right) v^2. \tag{6}$$

In general, one obtains two coupled equations for the two components of the random velocity associated with the orbital eccentricity and inclination (Hornung et al. 1985). However, during the early stages of planetesimal accumulation the ratio of rms inclination to rms eccentricity is likely to be nearly constant due to an approximate equipartition of energy between the planar and vertical orbital motions. Barge and Pellat (1990) have derived a value of this ratio of approximately 0.6 by simultaneously solving coupled equations for the "thermal" motions in the vertical and horizontal directions. This value may be compared with values close to 0.5 that were obtained by Wetherill (1980*b*) using a Monte Carlo simulation, and by Ida and Makino (1992*a*) using direct n-body calculations. Calculations of steady-state planetesimal velocities indicate that the difference between 0.5 and 0.6 is unimportant compared to the other approximations described above.

Thus, setting $c_z/c_r = 0.5$ in Eq. (2), a single equation is derived for the velocity evolution of a test body of mass m_i interacting with a swarm of field bodies of mass m_k. Following Stewart and Wetherill (1988), we split the equation into four separate contributions and add a fifth term that models the velocity evolution caused by gas drag in the solar nebula. The rate of change in v_i, the rms velocity of bodies of mass m_i, is given by:

$$\frac{dv_i}{dt} = A + B + C + D + E \tag{7}$$

where the five terms on the right-hand side are:

1. *"Viscous stirring" resulting from gravitational scattering*:

$$A = \frac{3}{4}\frac{\sqrt{\pi}G^2}{v_i V_{ik}^3}\rho_k \ln\Lambda[(9L-12\sqrt{3})(m_i+m_k)v_i{}^2+(5L-4\sqrt{3})(m_k v_k{}^2-m_i v_i{}^2)] \tag{8}$$

where ρ_k is the density (specific gravity) of the planetesimals of mass m_k, $V_{ik}{}^2 \equiv v_i{}^2+v_k{}^2, L \equiv \ln[(2+\sqrt{3})/(2-\sqrt{3})] \approx 2.634$, and $\Lambda \equiv \sin(\Psi_{\max}/2)/\sin(\Psi_{\min}/2)$. The angle Ψ is related to the impact parameter b according to the Rutherford scattering formula,

$$\sin\left(\frac{\Psi}{2}\right) = \left(1+\frac{b^2}{b_o^2}\right)^{-1/2} \tag{9}$$

where $b_o = G(m_i + m_k)V_{ik}{}^{-2}$. The minimum deflection angle, $\Psi_{\min}$, is calculated with an impact parameter $b = b_{\max} = \max(v_i/\Omega, v_k/\Omega)$. The maximum deflection angle, $\Psi_{\max}$, is calculated using the impact parameter corresponding to the 2-body gravitational capture cross section,

$$b = b_{\min} = R_g = (R_i + R_k)\left[1 + \frac{2b_o}{(R_i + R_k)}\right]^{1/2}. \tag{10}$$

2. *Viscous stirring caused by inelastic collisions*:

$$B = \frac{\sqrt{\pi}}{8}\left(\sqrt{3} - \frac{5L}{12}\right)\frac{V_{ik}}{v_i}\rho_k R_g{}^2 \frac{m_k(v_i{}^2 - v_k{}^2) + 2m_i v_i{}^2}{(m_i + m_k)^2}. \tag{11}$$

3. *Velocity damping due to energy dissipated by inelastic collisions*:

$$C = -\sqrt{\pi}\left(\frac{11\sqrt{3}}{18} + \frac{L}{24}\right)\frac{V_{ik}}{v_i}\rho_k R_g{}^2 \frac{m_k(v_i{}^2 - v_k{}^2) + 2m_i v_i{}^2}{(m_i + m_k)^2}. \tag{12}$$

4. *Energy transfer from large bodies to small ones via dynamical friction*:

$$D = \frac{4\sqrt{\pi}\, L\, G^2}{v_i\, V_{ik}{}^3}\rho_k \ln\Lambda(m_k v_k{}^2 - m_i v_i{}^2). \tag{13}$$

5. *Energy damping caused by gas drag*:

$$E = -\pi \left(\frac{C_D}{2m_i}\right) \rho_g R_i{}^2 v_i (v_i + \eta) \tag{14}$$

where C_D is the drag coefficient, ρ_g is the gas density, and η is the velocity of a body moving in a circular Keplerian orbit relative to the that of the nebular gas, which orbits at a slower velocity because it is partially supported by thermal pressure (cf. Adachi et al. 1976; Weidenschilling 1977*a*).

The viscous stirring terms are so denoted because their sum is proportional to the product of the shear stress and the rate of strain generated by the differential rotation of the local mean velocity. The physical source of the energy for the viscous stirring terms is the kinetic energy contained in the sheared mean flow. Mutual planetesimal interactions transform this energy into the random motion associated with orbital eccentricities and inclinations. In a uniformly rotating disk, these two terms would vanish. The dynamical friction term tends to drive the system toward an equipartition of random kinetic energy. In a broad distribution of planetesimal sizes, energy equipartition is never actually achieved because the viscous stirring terms tend to drive the system away from the equipartitioned state towards a state in which velocities are independent of mass. For the special case of $m_i = m_k$ and $v_i = v_k$, these expressions are similar to those found by Safronov (1972) and Kaula (1979*a*) using relaxation time estimates. The equation for gas drag is adapted from the theory of Adachi et al. (1976). In this formula, the drag coefficient C_D appropriate for the large Reynolds number flows occurring around planetesimals in the circumstellar nebulae is ~0.5 (Whipple 1972).

The steady-state velocities predicted by Eqs. (7–14) are displayed in Fig. 1 for several different mass distributions. The mass distributions are labeled by the exponent q in the differential power law $dn/dm \propto m^{-q}$; the planetesimal masses range from 10^{18} to 9.766×10^{24} g in each case. In all the cases plotted, the largest planetesimals have the smallest velocities as a result of the transfer of energy from large bodies to small bodies via dynamical friction. This result is distinctly different from what one obtains from Safronov's relaxation time theory, which omitted dynamical friction. The power law with exponent $q = 2$ is a transitional case because equal amounts of mass are distributed in each logarithmic mass interval in that case. Mass distributions with $q > 2$ have most of their mass in the smaller bodies and are therefore more efficient at draining away the energy from the larger bodies. For large values of q, the velocity distribution achieves its maximum value at an intermediate mass planetesimal where the viscous stirring by gravitational scattering is strong relative to both inelastic collisions, which dominate for small planetesimals, and dynamical friction, which slows the largest bodies. Somewhat surprisingly, omitting the gas drag term from Eq. (14) would hardly change the velocity curves in Fig. 1 at all. Apparently, the absence of gas drag is mostly compensated by a larger energy loss due to inelastic collisions if the steady-state velocities

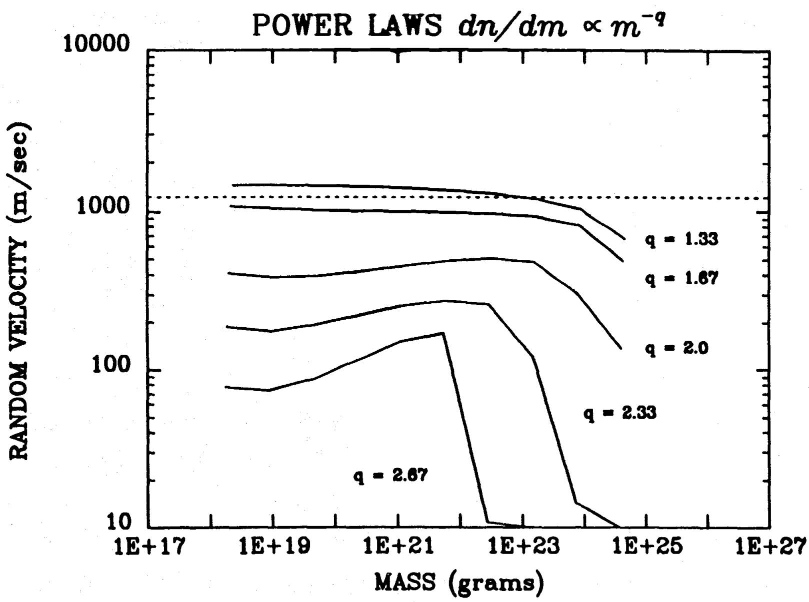

Figure 1. Steady-state random velocities calculated from Eqs. (7–14) for five different power law mass distributions. The curves are labeled by the exponent q in the differential power law $n(m)\mathrm{d}m \propto m^{-q}\mathrm{d}m$. The horizontal dotted line gives the escape velocity of the largest body. In all cases, the surface mass density of solids was 20 g/cm^2, the masses ranged from 10^{18} to 9.766×10^{24} g and the gas density was 1.18×10^{9} g/cm^3.

increase by only a few percent. We suspect that gas drag would have a more significant affect if planetesimals less massive than 10^{18} g were included in the simulations.

C. Relationship Between Random Velocities and Planetesimal Orbital Elements

Planetary accretion occurs within disks of planetesimals on nearly Keplerian orbits about a central star. The particle trajectories can be described using Keplerian orbital elements. Particle-in-a-box calculations approximate the relative orbital motions of planetesimals with a random (or relative) velocity. The relationship between random velocity and orbital elements is by no means straightforward; several different relationships have been used by various authors, causing a great deal of confusion in the field (especially because all of these different quantities are denoted by the symbol v). We attempt to clarify the situation below.

The appropriate conversion between orbital elements and random velocities depends on the specific aspect of the accretion problem being examined. Four different conversions have been used, each of which assumes the epicyclic approximation ($e, i << 1$):

1. The velocity of a planetesimal relative to the mean circular orbit in the disk midplane with the same semimajor axis as that of the planetesimal,

i.e., its epicyclic velocity plus a contribution due to inclination:

$$v_{ep} = (e^2 + i^2)^{1/2} v_K. \tag{15}$$

2. The velocity of a planetesimal relative to the *local* mean circular orbit, averaged over an epicycle:

$$v_{lc} = \left(\frac{5e^2}{8} + \frac{i^2}{2}\right)^{1/2} v_K. \tag{16}$$

This is the local rms velocity one would calculate from the velocity distribution equations (cf. to transform between Eqs. [1] and [2]) and is also the velocity which appears in Eqs. (7–14).

3. The velocity of a planetesimal relative to other planetesimals in the swarm, averaged over an epicycle *and* over a vertical oscillation:

$$v_{sw} = \left(\frac{5e^2}{4} + i^2\right)^{1/2} v_K. \tag{17}$$

This is the local rms relative velocity that one would calculate from the product of two velocity distributions:

$${v_{sw}}^2 = \int d^3v_1 d^3v_2 f_1(\mathbf{v}_1) f_2(\mathbf{v}_2)(\mathbf{v}_1 - \mathbf{v}_2)^2/n^2.$$

4. The weighted averaged approach velocity of a planetesimal to a protoplanet on a circular orbit averaged over an epicycle. The averaging is weighted by a factor of $1/v^2$ in order to produce the appropriate 2-body approximation to the gravitational enhancement in cross sections for the calculation of accretion rates (cf. Greenzweig and Lissauer 1990):

$$v_{cs} = (e^2 + i^2)^{1/2} \left(\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right)^{1/2} v_K \tag{18}$$

where $k \equiv (4(I^2+1)/3)^{-1/2}$, $I \equiv \sin i/e$, and where $\mathbf{K}(k) \equiv \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{-1/2} d\theta$ and $\mathbf{E}(k) \equiv \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\theta)^{1/2} d\theta$ are complete elliptic integrals of the first and second kinds.

III. COLLISION CROSS-SECTIONS AND GROWTH RATES

The size distribution of planetesimals evolves principally due to physical collisions among its members. Stresses caused by tidal forces during close encounters between planetesimals may also fragment very weak bodies, but such disruptive encounters require special circumstances (Boss et al. 1991), and thus can be neglected or folded into the formalism of fragmentation via physical collisions. The evolution of the size distribution of planetesimals can be studied using coagulation theory (cf. Sec. IV). The inputs required in the coagulation calculations are the collision frequency and physical assumptions regarding the outcome of collisions.

The velocity at which two bodies of radii R_1 and R_2 and masses m_1 and m_2 collide is given by:

$$v_c = (v^2 + v_e{}^2)^{1/2} \tag{19}$$

where v is the relative velocity of the two bodies far from encounter and v_e is the escape velocity from the point of contact:

$$v_e = \left(2G\frac{m_1 + m_2}{R_1 + R_2}\right)^{1/2}. \tag{20}$$

The rebound velocity is equal to ϵv_c, where $\epsilon \leq 1$ is the coefficient of restitution. If $\epsilon v_c \leq v_e$, then the two bodies remain bound gravitationally and soon re-collide and accrete. Net disruption requires both fragmentation, which depends on the internal strength of the bodies, and post-rebound velocities greater than the escape speed. As we saw in Sec. II, relative velocities of planetesimals are generally less than the escape velocity from the largest typical bodies in the swarm. Thus, unless ϵ is very close to unity, the largest members of the swarm are likely to accrete the overwhelming bulk of material with which they collide. Fragmentation is likely to be most important for very small planetesimals.

In this section, we shall give formulas for the calculation of the collision rates between planetesimals. For the case of the largest bodies in the swarms, which eventually are to grow to planetary size, the rate of collision with material is essentially identical to the accretion rate.

A. The Particle-in-a-Box Approximation

We wish to compute the accretion rate of a given body, which we shall refer to as the protoplanet, embedded within a uniform surface density and velocity dispersion disk of bodies which we shall call planetesimals. The simplest model for computing the collision rate of planetesimals ignores their motion about the central star entirely. This problem can be treated using methods of the kinetic theory of gases (Safronov 1972; Wetherill 1980*a*). Collisions occur when the separation between the centers of two particles becomes less than the sum of their radii, R_s. The accretion rate of a body in a swarm of planetesimals of density ρ_{sw} (not to be confused with the density or specific gravity of the individual bodies, ρ) at relative velocity v (= v_{cs}) is given by:

$$\dot{M}_{PiB} = \rho_{sw} v \pi R_s{}^2 \left[1 + \left(\frac{v_e}{v}\right)^2\right] = \rho_{sw} v \pi R_s{}^2\,(1 + 2\theta) \tag{21}$$

where the second term in the parentheses represents the gravitational enhancement of the accretion cross section. The Safronov number, $\theta \equiv (v_e/v)^2/2$, is frequently used to quantify the effects of gravitational focusing.

Note that in a disk of given surface mass density σ, the volume density of the planetesimal swarm is inversely proportional to v, assuming the mean ratio of horizontal to vertical motions (eccentricities to inclinations) remains fixed:

$$\rho_{sw} \approx \frac{\sigma}{2H} = \frac{\sigma}{2a \sin i} \approx \frac{\sigma\Omega}{2v_z} \approx \frac{\sqrt{3}}{2}\frac{\sigma\Omega}{v} \tag{22}$$

where $H = a \sin i$ is the half-thickness ($\approx$ scale height) of the disk and the factor of $\sqrt{3}$ assumes the velocity dispersion is isotropic. Substituting Eq. (22) into Eq. (21), we find that the mass accretion rate of a protoplanet depends on v only through the gravitational focusing factor:

$$\dot{M}_{PiB} = \frac{\sqrt{3}}{2}\sigma\Omega\pi R_s{}^2[1 + \left(\frac{v_e}{v}\right)^2]. \tag{23}$$

The exact value of the constant in front of the right-hand side of Eq. (23) depends on the velocity distribution; thus, various values have been quoted in the literature and used in estimating growth times for planets.

B. Formulas for the 2-Body Approximation Including Kepler Shear

The particle-in-a-box approximation discussed above works well *locally* as long as encounters are rapid compared to an orbital period, i.e., that relative velocities are high. Actual planetary encounters can be much more complicated than the simple particle-in-a-box model. In addition to the gravitational forces between the colliding bodies, the gravity of the star needs to be taken into account and the 3-body problem must be solved in order to calculate accretion rates. For sufficiently rapid relative velocities, the encounter is brief enough that the tidal effect of stellar gravity can be neglected during the interval in which the gravitational forces between the secondaries is important. Stellar gravity must, however, still be incorporated to determine the flux and velocities of approaching bodies. As the problem is broken into a set of 2-body calculations (which, unlike the 3-body problem, may be solved analytically), this method is known as the 2-body approximation.

The dynamics of planetesimal-protoplanet encounters are the same in the 2-body approximation as in the particle-in-a-box case. The additional problem is finding the average density of planetesimals which a protoplanet encounters and the appropriately weighted encounter velocities "at infinity" in terms of the orbital elements of the bodies. Intermediate steps in the derivation of the accretion rate are omitted in the discussion below, but may be found in Greenzweig and Lissauer (1990, henceforth GL90).

Several simplifying assumptions are required in order to produce a simple analytic expression for the 2-body accretion rate analogous to Eq. (23). First, we assume that orbital eccentricities and inclinations are sufficiently small that terms quadratic in e and $\sin i$ may be neglected (i.e., we make the epicyclic approximation); this assumption is valid for all but possibly the very final stages of accretion. On the other hand, the radial and vertical oscillations of the planetesimals during one orbital period, ae and $a \sin i$, are assumed to be large compared to the particle-in-a-box accretion radius of the protoplanet, $R_s(1+v_e{}^2/v^2)^{1/2}$; accretion rates for the planer case (i.e., where $i = 0$) for both zero and nonzero eccentricities are given by GL90. We also assume that the pre-encounter inclinations and eccentricities of the planetesimal orbits relative to that of the protoplanet are the same for all planetesimals. (If the protoplanet is on a circular orbit, as is often the case [cf. Sec. II], then the relative eccentricity is identical to the eccentricity of the planetesimals; see Eq. [10] of GL90 for the general definition of relative eccentricity.) Distributions in e, i and R_s may be accounted for by integrating the results presented below over these distributions (cf. Sec. III.D).

Under the assumptions stated above, the accretion rate of a protoplanet is given by (cf. Eqs. 30–32 of GL90):

$$\dot{M}_{2B} = P_{2B}(e, i)\sigma h^2 \frac{2\pi}{T} = \frac{\sigma R_s{}^2}{T} F(I) F_{2B} \tag{24}$$

where $P_{2B}(e, i)$ is the normalized 2-body collision probability as defined by Nakazawa et al. (1989), T is the protoplanet's orbital period, and

$$F(I) \equiv 4 \frac{\sqrt{(1+I^2)}}{I} \mathbf{E}(k) \tag{25}$$

where I is defined by Eq. (18) above. The 2-body gravitational enhancement factor, F_{2B}, is the ratio of a protoplanet's 2-body collision rate to that of a nongravitating protoplanet of identical size:

$$F_{2B} = 1 + \frac{v_e{}^2}{v^2} = 1 + 2\theta \tag{26}$$

where $v \equiv v_{cs}$.

C. The 3-Body Gravitational Enhancement Factor

The 3-body problem can be solved analytically only for a few special equilibrium cases; thus, numerical integrations are necessary to calculate the 3-body gravitational enhancement factor F_g. The general problem of computing F_g for a variety of planetesimal velocity dispersions over the entire range of different protoplanet masses, radii and orbital locations plausible during the planetary accretion epoch would be intractable were it not for a very useful set of scaling laws based on Hill's (1878) equations. A desirable feature of Hill's equations is their lack of dependence on the ratio of the protoplanet's mass to that of the star as long as $m_1/M_\star << 1$. The radius of a protoplanet's Hill sphere is

$$h = \left(\frac{m_1}{3M_\star}\right)^{1/3} a. \tag{27}$$

An encounter with a planetesimal whose mass is not negligible compared to that of the protoplanet is best described by replacing m_1 in Eq. (27) by $m_1 + m_2$. It is also useful to define the Hill eccentricity e_H, Hill inclination i_H, and relative Hill semimajor axis b_H as:

$$e_H \equiv \frac{ea}{h}, \; i_H \equiv \frac{ia}{h}, b_H \equiv \frac{a_2 - a_1}{h}. \tag{28}$$

and the scaled accretion radius of a protoplanet as:

$$r_H \equiv \frac{R_s}{h}. \tag{29}$$

Three-body integrations of planetesimal-protoplanet encounters can be scaled to protoplanets of different masses, sizes and separations from stars of any mass if e_H, i_H and r_H remain fixed (Nishida 1983; GL90). Note that r_H is the same for all protoplanets of a given density at a fixed distance from any particular star. A detailed analytic development of the Hill scaling of the accretion problem is presented by GL90.

Following Ida and Nakazawa (1989), we define the collision probability, $P(e_H, i_H, r_H)$, as the ratio of the rate at which bodies hit a protoplanet in a uniform surface density disk to the flux of bodies with semimajor axes in the range $-1 < b_H < 1$ which would pass the planets if their orbits about the star were unperturbed. The value of $P(e_H, i_H, r_H)$ may be calculated from numerical experiments (Ida and Nakazawa 1989; GL90). The mass accretion rate may be obtained from P using the relationship:

$$\dot{M}_{3B} = P(e_H, i_H, r_H)\sigma h^2 \frac{2\pi}{T} \tag{30}$$

and F_g may be determined by dividing $\dot{M}_{3B}$ by the accretion rate for a non-gravitating planet of the same size.

Numerical results for $P(e_H, i_H, 0.005)$, which represents, e.g., a protoplanet of density $\rho = 3.4$ g cm^{-3} orbiting 1 AU from a 1 $M_\odot$ star, are presented for a variety of different values of e_H and i_H by Ida and Nakazawa (1989) and GL90. Figure 2 shows the dependence of F_g on random velocities for $r_H = 0.005$ and $e_H = 2i_H$. Similar plots for other values of r_H are presented in GL90. Note that the 2-body approximation is valid for $v/v_e \gtrsim 0.1$, 3-body accretion rates exceed those given by the 2-body formula by up to a factor of 2 when $0.02 \lesssim v/v_e \lesssim 0.1$, and F_g rises less steeply than the 2-body formula at smaller v/v_e, approaching the value of $\sim 1.7 \times 10^4$ as $v \to 0$.

D. Cross Sections for a Gaussian Velocity Distribution

The formulas quoted above are valid for uniform surface density disks in which all planetesimals either approach the protoplanet at the same speed (particle-in-a-box case) or have the same unperturbed values of e and i (2-body and 3-body cases). While these homogeneous cases provide insight

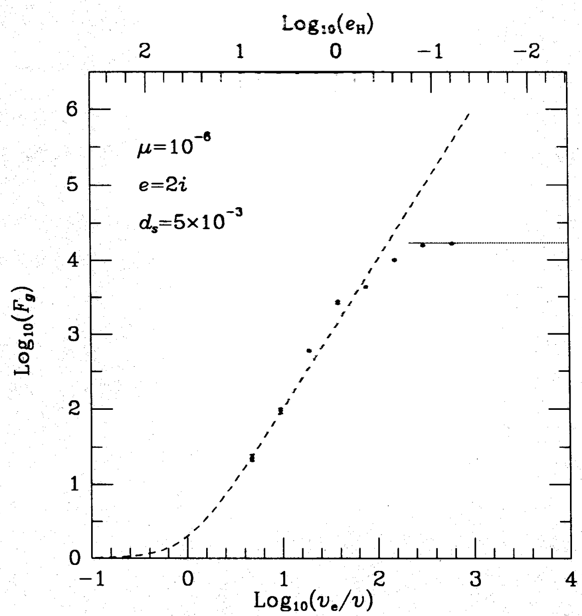

Figure 2. The 3-body gravitational enhancement of the accretion cross-section of a protoplanet on a circular orbit plotted as a function of planetesimal random velocities (v_{cs}). The radius of the protoplanet is r_H = 0.005, which represents, e.g., a protoplanet of density ρ = 3.40 g cm^{-3} orbiting 1 AU from a star of mass 1 $M_\odot$. The ratio of the mass of the protoplanet to that of the star used in the numerical calculations is $\mu = 10^{-6}$; however, the results should be valid for any $\mu << 1$. Individual points correspond to results for separate runs; in each run all of the planetesimals have identical eccentricities and inclinations with $e_H = 2i_H$, and are distributed in semimajor axis as a uniform surface density swarm. Error bars represent statistical ($\sqrt{n}$) uncertainties in the numerical experiments. The dotted line represents the limiting case of planar circular orbits. The dashed curve is the 2-body approximation given by Eq. (24). Figure from GL90.

into the dynamics of the orbits leading to collisions, they do not represent a realistic disk of planetesimals. Planetesimal motions are much better approximated by a Rayleigh distribution in e and i (Eq. 1) or, equivalently, a triaxial Gaussian distribution in local velocities (Eq. 2). Using the particle-in-a-box approximation, Vityazev and Pechernikova (1981) found that when $v << v_e$, the average eccentricity of *colliding* planetesimals is $\sim 3^{-1/2}\langle e^2\rangle^{1/2}$; this implies a factor of ~3 enhancement in accretion rates. Greenzweig and Lissauer (1992) compute formulas for the accretion rate of a protoplanet embedded in a uniform surface density disk of planetesimals with such a velocity distribution. They find that in the 2-body approximation, accretion rates when $v << v_e$ are enhanced compared to the case where all planetesimals have e and i equal to the rms of the Rayleigh distribution by a factor which depends weakly on $I (\equiv \sin i/e)$ and always exceeds 2.6. Numerical 3-body integrations show comparable enhancements, except when $\langle {e_H}^2\rangle^{1/2} < 1$ (Greenzweig and Lissauer 1992); at very low eccentricities, the enhancement disappears due to the flattening of the accretion rate as $e \to 0$ (cf. Fig. 2). Qualitatively similar results have been reported by Ida and Nakazawa (1988) and Ohtsuki and Ida (1990).

IV. EARLY STAGES OF PLANETESIMAL ACCUMULATION

The velocity dependence of the collision cross section described above leads to a strong coupling between the evolution of the velocity and size distributions of planetesimals. The study of planetary accumulation therefore requires a simultaneous calculation of the velocity evolution and size evolution of the planetesimal swarm. The early stages of planetesimal evolution has been modeled by several authors (Greenberg et al. 1978,1984; Nakagawa et al. 1983; Ohtsuki et al. 1988; Wetherill and Stewart 1989). All of these papers use some variant of the particle-in-a-box approximation, but they differ in their treatment of the velocity evolution and their method of simulating the planetesimal size evolution. The most important result that has emerged from this work is that the evolving planetesimal size distribution can follow two qualitatively distinct paths that are characterized by very different time scales. The slower evolutionary path exhibits an orderly growth of the entire size distribution so that all the planetesimals remain tied to the continuous size distribution throughout the early stages of planetesimal accumulation. The term *runaway accretion* refers to a different evolutionary path where the largest planetesimal in the local region grows much more rapidly than the remainder of the population and therefore becomes detached from the continuous size distribution. Recent modeling efforts have focused on the task of delineating the circumstances under which *runaway* accretion may or may not occur (Ohtsuki et al. 1990; Ohtsuki and Ida 1990; Wetherill 1989,1990*a*). In this section we review the causes of runaway accretion and discuss the relevance of this process to planet formation.

A. Solutions of the Coagulation Equation

The origin of runaway accretion can be best understood in the context of various solutions to the discrete form of the coagulation equation,

$$\frac{\mathrm{d}n_k}{\mathrm{d}t} = \frac{1}{2}\sum_{i+j=k} A_{ij}n_in_j - n_k\sum_i A_{ik}n_i \tag{31}$$

which describes the time evolution of the number of bodies n_k with mass m_k. The merger of smaller bodies increases n_k, whereas the incorporation of bodies of mass m_k into larger bodies causes n_k to decrease. Analytical solutions to Eq. (31) are known for a few simple forms of the collision probability A_{ik} (Safronov 1972; Trubnikov 1971; Wetherill 1990*a*). Although none of these special cases accurately represent the velocity-dependent collision probabilities described in Sec. III, the few analytical solutions to the coagulation equation provide a rigorous test for the accuracy of any numerical algorithm used to solve the coagulation equation. Two such solutions which exhibit orderly growth rather than runaway accretion are the cases when A_{ij} equals a constant and when A_{ij} is proportional to the sum of the masses, $m_i + m_j$. Ohtsuki et al. (1990) used these two solutions to show how numerical algorithms that employ fixed mass-coordinate divisions to represent the mass distribution can produce erroneous results. They conclude that numerical calculations of planetesimal accumulation which use a mass ratio between neighboring mass coordinates that exceeds $\sqrt{2}$ can substantially overestimate the rate of planetary growth.

In contrast to the fixed mass-coordinate algorithms used by most previous workers, Wetherill and Stewart (1989) represented the size distribution with a number of moving "batches," each containing a large number of bodies of the same mass. In this Lagrangian-type scheme, the mass value characterizing each batch is allowed to grow by the sweep-up of smaller bodies as well as by merger with other bodies in the same batch. Wetherill (1990*a*) has presented test calculations using this method which show better agreement with the two analytical solutions described above than was obtained by Ohtsuki et al.

A more stringent test of any numerical procedure for solving the coagulation equation is the ability to reproduce the solution when A_{ij} is proportional to the product of the masses $m_i\, m_j$ because this is a case known to exhibit runaway growth of the largest body. As pointed out by Wetherill (1990*a*), the analytical solution for this case is best understood by replacing Eq. (31) with a set of equations which explicitly separate out the evolution of the largest body in the distribution:

$$\frac{\mathrm{d}n_k}{\mathrm{d}t} = \frac{\gamma}{2} \sum_{i+j=k} ij\, n_i n_j - \gamma k n_k \sum_i i n_i - \gamma k_R k n_k \tag{32}$$

$$\frac{\mathrm{d}k_R}{\mathrm{d}t} = \gamma k_R \sum_k k^2 n_k \tag{33}$$

where the masses have been normalized such that $A_{ij} = \gamma ij$ and k_R is the normalized mass of the largest body. The last term on the right-hand side of Eq. (32) represents the mass lost from bin "k" as a result of collisions with the largest body. These equations are consistent with a time independent total mass n_o where

$$n_\mathrm{o} = k_R + \sum_k k n_k. \tag{34}$$

The solution of Eq. (32) was found by Trubnikov (1971) to be

$$n_k(\eta) = \frac{n_\mathrm{o}(k\eta)^{k-1} e^{-k\eta}}{k!\, k} \tag{35}$$

where $\eta = n_\mathrm{o}\gamma t$ is a dimensionless time variable. The time evolution of the runaway body is obtained by substituting the solution (35) into Eq. (34) and solving for k_R.

Wetherill (1990*a*) has utilized the above solution to verify the accuracy of the numerical procedure used by Wetherill and Stewart (1989) in their calculation of planetesimal evolution. A crucial attribute of Wetherill and Stewart's calculation is the probabilistic procedure used to simulate abrupt changes in the mass of the largest body. During a time step of length Δt, the number of merging collisions between bodies in batch "i" and bodies in batch "j," ν_{ij}, is given by the formula

$$\nu_{ij} = n_i n_j A_{ij} \Delta t. \tag{36}$$

When "i" and "j" represent very large bodies, ν_{ij} often falls between 0 and 1. It is physically reasonable to interpret a fractional value of ν_{ij} as a collision probability per time step because collisions between very large bodies do not really occur during every time step. The procedure used by Wetherill and Stewart is therefore to create a new batch containing one body only when $m_i + m_j$ exceeds the mass of the largest body by a factor δ (where $1.07 < \delta < 1.15$) *and* ν_{ij} exceeds a random number between 0 and 1. Numerical calculations using this procedure are able to reproduce the runaway growth predicted by Eqs. (34) and (35) quite accurately, aside from a small time lag that depends on the value chosen for δ (Wetherill 1990*a*). Alternative numerical procedures which fail to simulate the abrupt changes in the size of the largest body can artificially suppress runaway growth. Taken together, these studies of the coagulation equation provide an important lesson: it is always advisable to test one's numerical procedures against known solutions of the coagulation equation before drawing any conclusions about the physical causes of runaway accretion of the planets.

B. The Causes of Runaway Accretion

The three analytic solutions of the coagulation equation described above indicate a bifurcation between solutions which display orderly growth and solutions which display runaway growth of the largest body. The decisive factor that determines which branch the solution will take during planetesimal accumulation is the velocity dependence of the collision rate described in Sec. III. In general, a velocity distribution which exhibits smaller velocities for the larger bodies (as is shown in Fig. 1) will increase the effective mass dependence of the collision rate and will therefore increase the likelihood of runaway accretion. Wetherill and Stewart (1989) presented a series of calculations which serve to delineate the conditions required for runaway accretion to occur. In those calculations which omitted the energy equipartition terms proportional to $(m_k v_k^2 - m_i v_i^2)$ in Eqs. (7–14), orderly growth of the planetesimal size distribution was found. These results show qualitative agreement with the earlier results of Safronov (1972) and Nakagawa et al. (1983), who also neglected the energy equipartitioning caused by gravitational scattering.

When Wetherill and Stewart (1989) included the energy equipartition terms in their velocity evolution equations, they found runaway growth of the largest planetesimal to occur. Complementary calculations by Ohtsuki and Ida (1990) confirm the result that runaway accretion occurs when the random velocities decrease with increasing planetesimal mass. Another mechanism which may decrease the velocities of the largest bodies is the enhanced gas drag that would result from gravitational concentration of nebular gas around massive planetesimals (Takeda et al. 1985). Ohtsuki et al. (1988) present calculations including this enhancement of gas drag which also display runaway accretion. These recent investigations were motivated to large degree by the pioneering work of Greenberg et al. (1978) that first reported runaway accretion. However, Wetherill and Stewart were unable to reproduce runaway growth when using the same physical assumptions that were stated by Greenberg et al., and the validity of those early results remains controversial (cf. Spaute et al. 1991; Kolvoord and Greenberg 1992).

As energy equipartition via gravitational scattering plays a pivotal role in the bifurcation between orderly and runaway growth, is important to establish the limits of validity of the velocity evolution equations discussed in Sec. II. Ida (1990) has presented an extensive series of 3-body orbit integrations in order to determine directly the ability of a protoplanet on a circular orbit to alter the velocity distribution of the surrounding planetesimal swarm as well as the ability of a planetesimal swarm to damp the eccentricity and inclination of a protoplanet via dynamical friction. Ida finds that the energy equipartition rate agrees well with the 2-body results at high velocities. However, when e_H of the smaller planetesimals drops below 2, the rate of equipartition stops increasing with decreasing velocity (unlike the 2-body case, where the rate increases indefinitely). Ida's work therefore implies that dynamical friction continues to operate at low velocities, but the energy equipartition rate attains a maximum asymptotic value that is given by setting $e_H = 2$ in the 2-body formula. When the velocity evolution Eqs. (7–14) are modified to include the limiting equipartition rate, one finds that the steady-state velocities of the largest planetesimals are somewhat greater than shown in Fig. 1. Nevertheless, the velocities of the largest bodies are still found to be substantially smaller than $e_H = 2$ in size distributions where most of the mass is contained in the smaller planetesimals. In light of this result, it appears likely that runaway growth of the largest planetesimals will occur in spite of the limiting equipartition rate found by Ida.

Although the causes of runaway accretion are fairly well established, the precise rate of runaway growth remains somewhat uncertain because the statistical arguments used to derive the velocity evolution equations begin to break down once runaway growth has begun. The Fokker-Planck operator used to derive Eqs. (7–14) implicitly assumes that successive gravitational encounters are uncorrelated, so that random velocities evolve in a random-walk fashion. However, once a protoplanet grows much more massive than any other body in the local accretion zone, each successive gravitational encounter with this protoplanet tends to be correlated with the preceding encounter. In the limit that one can neglect mutual interactions among the smaller planetesimals, the smaller planetesimals will always approach the protoplanet with the same velocity owing to the conservation of their Jacobi constant with respect to the protoplanet. Wetherill and Stewart (1989) placed limits on the importance of this effect by presenting an accretion calculation which omitted perturbations by the largest body in the zone. The rate of runaway accretion was accelerated in that case, due to the reduced random velocities.

When a single protoplanet is the dominant perturber in a zone, the velocity dispersion ceases to be isotropic. Because of the coupling between azimuthal and radial motions, the protoplanet is able to excite eccentricities in planetesimals initially on circular orbits, but due to the symmetry of the equations of motion about the plane of the protoplanet's orbit, the protoplanet cannot excite inclinations in coplanar planetesimals. Numerical studies of planetesimals with initially small e_H and i_H imply much more rapid growth of eccentricity than inclination (Ida 1990; GL90). The rate of planetary growth can be greatly accelerated in such hot, flat disks. (For details, see the discussion surrounding Eq. [50] in GL90.)

Realistically, the precise rate of runaway growth is determined by the relative frequency of encounters with the protoplanet compared to the rate of velocity evolution due to gas drag and mutual interactions among the smaller planetesimals. Hayashi et al. (1977) have stressed the importance of calculating the evolution of both semimajor axes and eccentricities in order to determine the long-term evolution of the Jacobi "constant" when all of these processes are active. More detailed simulations of this problem are needed to better constrain the maximum accretion rate during runaway growth.

C. The Problem of the Asteroid Belt

The absence of a large terrestrial planet in the region between the orbits of Mars and Jupiter provides a strong constraint for models of solar system formation. Numerical simulations of planetesimal accumulation in the primordial asteroid belt yield runaway growth of a protoplanet more massive than 10^{27} g on a time scale of several times 10^5 yr if the influence of Jupiter is neglected (Wetherill 1989). Gravitational perturbations by Jupiter could have curtailed runaway in the asteroid belt only if Jupiter's core had also formed on a time scale of several times 10^5 yr. The early runaway growth of Jupiter's core has therefore emerged as a desirable feature in scenarios of planet formation.

The accumulation of Jupiter's core on a time scale of <1 Myr appears to be possible if the surface density of condensable materials at Jupiter's orbit was ≥ 20 g cm^{-2} (Lissauer 1987; Wetherill 1989). This large a surface density is not predicted by standard minimum-mass solar-nebula models. (A minimum-mass solar nebula is a model circumsolar disk which contains only the amount of condensable material currently present in the planets, augmented with volatiles to solar composition and spread out to give a smooth density distribution in radius. The total mass of the protoplanetary disk in such models is 0.01 to 0.02 $M_\odot$.) One possibility is that the surface density of solids may have been enhanced at Jupiter's orbit due to water ice condensation and by diffusive transport of condensable water vapor from the inner solar system (Stevenson and Lunine 1988). However, we believe that the more likely explanation is that the surface mass density of the protoplanetary disk in the region where the giant planets accreted was a factor of several greater than that predicted by minimum-mass models. This "excess" mass not only could account for the rapid formation of the cores of the giant planets (Sec. V; cf. Lissauer 1987), but also offers a source of condensable material large enough to have produced the $\sim$50 $M_\oplus$ of comets believed to exist in the Oort cloud at the present epoch, even after losses over the age of the solar system have been accounted for (Fernandez and Ip 1984; Duncan et al. 1987; Weissman 1990). Moreover, disk stability analysis, which suggests a preferred mass of protoplanetary disks equal to $\sim$1/3 that of the central star (Chapter by Adams and Lin; cf. Shu et al. 1990), and observations of relatively massive disks around many young stars (Chapter by Beckwith and Sargent), reinforce our opinion that minimum-mass models of the solar nebula are headed towards the dustbin of history.

Although Jovian perturbations are widely invoked to explain the asteroid belt, the precise mechanism that halted planet formation is still a subject of some dispute. The only way to stop planet growth is to increase the planetesimal velocities to a value substantially greater than the surface escape velocity of the largest bodies in a local region. The difficulty with invoking gravitational perturbations by Jupiter for this purpose is that the resultant eccentricity pumping is only appreciable at narrow resonance locations. Several authors have discussed how the dissipation of the nebular gas could have shifted the resonance locations, thereby causing Jupiter's resonances to sweep through the entire asteroid belt (Heppenheimer 1980; Torbett and Smoluchowski 1980; Ward 1981). Detailed models which predict the time scale for removal of the gas are required to evaluate this suggestion properly. Changes in Jupiter's semimajor axis due to accretion of gas and/or ejection of planetesimals from the solar system could also have caused Jovian resonances to have swept across the asteroid belt (Safronov and Gusseinov 1989).

Another possible way to pump up velocities in the asteroid belt is to postulate a population of Jupiter-zone planetesimals that are scattered into the asteroid belt once Jupiter becomes sufficiently massive. Recent Monte Carlo calculations of this scenario show that the necessary large velocities are produced in the asteroid belt if the Jupiter zone planetesimals are as massive as the Earth and if one of these bodies becomes trapped in the asteroid belt for an extended period of time (Ip 1987; Wetherill 1989). This result is somewhat unsatisfying, because one must then require this Earth-size body to end up near a strong Jupiter resonance in order to provide a mechanism for its removal. It is also not clear that such a population of Jupiter-zone planetesimals would be created during the rapid runaway growth of Jupiter's core.

One variant of the previous model is that a planet-sized body did form at $\sim$3 AU from the Sun. This body could then have excited the eccentricities of the current asteroids prior to being ejected from the solar system by resonant perturbations from Jupiter. In this case, asteroid belts might not be a common feature among planetary systems otherwise much like our own. More detailed simulations of planetesimal accumulation which accurately model Jupiter perturbations should help narrow the possibilities (cf. Wetherill 1991*c*).

V. LIMITS TO RUNAWAY GROWTH

Runaway accretion with high F_g requires low random velocities, and thus small radial excursions, $2ae$, of planetesimals. This implies that a protoplanet's feeding zone is limited to the annulus of planetesimals which it can gravitationally perturb into intersecting orbits. Thus, rapid runaway growth must cease when a protoplanet has consumed most of the planetesimals within its gravitational reach (Lissauer 1987).

For the case of a protoplanet on a circular orbit, the standard theory of the restricted 3-body problem places an upper bound on the initial semimajor axis, b_H, that may lead to collision. Neglecting gas drag and interactions with other planetesimals, a planetesimal whose orbital elements satisfy the inequality:

$$\frac{3}{4}{b_H}^2 - {e_H}^2 - {i_H}^2 \geq 9 \tag{37}$$

cannot enter the protoplanet's Hill sphere (see, e.g., Artymowicz 1987). For example, a particle which initially has $e_H = i_H = 0$ and $b_H > 2\sqrt{3} \approx 3.5$ remains in superior orbit to the protoplanet, although it's path may be perturbed (this perturbation preserves the left-hand side of Eq. (37), which is a version of the Jacobi constant). For a single encounter (one synodic period), GL90 find that planetesimals with initial $e_H = i_H = 0$ and $b_H > 2.6$ do not approach closer than 0.1 h from the protoplanet; however, Kary et al. (1993) show that most planetesimals with initial $e_H = i_H = 0$ and $1.3 < b_H < 3.2$ come within 0.1 h of the protoplanet during 20 synodic periods. For nonzero "initial" eccentricity and/or inclination, the accretion zone expands slightly, but for e and i low enough for F_g to be large, planetesimals with $|b_H| > 4$ cannot be accreted. Thus, the accretion zone of a protoplanet embedded in a disk of low random-velocity planetesimals extends over the region:

$$|b_H| < B \tag{38}$$

where B depends on the magnitude of other perturbations on the planetesimals, and typically is ~3.5 to 4 in a quiescent disk.

The size of a protoplanet's Hill sphere expands as it accretes matter. The mass of a protoplanet which has accreted all of the planetesimals within an annulus of width $2\Delta r$ is:

$$m = \int_{r-\Delta r}^{r+\Delta r} 2\pi r' \sigma(r') \mathrm{d}r' \approx 4\pi r \Delta r \sigma(r). \tag{39}$$

Setting $\Delta r = Bh = Br(m/3M_\star)^{1/3}$, we obtain the isolation mass to which a protoplanet orbiting at a distance r from a star of mass $M_\star$ may grow:

$$m = \frac{(4\pi B r^2 \sigma)^{3/2}}{(3M_\star)^{1/2}} = 2.10 \times 10^{-3} \left(\frac{Br^2\sigma}{2\sqrt{3}}\right)^{3/2} \left(\frac{\mathrm{M}_\odot}{M_\star}\right)^{1/2} \mathrm{M}_\oplus \qquad (40)$$

where the mass of the Earth $\mathrm{M}_\oplus = 5.98 \times 10^{27}$ g, r is expressed in AU and σ in g cm^{-2} (Lissauer 1987). For example, assuming $B = 2\sqrt{3}$, a minimum-mass solar nebula with $\sigma = 10$ g cm^{-2} at 1 AU implies protoplanet isolation at 0.066 $\mathrm{M}_\oplus$; whereas $\sigma = 3$ g cm^{-2} at 5 AU implies protoplanet isolation at 1.36 $\mathrm{M}_\oplus$.

Runaway growth can persist beyond the isolation mass given by Eq. (40) only if additional mass can diffuse into the protoplanet's accretion zone. Three plausible mechanisms for such diffusion are scattering between planetesimals, perturbations by protoplanets in neighboring accretion zones and gas drag. The process of radial drift due to scattering within the vicinity of a protoplanet has not yet been analyzed quantitatively and remains a major open question.

Drift due to gas drag has been modeled in more detail. Weidenschilling and Davis (1985) suggested that a protoplanet can enhance the effects of gas drag on material orbiting just outside its accretion zone in the following manner: As planetesimals drift slowly inwards due to gas drag, they eventually encounter and are trapped into small integer commensurabilities (resonances) with the protoplanet. The eccentricities induced at such resonances lead to high-velocity collisions which grind planetesimals into small debris. Small planetesimals are very strongly affected by gas drag; they cannot be stopped by protoplanet resonances, and thus rapidly drift into the protoplanet's accretion zone.

Alternatively, radial motion of the protoplanet may bring it into zones not depleted of planetesimals. Gravitational torques due to excitation of spiral density waves in the gaseous component of the protoplanetary disk have the potential of inducing rapid radial migration of protoplanets; however, as the migration rate depends on the difference between comparable positive and negative torques (due to excitation of waves at resonances interior and exterior to the protoplanet's orbit, respectively), the magnitude of this effect is extremely difficult to quantify (Goldreich and Tremaine 1980; Ward 1986). Recently it has been suggested that gravitational focusing of gas could vastly increase the rate of inward drift of protoplanets due to gas drag (Takeda et al. 1985; Ohtsuki et al. 1988). This result would have profound implications for models of planetary accretion. However, the fluid calculations assumed low Reynolds number, which may be appropriate for a turbulent protoplanetary disk, but not for a laminar one. The effects of the stellar gravitational field upon the flow pattern also must be examined.

Radial drift of planetesimals relative to a protoplanet increases the protoplanet's isolation mass only if the protoplanet is able to efficiently accrete those planetesimals which approach its orbit. The case of planetesimals decaying inwards (due to gas drag) towards a planet on a circular orbit has been studied by Kary et al. (1993). They find that unless the planet's accretion radius is $\gtrsim$0.01 h (which would imply a distended thick atmosphere), a majority of the planetesimals miss the planet, and continue to drift inwards towards the star. Thus, radial drift is not as promising a mechanism to increase a planet's isolation mass as was previously believed.

VI. FINAL STAGES OF PLANETESIMAL ACCUMULATION

The self-limiting nature of runaway growth strongly implies that massive protoplanets form at regular intervals in semimajor axis throughout the inner solar system. The mutual accumulation of these protoplanets into a small number of widely spaced planets necessarily requires a stage characterized by large orbital eccentricities, significant radial mixing, and giant impacts. Mutual gravitational scattering can pump up the relative velocities of the protoplanets to values comparable to the surface escape velocity of the largest protoplanet, which is sufficient to ensure their mutual accumulation into planets. The large velocities imply small collision cross sections and hence long accretion times. In the outer solar system, the limits of runaway growth are less severe; it is feasible that runaway growth of Jupiter's core continued until it attained the necessary mass to rapidly capture its massive gas envelope (Lissauer 1987).

A. Simulations of Terrestrial Planet Formation

The transition from runaway growth in isolated accretion zones to the mutual accumulation of protoplanets in large eccentricity orbits has so far only been studied qualitatively. The limiting runaway mass given by Eq. (40) suggests that runaway growth in the inner solar system will yield protoplanets of mass $\sim 10^{26}$ g, with their semimajor axes spaced 0.01 to 0.02 AU apart. At this stage, most of the original mass will be contained in the large protoplanets, so their random velocities will no longer be strongly damped by energy equipartition with the smaller planetesimals. Even if the protoplanets form in circular orbits, mutual gravitational perturbations among several bodies can eventually induce eccentricities of order 0.01 ($e_H \approx 5$), which is sufficient to enable their orbits to cross so the protoplanets can suffer close gravitational encounters. Stagnation of protoplanets in isolated orbits is unlikely because the width of the accretion zone at the end of runaway growth roughly coincides with the maximum orbit separation from which neighboring pairs of protoplanets can significantly perturb each other.

Once the protoplanets have perturbed one another into crossing orbits, their subsequent orbital evolution is governed by close gravitational encounters and violent, highly inelastic collisions. Wetherill (1980*a*,1985,1986,1988, 1990*b*) has described numerous simulations of this final stage of accretion, exploring a wide range of initial conditions. Surprisingly, the results of these simulations are not strongly affected by the earlier runaway growth stage. Regardless of whether the simulations start from a swarm of 500 bodies of mass 10^{25} g or just 30 protoplanets of mass several times 10^{26} g, the end result is the formation of 2 to 5 terrestrial planets on a time scale of about 100 Myr. An important feature of Wetherill's simulations is that planetesimal orbits execute a random walk in semimajor axis due to successive gravitational encounters. The resulting widespread mixing of material throughout the terrestrial planet region greatly diminishes any chemical gradients that may have existed during the early stages of planetesimal formation. Although, some correlations between the final heliocentric distance of a planet and the region where most of its constituents originated are preserved in the simulations (Wetherill 1988). Nevertheless, Mercury's high iron abundance is therefore less likely to arise from chemical fractionation in the solar nebula and more likely to be caused by a catastrophic giant impact during the final stages of accretion (see Sec. VI.B below).

Although radial mixing does occur, the total mass and orbital angular momentum of the swarm is nearly conserved during its accumulation into planets. The energy lost as heat in collisions, however, amounts to a few percent of the total orbital energy, resulting in a significant spreading of the disk. Thus, in order to end up with the same angular momentum distribution as is observed for Mercury, Venus, Earth and Mars, one must confine the initial swarm of protoplanets to a narrow annulus with most of the mass between 0.7 and 1.1 AU (Wetherill 1988). More plausible initial mass distributions which vary smoothly with heliocentric distance are incapable of reproducing the observed angular momentum distribution of the inner planets unless the planetesimal swarm can be supplied with extra "free" energy from an external source, e.g., Jupiter. By free energy, we mean energy in excess of that required for a circular orbit at a given semimajor axis:

$$E_f \equiv E - \Omega L \tag{41}$$

where E is orbital energy and L is the magnitude of the orbital angular momentum. Thus, the free energy may be increased by the removal of energy and angular momentum, provided $\Delta E/\Delta L < \Omega$.

The problem of removing excess angular momentum from the terrestrial planets is intimately tied to the larger problem of explaining the extreme depletion of mass between the orbits of Mars and Jupiter. A large radial redistribution of mass is apparently required, but the candidate mechanisms proposed to accomplish this redistribution are poorly constrained at present. Whatever process pumped up the velocities of the asteroids must have yielded a vast quantity of small collision fragments that would have been susceptible to gas drag. These collision fragments would tend to spiral in toward the Sun, thereby enhancing the density of solids in the terrestrial planet zone. Large protoplanets may also experience significant orbital decay by exciting spiral density waves in the gaseous disk (Ward 1986,1988,1989*b*). Ward calculates that protoplanets more massive than 10^{27} g suffer significant orbital evolution inwards towards the protostar on a 1 to 10 Myr time scale via this process. Numerical simulations of planetesimal accumulation which include orbital migration due to density wave torques also show significant redistribution of mass and angular momentum in the terrestrial planet zone (Wetherill 1990*b*,1991*b*). More detailed studies are needed to determine if density wave torques can fully resolve the angular momentum problem in the inner solar system.

B. Giant Impacts

The mutual accumulation of numerous protoplanets into a small number of planets must have entailed many collisions between protoplanets of comparable size. As, in the post-runaway era, the random velocities keep pace with the escape velocity of the larger protoplanets, collisions between smaller bodies result in disruption instead of aggregation. The largest bodies are resistant to collisional disruption because their gravitational binding energy exceeds the kinetic energy of the collision. When the largest protoplanet reached a mass comparable to the Earth's mass, bodies near the size of Mercury became marginal cases, with collisions just as likely to strip off material as to add to the final mass of the planet. This result led Wetherill (1988) and Vityazev et al. (1988) to suggest that Mercury's silicate mantle was stripped off in a giant impact, leaving behind an iron-rich core (cf. Benz et al. 1988). Wetherill's simulations also lend support to the giant impact hypothesis for the origin of the Earth's Moon (e.g., Stevenson 1987); during the final stage of accumulation, an Earth-size planet is typically found to collide with several objects as large as the Moon and frequently one body as massive as Mars. The obliquities of the rotation axes of the planets provide independent evidence of the occurrence of giant impacts during the accretionary epoch (Safronov 1966, Lissauer and Safronov 1991).

C. Accretion Time Scales

Growth of planet-sized bodies from kilometer-sized planetesimals involves an increase in radius of 4 to 5 orders of magnitude. The planetary accretion rates given by Eqs. (19–26) all vary as the surface area of the planet $R_s{}^2$. Thus, for all other parameters constant, a planet's radius grows at a (statistically) uniform rate, and most of the growth time is spent in the last decade of radial expansion. The late phases of planetary growth are therefore crucial to determining the overall length of the accretionary epoch. Thus, a stage of runaway growth which ends in isolated protoplanets which must pump up velocities in order to collide and continue their agglomeration to planetary size, leads to accretion time scales similar to those models in which runaway never occurs.

Of course, protoplanet radius is not the only parameter in the equations which varies with time. Runaway accretion starts slowly (small F_g) and accelerates with time; thus it is possible that the "initial" size distribution of planetesimals significantly influenced the growth times of the planets. For example, a "protoplanet" initially of radius 10 km, at 5.2 AU from a 1 $M_\odot$ star, in a disk of surface mass density $\sigma = 15$ g cm^{-2} whose velocity dispersion is controlled by planetesimals 5 km in radius, would require $\sim 2 \times 10^5$ yr to grow large enough that $F_g = 3000$ (at which point $e_H = 2$ and protoplanet perturbations become an important factor in exciting noncircular motions of planetesimals; cf. GL90). This protoplanet would take a subsequent $\sim 6 \times 10^5$ yr to attain a mass of 15 $M_\oplus$ (Lissauer 1987). A protoplanet in an equivalently skewed initial distribution of larger planetesimals would take much longer to reach $F_g = 3000$, whereas the runaway would proceed much faster in a swarm of smaller planetesimals, because the radius doubling time for the protoplanet would be shorter.

VII. CONCLUSIONS

The planetesimal hypothesis provides a viable theory of the growth of the terrestrial planets, the cores of the giant planets and the smaller bodies present in the solar system. The formation of solid bodies of planetary size should be a common event, at least around stars which do not have binary companions orbiting at planetary distances. The formation of giant planets, which contain significant fractions of H_2 and He, requires rapid growth of planetary cores, so that gravitational trapping of gas can occur prior to the dispersal of the gas from the protoplanetary disk. According to the scenario which we have outlined, the largest bodies in any given zone are the most efficient accreters, in the sense that they double in mass the fastest. Such runaway accretion of a few large solid protoplanets can lead to core formation in ~ 1 Myr, provided disk masses are a few times as large as those given by minimum-mass models of the solar nebula. Thus, it appears possible that giant planets may also be common, although this conclusion must be regarded as much more tentative.

The ultimate sizes and spacings of solid planets are determined by their ability to gravitationally perturb each other into crossing orbits. Such perturbations often are due to weak resonant forcing, and occur on time scales much longer than the bulk of planetesimal interactions discussed herein. These interactions are not yet fully understood, although a great deal of progress has been made in recent years (Wisdom 1983; Chapter by Duncan and Quinn). Although quantitative formulas for scaling planetary sizes and spacings will require a better grasp of these processes, a few qualitative remarks can be made. First, a more massive protoplanetary disk will probably produce larger but fewer planets. Second, stochastic processes are important in planetary accretion, so nearly identical initial conditions could produce quite different outcomes, e.g., the fact that there are 4 terrestrial planets in our solar system as opposed to 3 or 5 or 6 is probably just the luck of the draw (cf. Wetherill 1988). Third, migration of some planetesimals over significant distances within the protoplanetary disk probably occurred, leading to a radial mixing of material which condensed in differing regions of the solar nebula. Fourth, although the spacing of giant planets is probably determined by similar processes to that of solid planets, their ultimate sizes may depend more on such factors as how fast they grew relative to the dispersal of the nebula and their potential ability to halt their own growth by gravitationally truncating the gaseous disk (Lin and Papaloizou 1979*a*).

Many aspects of planetary growth remain poorly understood. The asteroid belt currently contains much less than a planetary mass of material, and that material is spread over countless bodies which move at high velocities relative to each other. Models of the protoplanetary disk suggest this is unlikely to be due to an initial absence of condensed material. Perturbations by Jupiter and/or Jupiter-scattered protoplanets have been invoked for preventing planetary growth in the asteroid zone by increasing relative velocities of planetesimals, and for clearing material from that region; however, many problems remain with these scenarios (cf. Sec. IV.C). Planetesimal dynamics in the Uranus-Neptune region are complicated by the fact that bodies in this region are not tightly bound to the solar system: The difference between circular orbit velocity and escape speed from the solar system at 30 AU is less than the escape speed from the Moon. Using a planetary accumulation model analogous to those which Wetherill has successfully applied to the late stages of growth of the terrestrial planets (cf. Ipatov 1987), Ipatov (1989) finds that the mass of material ejected into hyperbolic orbits from the Uranus-Neptune region during the accretionary epoch exceeds the amount of solid matter incorporated into these planets by approximately an order of magnitude. Moreover, much of the material remaining in heliocentric orbit has spread to tens of AU beyond the original outer boundary of the disk. However, Ipatov's calculations start with planetesimals of identical size and neglect dynamical friction; these assumptions suppress runaway growth and thus lengthen the accretion time scales to unreasonably large values. Further studies of planetesimal accumulation in the outer regions of protoplanetary disks are needed to answer the many interesting questions raised by this study (Lissauer et al., in press).

Acknowledgments. We thank Y. Greenzweig, D. Kary, K. Ohtsuki, V. Safronov, A. Vityazev and G. Wetherill for their valuable comments. This work was supported in part by the NASA Planetary Geology and Geophysics Program under grants at the State University of New York at Stony Brook and at the University of Colorado. JJL is an Alfred P. Sloan Research Fellow.