EDUCATION NOTES/RUBRIQUE PÉDAGOGIQUE

EDITED BY ROY L. BISHOP
Physics Department, Acadia University, Wolfville, N.S. B0P 1X0

STELLAR MAGNITUDES AND PHOTON FLUXES

Difficulty in acquiring an intuitive grasp of the concept of stellar magnitude is not uncommon. Statements to the effect that "magnitude is a logarithmic measure of brightness" appear in virtually every elementary text, but explicit definitions of brightness (flux, intensity, luminosity) often do not appear.

An intuitively appealing definition of brightness is "How many photons from a given star are incident per second per unit area on my detector (telescope, eyeball, film, photometer, CCD ...)?" The purpose of this note is to derive an explicit relation between brightness as defined in this way and the apparent magnitude and surface temperature of a star. The argument requires knowledge only of the Planck blackbody function, Stefan's law, and the distance modulus equation. Of course, the latter already has built-in the concept of magnitude as a logarithmic measure of intensity and the inverse-square law; the idea here is not to define magnitude from first principles, but to give a recipe by which something physically imaginable – photon numbers – can be computed from a magnitude.

We begin with Planck's formula for the energy emitted per second (that is, power) per unit area from a surface of absolute temperature T between wavelengths λ and $\lambda + d\lambda$:

$$B_\lambda(T)d\lambda = \frac{2\pi hc^2}{\lambda^5(e^{hc/\lambda kT} - 1)} d\lambda \text{ [W/m}^2\text{]} \tag{1}$$

(square brackets following equations enclose the relevant units; SI (Système International) units are maintained unless otherwise indicated). Dividing by the energy of a photon, $E = hc/\lambda$, and integrating over all wavelengths converts equation (1) into an expression for emitted photon flux F_e, the total number of photons emitted per second per square metre of surface area:

$$F_e = 2\pi c \int_0^\infty \frac{d\lambda}{\lambda^4(e^{hc/\lambda kT} - 1)} = \left(\frac{2\pi k^3}{h^3c^2}\right) T^3 \int_0^\infty \frac{x^2}{e^x - 1} dx, \text{ [m}^{-2}\text{ s}^{-1}\text{]} \tag{2}$$

where the last step follows on defining $x = (hc/\lambda kT)$. The integrals appearing

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in equation (2) can only be evaluated numerically; that over x evaluates (Massa 1986) to approximately 2.404. Hence

$$F_e = 2.404\left(\frac{2\pi k^3}{h^3c^2}\right)T^3 = \alpha T^3,\ [\mathrm{m}^{-2}\ \mathrm{s}^{-1}] \tag{3}$$

with α evaluating to about 1.52×10^{15} $(\mathrm{K}^3\ \mathrm{m}^2\ \mathrm{s})^{-1}$.

Equation (3) indicates that stellar *photon* flux is proportional to T^3, as opposed to stellar *energy* flux, which we know from Stefan's law to be proportional to T^4. This is because the average photon energy is proportional to T.

Now, consider a star of absolute bolometric magnitude M at distance d parsecs from Earth. The luminosity of this star is given by

$$L = L_0 10^{0.4(M_0-M)}\ [\mathrm{W}], \tag{4}$$

where L_0 is the luminosity (in watts) and M_0 the absolute bolometric magnitude of the Sun. Furthermore, assuming that a star acts like an ideal blackbody, Stefan's law tells us that the surface area of a star of radius r is related to its luminosity and surface temperature via

$$\text{Surface area} = 4\pi r^2 = L/\sigma T^4\ [\mathrm{m}^2]. \tag{5}$$

If equation (4) is substituted into (5) and the result is multiplied by (3), one obtains an expression for the total number of photons emitted by the star per second, that is, the rate R of photon emission:

$$R = \frac{\alpha L_0}{\sigma}\ \frac{10^{0.4(M_0-M)}}{T}\ [\mathrm{s}^{-1}]. \tag{6}$$

The apparent bolometric magnitude m of the star is related to d and M via the distance modulus equation:

$$d = 10^{0.2(m-M+5)}\ [\mathrm{pc}] = \beta 10^{0.2(m-M+5)}\ [\mathrm{m}] \tag{7}$$

where β is the number of metres in a parsec (3.086×10^{16}). The number of photons incident at Earth per second per square metre (incident photon flux F_i) is given by equation (6) divided by the area of the sphere over which the emitted photons are distributed after travelling distance d, that is, $4\pi d^2$. Using equation (7) for d gives

$$F_i = \frac{\alpha L_0}{400\pi\beta^2\sigma}\ \frac{10^{0.4(M_0-m)}}{T}\ [\mathrm{m}^{-2}\ \mathrm{s}^{-1}]. \tag{8}$$

Note that the absolute magnitude of the star has cancelled out: only observable quantities (T, from the colour of the star, and m) remain. Stefan's constant σ can be expressed as $2\pi^5 k^4/15h^3c^2$ (Zemansky and Dittman 1981), and with incorporation of the definition of α from equation (3), equation (8) reduces to

$$F_i = \left[\frac{15(2.404)L_0}{400\pi^5\beta^2 k}\right] \frac{10^{0.4(M_0-m)}}{T} \; [\mathrm{m^{-2}\ s^{-1}}]. \tag{9}$$

The only physical constant explicitly appearing in our result is Boltzmann's constant. From the OBSERVER'S HANDBOOK we find $L_0 = 3.85 \times 10^{26}$W and $M_0 = 4.75$, hence

$$F_i = 6.85 \times 10^{14} \frac{10^{-0.4m}}{T} \; [\mathrm{m^{-2}\ s^{-1}}]. \tag{10}$$

For practical purposes we are probably more interested in how many of these photons are detectable by the human eye. The range of wavelength sensitivity of the eye is a matter of personal physiology, but for sake of argument we take it to be from 400 to 625 nm (a range referred to in the following as the visual or V-band). These limits represent the wavelengths at which the sensitivity of a dark-adapted eye falls to a value of about 0.01 relative to the wavelength of maximum sensitivity (about 500 nm) according to figure 5 of Hughes (1983).

Figure 1 shows the logarithm of the fraction of emitted photons with wavelengths in the V-band as a function of the logarithm of stellar surface temperature. The curve reaches a maximum at around $T = 10\,000$ K (roughly the surface temperature of an A0 star), where the fraction is about 0.258. For this T and $m = 6$, equation (10) predicts some 27 300 photons incident per square centimetre per second. Multiplying by a factor of 0.258 and assuming a dark-adapted pupil to be of diameter 0.7 cm yields some 2700 V-band photons entering the eye per second from a 6th mag A0 star – a number easy to keep in mind. For a zeroth-magnitude star the corresponding number is about 670 000. Hughes remarks that stars of magnitude 8.5 can be seen with the naked eye if the majority of extraneous light is excluded; in this case we would have only some 270 photons per second – a testament to the near "quantum-noise" sensitivity of the eye. In reality, the eye must be much more sensitive than these numbers would indicate, as we have taken no account of losses due to reflection, scattering and absorption within the eye, or of the properties of rod cells.

This model gives results in reasonable accord with actual measurements. Code (1960) quotes results indicating that the energy flux at the top of Earth's atmosphere due to a star of $V = B - V = 0$ (that is, a zeroth-magnitude A star with $T \approx 10^4$ K) is 3.8×10^{-11} W m^{-2} nm^{-1} wavelength interval at 540 nm, equivalent to about 10 300 photons s^{-1} cm^{-2} nm^{-1}. Equation (10) and figure 1 predict

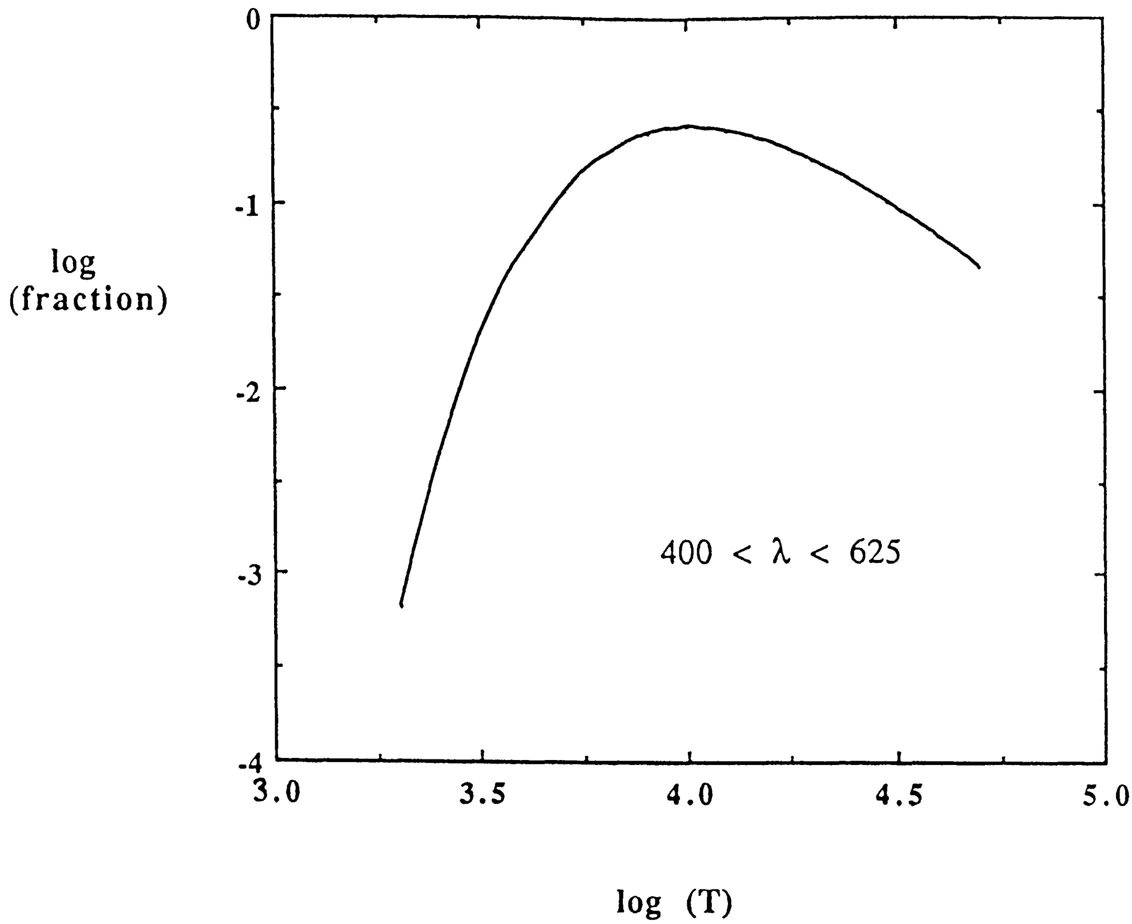

FIG. 1—Fraction of photons emitted by a star with $400 \leqq \lambda \leqq 625$ nm as a function of stellar surface temperature T. Note the (base 10) logarithmic scales.

some 1.8×10^6 photons incident per cm^2 per s; dividing by 225 nm gives 8000 photons s^{-1} cm^{-2} nm^{-1}. This is not a bad result: stars are not perfect blackbodies and our estimate will be somewhat low due to bolometric magnitudes being brighter than their visual counterparts – about 0.4 mag in this case.

It may seem surprising that equation (10) predicts lower photon fluxes for progressively hotter stars of a given apparent magnitude. We can understand this as follows. If a star is to maintain a fixed energy output, then equation (5) says that its surface area must be proportional to $1/T^4$. At the same time, equation (3) indicates that photon output per unit surface area grows only as T^3: the overall effect is the $1/T$ dependence seen in (10). Unfortunately, we can get no information on r or d from measurements of m and T alone – an absolute magnitude is necessary to relate them.

B. Cameron Reed,
Department of Physics,
Alma College,
Alma, Michigan 48801,
U.S.A.

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