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A COASTING COSMOLOGY

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ABSTRACT

A Friedmann-Robertson-Walker cosmology with energy density decreasing in expansion as R^{-2}, where R is the Robertson-Walker scale factor, is studied. In such a model the universe expands with constant velocity; hence the term coasting cosmology. Observational consequences of such a model include the age of the universe, the luminosity distance–redshift relation (the Hubble diagram), the angular diameter distance–redshift relation, and the galaxy number count as a function of redshift. These observations are used to limit the parameters of the model. Among the interesting consequences of the model are the possibility of an ever-expanding closed universe, a model universe with multiple images at different redshifts of the same object, a universe with $\Omega - 1 \neq 0$ stable in expansion, and a closed universe with radius smaller than H_0^{-1}.

Subject heading: cosmology

I. INTRODUCTION

The standard Friedmann-Robertson-Walker (FRW) cosmology includes two components in the stress-energy tensor; radiation with an equation of state $p_R = \rho_R/3$, and matter with an equation of state $p_M = 0$. In this paper I will consider a cosmological model with stress-energy tensor dominated by a new type of matter, called K-matter, with equation of state $p_K = -\rho_K/3$.

In the remainder of the introduction I will review the dynamical equations of the FRW cosmology (for a more complete review, see, e.g., Weinberg 1972). In the next section I will discuss kinematic tests of the model. The final section contains the conclusions and discusses some motivation for the model.

Recall that the standard FRW metric is spatially homogeneous and isotropic and can be written in the form

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right), \quad (1)$$

where (r, θ, ϕ) are comoving coordinates, and $R(t)$ is the cosmic scale factor. I will rescale the coordinates so that k is $+1$, -1 or 0, corresponding to constant-time spatial sections of constant positive curvature, constant negative curvature, or vanishing curvature, respectively. With such a rescaling, the coordinate r in equation (1) is dimensionless, and $R(t)$ has dimensions of length. In the case $k = +1$, the spatial geometry is that of the three-sphere of radius $R(t)$, and r ranges from 0 to 1.

To be consistent with the symmetries of the metric, the stress-energy tensor, $T_{\mu\nu}$, must be diagonal, and the nonzero spatial parts of the metric must be equal. The simplest realization of such a stress-energy tensor is that of a perfect fluid characterized by an energy density ρ and a pressure p:

$$T^{\mu}_{\nu} = \text{diag}\,(\rho, -p, -p, -p)\,. \quad (2)$$

The conservation of stress-energy equation is $T^{\mu\nu}_{;\nu} = 0$. The $\mu = 0$ component of this equation gives the familiar first law of thermodynamics:

$$d(\rho R^3) = p d(R^3)\,. \quad (3)$$

For an equation of state $p = w\rho$, the energy density evolves as $\rho \propto R^{-3(1+w)}$. For radiation, $w = \frac{1}{3}$ and $\rho_R \propto R^{-4}$, while for matter $w = 0$ and $\rho_M \propto R^{-3}$. In a universe with both radiation and matter, the "early" universe will be radiation dominated ($\rho_R \gg \rho_M$) and the "late" universe will be matter dominated ($\rho_M \gg \rho_R$). For K-matter with equation of state $w = -\frac{1}{3}$, $\rho_K \propto R^{-2}$. For a model universe with radiation, matter, and K-matter, the "intermediate" universe will be matter dominated, while the late universe will be K-dominated.

The evolution of $R(t)$ follows from the Einstein equations. The 0–0 component of the Einstein equation gives the Friedmann equation

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho\,, \quad (4)$$

and the i–i component gives

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p\,. \quad (5)$$

The difference of equations (5) and (4) provides an equation for the acceleration, $\ddot{R}$:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)\,. \quad (6)$$

Note that for matter and radiation, $\rho + 3p > 0$, so the universe decelerates in expansion, while for K-matter, $\rho + 3p = 0$, and the acceleration vanishes.

The expansion rate of the universe is described by the Hubble parameter, $H \equiv \dot{R}/R$. The Friedmann equation can be expressed in the forms

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2}\,; \quad \frac{k}{R^2H^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1\,, \quad (7)$$

where Ω is the ratio of the density to the critical density ρ_C:

$$\Omega \equiv \frac{\rho}{\rho_C}\,, \quad \rho_C \equiv \frac{3H^2}{8\pi G}\,. \quad (8)$$

The i different components to the energy density will often be parameterized by the individual Ω_i. If the energy density of the universe is dominated by K-matter the energy density term scales in expansion in the same manner as the curvature term k/R^2.

It will prove convenient to define a dimensionless parameter K by

$$\frac{8\pi G}{3}\rho_K \equiv \frac{K}{R^2}\,. \tag{9}$$

Since $\rho_K \propto R^{-2}$, K is constant. In terms of K, the Friedmann equation is

$$\left(\frac{\dot{R}}{R}\right)^2 \equiv H^2 = \frac{K}{R^2} - \frac{k}{R^2}\,, \tag{10}$$

and $\Omega = 1 + k/R^2H^2 = 1 + k/\dot{R}^2$ is a constant, given by

$$\Omega = \frac{K}{K-k}\,. \tag{11}$$

From equation (10) it is clear that K-matter acts as an effective curvature term (hence the notation "K").

Only the case $K \geq 1$ will be of interest for the closed model. If $k = +1$ (a closed model) and $K \leq 1$, the universe will recollapse due to the curvature, and K-matter will not have much of an effect. If $k = -1$ (an open model) and $K \leq 1$, K-matter will have a small, negligible effect.

Of course since the deceleration vanishes, $\dot{R}$ is constant in time. A universe dominated by K-matter will either expand with constant velocity or collapse with constant velocity. In a universe with radiation, matter, and K-matter, a collapsing universe will become dominated in turn by matter and then radiation, while an expanding universe once dominated by K-matter will remain so. Therefore the expanding model will be studied.

The *present* expansion rate of the universe is defined to be the Hubble *constant*, H_0. The Hubble *parameter*, $H \equiv \dot{R}/R$, in general evolves in time. From equation (10), for the K-dominated universe

$$H^2R^2 = H_0^2R_0^2 = K - k = \text{const}\,. \tag{12}$$

In a universe dominated by radiation or matter, H^2R^2 is not constant, but decreases in time. Therefore as seen from equation (7), $\Omega - 1$ in general grows in time, and Ω exactly 1 is the only stable solution. However, since H^2R^2 is constant in a K-dominated universe, any value of Ω is stable. Of course as discussed above, for the $k = 1$ case, $K > 1$ and $\Omega_K \geq 1$. As $K \to 1, \Omega_K \to \infty$ and as $K \to \infty, \Omega_K \to 1$.

II. EVOLUTION OF A K-DOMINATED UNIVERSE

a) Age of the Universe

The age of a K-dominated universe is found by integrating the Friedmann equation (10). Obviously $\dot{R}$ is a constant, and the solution for $R(t)$ is straightforward

$$\dot{R} = (K-k)\;;\quad R = (K-k)^{1/2} = R_0H_0t\,. \tag{13}$$

The above equation for $\dot{R}$ is a special case of the more general result

$$\dot{R}^2 = R_0^2H_0^2\left[1 - \Omega_0 + \Omega_0\left(\frac{R_0}{R}\right)^{(1+3w)}\right], \tag{14}$$

which follows from the Friedmann equation and the fact that $\rho \propto R^{-3(1+w)}$.

With the usual definition of the redshift z (here and below the subscript 0 on a parameter will refer to its present value)

$$1 + z \equiv \frac{R_0}{R}\,, \tag{15}$$

the age of a K-dominated universe is

$$t = \frac{H_0^{-1}}{(1+z)}\,. \tag{16}$$

This gives for the present age of the universe $t_0 = H_0^{-1} = 9.776 \times 10^9\, h^{-1}$ yr, where the Hubble constant is taken to be $H_0 = 100h$ km s^{-1} Mpc^{-1}. This age is 50% larger than the age of a matter-dominated universe and twice the age of a radiation-dominated universe. Of course in a realistic model with radiation, matter, and K-matter, the age must be adjusted to reflect the fact that equation (16) is only valid for the epoch of K-domination. Since the age of the universe is independent of K, the age does not restrict the model.

b) Horizons and r(z)

One of the most fundamental parameters in classical cosmology is the fraction of the universe in causal contact. Consider a comoving observer at coordinates ($r_0 = 0$, θ_0, ϕ_0) at time t. A light signal satisfies the geodesic equation of motion $ds^2 = 0$. Geodesics intersecting $r_0 = 0$ are lines of constant θ and ϕ. Thus the geodesic equation becomes $0 = dt^2 - R^2(t)dr^2/(1 - kr^2)$, and a light signal emitted from (r_H, θ, ϕ) at time $t = 0$ will reach the observer at time t determined by

$$\int_0^1 \frac{dt'}{R(t')} = \int_0^{r_H} \frac{dr}{(1-kr^2)^{1/2}}\,. \tag{17}$$

Since the proper distance to the horizon is

$$d_H(t) = \int_0^{r_H} (g_{rr})^{1/2}\,dr\,,$$

the distance to the horizon can be expressed as

$$d_H(t) = R(t)\int_0^t \frac{dt'}{R(t')} = R(t)\int_0^{R(t)} \frac{dR(t')}{\dot{R}(t')R(t')}\,. \tag{18}$$

For a general w, using equation (14) and defining $x \equiv R(t)/R(t_0)$,

$$\begin{aligned} d_H(t) &= \frac{1}{H_0(1+z)}\int_0^{(1+z)^{-1}} \frac{dx}{[x^2(1-\Omega_0) + \Omega_0 x^{1+3w}]^{1/2}} \\ &= \frac{1}{H_0(1+z)}\int_0^{(1+z)^{-1}} \frac{dx}{x}\quad\left(w = -\frac{1}{3}\right). \end{aligned} \tag{19}$$

In the K-dominated universe the distance to the horizon diverges. Of course in a realistic cosmology, the lower limit must be modified to reflect the fact that the early universe was radiation dominated and the intermediate universe was matter dominated.

The fact that the horizon distance formally diverges implies that if the universe is dominated by K-matter, it is possible to see around the three-sphere in the $k = +1$ model. This curious circumstance is due to the fact that if $K > 1$, a closed universe expands forever. Seeing around the universe will have consequences in observable parameters such as the luminosity distance-redshift relation discussed below. The phenomena of seeing around the universe cannot occur in a matter-dominated universe, as the antipodal point is visible only at the point of maximum expansion of the universe, and the universe is circumnavigated by a photon only upon completion of the entire expansion-collapse cycle of the closed model

By changing the limits of integration in equation (19) it is possible to express any coordinate $r_1 < r_H$ in terms of Ω_0 and z:

$$\int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \int_{R_1}^{R_0} \frac{dR(t')}{\dot{R}(t')R(t')} = \frac{1}{R_0 H_0} \int_0^{(1+z)^{-1}} \times \frac{dx}{[x^2(1-\Omega_0) + \Omega_0 x^{(1-3w)}]^{1/2}} . \quad (20)$$

For a matter-dominated universe ($w = 0$) the solution for any choice of k is

$$r_1(z) = \frac{q_0 z + (q_0 - 1)[(2q_0 z + 1)^{1/2} - 1]}{H_0 R_0 q_0^2 (1+z)} , \quad (21)$$

where the deceleration parameter q_0 is related to Ω_0 by $2q_0 = \Omega_0$ for the matter-dominated model. For a K-dominated universe

$$r_1(z) = \begin{cases} (R_0 H_0)^{-1} \ln(1+z) , & k = 0 , \\ |\sin[(R_0 H_0)^{-1} \ln(1+z)]| & k = +1 , \\ \sinh[(R_0 H_0)^{-1} \ln(1+z)] , & k = -1 . \end{cases} \quad (22)$$

Note that the $k = 0$ result can be obtained from the $k = \pm 1$ results by taking the limit $H_0 R_0 = (K-1)^{1/2} \to \infty$.

Equation (22) will be used extensively in the next section. Of particular interest is the periodic nature of $r_1(z)$ for the $k = +1$ geometry. As z starts from zero and increases, the three-sphere is circumnavigated. Extrema of $r_1(z)$ occur at the points $(1 + z_n) = \exp(n\pi R_0 H_0/2)$, with odd n corresponding to maxima $r_1(z_{n=2m+1}) = 1$ and even n corresponding to minima $r_1(z_{n=2m}) = 0$. Recall that the coordinate system was chosen such that the point $z = 0$, $r_1 = 0$ is the "north pole" of the three-sphere.[1] As z increases from 0 to z_1, r_1 increases, reaching its maximum $r_1 = 1$ at $z = z_1$. The point z_1 corresponds to the "equator" of the three-sphere. As z increases in the range $z_1 \leq z \leq z_2$, r_1 decreases, reaching its minimum $r_1 = 0$ when $z = z_2$. The point z_2 is the "south pole" of the three-sphere. Further increase in z in the range $z_2 \leq z \leq z_3$ leads to an increase in r_1, reaching a second maximum $r_1 = 1$ at $z = z_3$ when the equator is again reached. Increasing z in the interval $z_3 \leqslant z \leqslant z_4$ leads to decreasing r_1 as the north pole is approached at $z = z_4$. As z is increased further, the three-sphere is again circumnavigated. As an observer looks out into the Universe, objects with redshift $z_{n=4m+1}$ are located on the equator, objects with redshift $z_{n=4m+2}$ are located on the south pole, objects with redshift $z_{n=4m+3}$ are also located on the equator, but are viewed by looking past the south pole, and objects with redshift $z_{n=4m}$ are at the north pole. The integer $m = 0, 1, 2, \ldots$ indicates how many times the photon has circumnavigated the universe before detection. In principle, every object will be seen with multiple redshifts, but in practice if the object (or observer) has a nonzero peculiar velocity, the object will appear with different redshifts in different locations.

III. KINEMATIC TESTS OF THE COASTING MODEL

a) Luminosity Distance–Redshift

The luminosity distance is defined as the ratio of the detected energy flux, $\mathscr{F}$, and the luminosity of the source, $\mathscr{L}$:

$$d_L^2 = \frac{\mathscr{L}}{4\pi\mathscr{F}} . \quad (23)$$

In the absence of expansion, the luminosity distance is simply the physical distance to the source. In the FRW cosmology, the luminosity distance to a source at coordinate (r_1, θ, ϕ) is (assuming for convenience that the observer is located at $r = 0$)

$$d_L^2 = R_0^2 r_1^2 (1+z)^2 . \quad (24)$$

The factors are easy to understand. At the time of detection (t_0), a two-sphere surrounding the source encompassing the observer has area $4\pi R_0^2 r_1^2$. Thus the factor $R_0^2 r_1^2$ is simply the "inverse-square law." In traveling from the source to the detector, each photon suffers a decrease in energy of $(1 + z)$, and the time between arrival of successive photons is increased by a factor of $(1 + z)$. Therefore the factor of $(1 + z)^2$ decrease in the detected flux is due to the redshift of the radiation between the time of emission and the time of detection. Equation (24) follows simply from the form of the metric, and is independent of the form of the stress tensor.

Using equation (21) for $r_1(z)$, the luminosity distance in the standard matter-dominated universe is given by

$$H_0 d_L = q_0^{-2}\{zq_0 + (q_0 - 1)[(2q_0 z + 1)^{1/2} - 1]\} . \quad (25)$$

Expansion of this expression for small z yields

$$H_0 d_L = z + \tfrac{1}{2}(1 - q_0 z^2 + \cdots . \quad (26)$$

Of course the familiar linear "Hubble law" only obtains for small z. This luminosity distance as a function of redshift is shown as the dashed curves in Figure 1 for three values of q_0 ($q_0 = 0, 2$, and 5).

Using equation (22) for a K-dominated model, the luminosity distance as a function of z is

$$H_0 d_L = \begin{cases} H_0 R_0 (1+z) |\sin[(H_0 R_0)^{-1} \ln(1+z)]| & k = +1 , \\ H_0 R_0 (1+z) \sinh[(H_0 R_0)^{-1} \ln(1+z)] & k = -1 . \end{cases} \quad (27)$$

Expansion of the above expressions for small z gives

$$H_0 d_L = z + \tfrac{1}{2} z^2 + \cdots \quad (28)$$

The luminosity distance as a function of redshift for a K-dominated universe is shown in Figure 1 as the solid curves for the indicated values of K. Figure 1*a* is for the closed $k = +1$ geometry while Figure 1*b* is for the open $k = -1$ geometry. As with all kinematic tests, different cosmological models have similar behavior at $z \ll 1$ (as can be easily seen by the above small-z expansions), and the greatest discrimination among different models comes from observations at the largest redshift.

Construction of a luminosity distance–redshift (or Hubble) diagram is the most fundamental program in observational cosmology. There have been countless surveys in the literature. Of course at high redshift ($z \gtrsim 1$) evolutionary effects on galactic luminosity obfuscate the program. The data point and concomitant error bar in Figure 1 indicates the range of possible data at the largest z where evolutionary effects are not expected to swamp the results. For instance the data of Kristian, Sandage, and Westphal (1978) extend to $z \simeq 0.75$ and are consistent with matter-dominated universes with $0 \lesssim q_0 \lesssim 5$ (see Rowan-Robertson 1985). If the data point and error bar is taken seriously, the following limits to K obtain

$$K \gtrsim \begin{cases} 1.08 & k = +1 , \\ 1.5 & k = -1 . \end{cases} \quad (29)$$

The limits to K are summarized in Table 1.

[1] The terms north pole, equator, and south pole refer to the geometry of the three-sphere, and of course having nothing to do with any particular direction in the sky, e.g., north galactic pole, and so on.

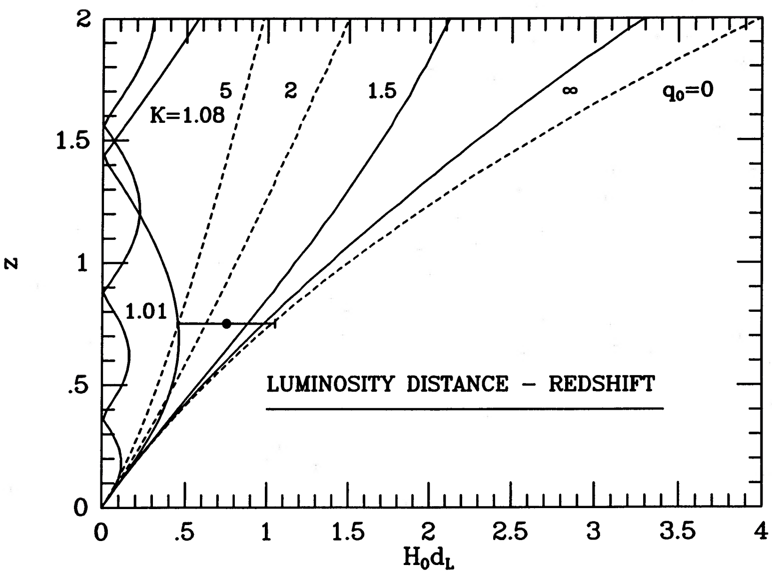

FIG. 1*a*

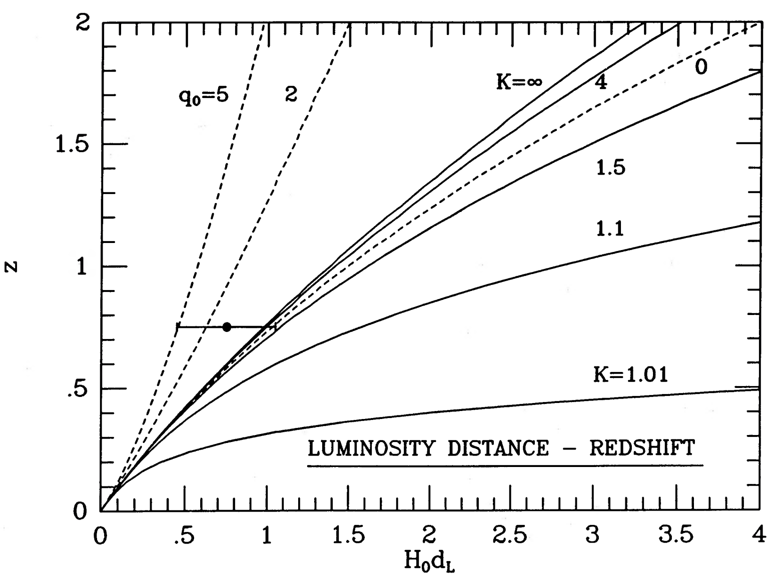

FIG. 1*b*

FIG. 1.—The luminosity distance-redshift relation for a matter-dominated universe (*dashed curves*) and a K-dominated universe (*solid curves*). Results for the matter-dominated model are given for $q_0 = 0$, 2, and 5. (*a*) The K-dominated results for a closed model with $K = 1.01$, 1.08, 1.5 and ∞. (*b*) The K-dominated results an open model with $K = 1.01$, 1.1, 1.5, 4, and ∞. The data point and error bar represents the results of Kristian, Sandage, and Westphal (1978).

The form of the Hubble diagram for small K in the $k = +1$ case is particularly unusual. It is due to the fact that since $R_0^2 H_0^2 = K - 1$, as $K \to 1$ the present radius of the three-sphere, R_0 becomes small: $R_0 = H_0^{-1}(K-1)^{1/2} \simeq 3000h^{-1}$ $(K-1)^{1/2}$ Mpc. If $K = 1.01$, for example, the radius of the three-sphere is only 300 Mpc. If we rotate the coordinate system to place ourselves at the north pole of the three-sphere, then the equator is at $z = z_1 = \exp(\pi R_0 H_0/2) - 1 = \exp(\pi(K-1)^{1/2}/2) - 1 = 0.1701$, the south pole at $z = z_2 = \exp(\pi R_0 H_0) - 1 = \exp[\pi(K-1)^{1/2}] - 1 = 0.3691$, the equator viewed by looking around the south pole is at $z = z_3 = \exp(3\pi R_0 H_0/2) - 1 = \exp[3\pi(K-1)^{1/2}/2] - 1 = 0.6020$, and finally if we look out to $z = z_4 = \exp(2\pi R_0 H_0) - 1 = \exp[2\pi(K-1)^{1/2}] - 1 = 0.8745$ we are looking around the universe and observing our own position. The effect of the closed geometry is to focus the light from any object in the opposite hemisphere. Light emitted from the south pole converges at the north pole, giving an infinite flux and therefore a vanishing luminosity distance. The light from an object at $z = z_6$ is also at the south pole, but it has traveled

TABLE 1
LIMITS TO K

Method	$k = +1$	$k = -1$
Luminosity distance–redshift	$K \gtrsim 1.08$	$K \gtrsim 1.5$
Number count–redshift	$K \gtrsim 1.2$	$K \gtrsim 4.0$
Angular diameter–redshift	$K \gtrsim 1.2$	$K \gtrsim 1.5$
Nonobservation of south pole	$K \gtrsim 1.2$	...

1.5 times around the universe before converging on the north pole. The $(1+z)^2$ factor in equation (24) results in the growth of the envelope of maxima of $H_0 d_L$.

The convergence property of the $k=1$ cosmology allows a limit to be placed on K. If convergence occurs, very bright high-redshift objects would be seen.[2] The lack of any obvious candidates implies that the south pole has not been observed. Since the south pole is at a redshift of $z = z_2 = \exp(\pi R_0 H_0) - 1 = \exp[\pi(K-1)^{1/2}] - 1$, with the choice $z_2 \gtrsim 3$, then $K \gtrsim 1.2$. The limit to K is not very sensitive to the minimum value chosen for z_2.

b) *Number Count–Redshift*

Another kinematic test is the galaxy number count per redshift interval. The number of galaxies in a comoving volume element in an angular solid area $d\Omega$ with redshift between z and $z + dz$ is sensitive to the number of galaxies, n, in a comoving volume element dV_C, and the spatial curvature:

$$dN_{\rm gal} = n\,dV_C = n\frac{r^2}{(1-kr^2)^{1/2}}\,dr\,d\Omega\,. \tag{30}$$

Using the exact solution for $r(z)$ in a matter-dominated model ([eq. [21])

$$\frac{1}{n}\frac{1}{z^2}\frac{dN_{\rm gal}}{dz\,d\Omega} = \frac{(H_0 R_0)^{-3}}{(1+z)^3 z^2 q_0^4} \times \frac{\{zq_0 + (q_0-1)[(2q_0 z+1)^{1/2}-1]\}^2}{[1-2q_0+2q_0(1+z)]^{1/2}}\,. \tag{31}$$

The small z expansion of the above result gives

$$\frac{1}{n}\frac{1}{z^2}\frac{dN_{\rm gal}}{dz\,d\Omega} = (H_0 R_0)^{-3}[1 - 2(q_0+1)z + \cdots]\,. \tag{32}$$

For the K-dominated universe, equation (22) leads to the result

$$\frac{1}{n}\frac{1}{z^2}\frac{dN_{\rm gal}}{dz\,d\Omega} = \begin{cases} [R_0 H_0(1+z)z^2]^{-1}\sin^2[(H_0 R_0)^{-1}\ln(1+z)] & k=+1 \\ [R_0 H_0(1+z)z^2]^{-1}\sinh^2[(H_0 R_0)^{-1}\ln(1+z)] & k=-1\,, \end{cases} \tag{33}$$

with small z expansion

$$\frac{1}{n}\frac{1}{z^2}\frac{dN_{\rm gal}}{dz\,d\Omega} = (H_0 R_0)^{-3}(1 - 2z + \cdots)\,. \tag{34}$$

Again, as expected, the small z expansions are identical in the two models.

The number count–redshift results for a K-dominated universe are shown in Figure 2 as the solid curves for the indicated values of K. Figure 2*a* is for the closed $k=+1$ geometry while Figure 2*b* is for the open $k=-1$ geometry. Also shown in Figure 2 by the dashed curves are the results for the matter-dominated model for $q_0 = 0$, 0.1, and 0.5. The data point and error bars are taken from the recent analysis of Loh and Spillar (1986). Loh and Spillar also have data points at $z = 0.5$ and $z = 0.25$, but as before, the largest z data point is the most sensitive for the purpose at hand.

Using the results of Loh and Spillar, the limits to K are

$$K \gtrsim \begin{cases} 1.2 & k=+1 \\ 4.0 & k=-1\,. \end{cases} \tag{35}$$

The limits to K are summarized in Table 1.

b) *Angular Diameter–Redshift*

The final kinematic test that will be considered here is the angular diameter–redshift test. The angular diameter of galaxy clusters have been measured out to $z \simeq 1$ (Bruzual and Spinrad 1978). Consider a source at $r = r_1$ which emitted light at $t = t_1$ (again taking our position as the north pole of the three-sphere). The observed angular diameter of the source, δ, is related to the proper diameter of the source, D, by

$$\delta = \frac{D}{R(t_1)r_1}\,. \tag{36}$$

The angular diameter distance, d_A is defined to be

$$d_A = \frac{D}{\delta} = R(t_1)r_1 = \frac{R_0 r_1}{1+z}\,. \tag{37}$$

For the matter-dominated model,

$$H_0 d_A = q_0^{-2}(1+z)^{-2}\{zq_0 + (q_0-1)[(2q_0 z+1)^{1/2}-1]\}\,, \tag{38}$$

which has small z expansion

$$H_0 d_A = z - \tfrac{1}{2}(3+q_0)z^2 + \cdots\,. \tag{39}$$

For the K-dominated model,

$$H_0 d_A = \begin{cases} R_0(1+z)^{-1}|\sin[(H_0 R_0)^{-1}\ln(1+z)]| & k=+1 \\ R_0(1+z)^{-1}\sinh[(H_0 R_0)^{-1}\ln(1+z)] & k=-1\,, \end{cases} \tag{40}$$

with small-z expansion

$$H_0 d_A = z - \tfrac{3}{2}z^2 + \cdots\,. \tag{41}$$

The angular diameter–redshift results for a K-dominated universe are shown in Figure 3 as the solid curves for the indicated values of K. Figure 3*a* is for the closed $k=+1$ geometry while Figure 3*b* is for the open $k=-1$ geometry. Also shown in Figure 3 by the dashed curves are the results for the matter-dominated model for $q_0 = 0$, 0.5, and 1.0. The data point and error bars are taken from Bruzual and Spinrad. Again, data at smaller z are not important.

The angular diameter–redshift limits to K are

$$K \gtrsim \begin{cases} 1.2 & k=+1 \\ 1.5 & k=-1\,. \end{cases} \tag{42}$$

The limits to K are included in Table 1.

IV. CONCLUSIONS AND SUMMARY

A universe dominated by K-matter has several interesting properties. (1) A closed universe may expand forever. (2) A model with $\Omega \neq 1$ is a stable model, i.e., $\Omega - 1$ is constant in time. (3) In the case of the closed universe model, the radius of the universe may be small. (4) The closed universe model admits the possibility of multiple images at different redshifts of the same object. (5) In the closed model, light from distant

[2] The fact that "nearby" $z \lesssim 1$ quasars are seen removes the temptation of identifying such objects as quasars.

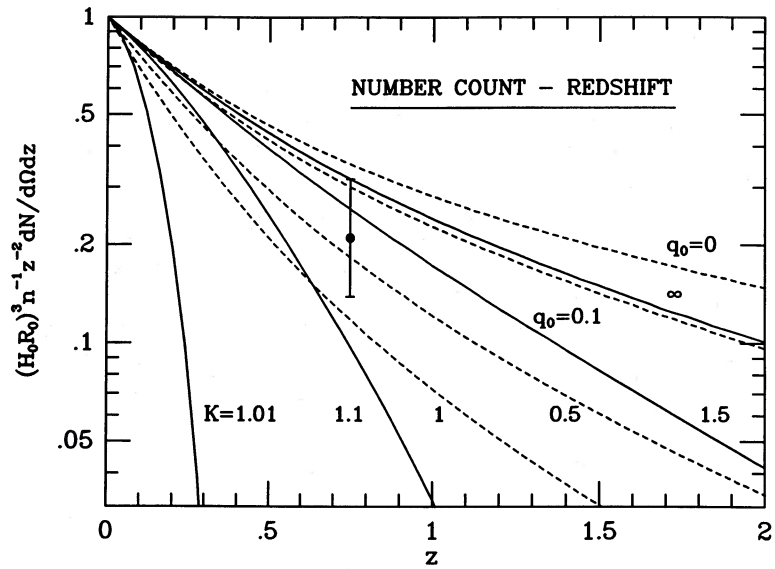

FIG. 2*a*

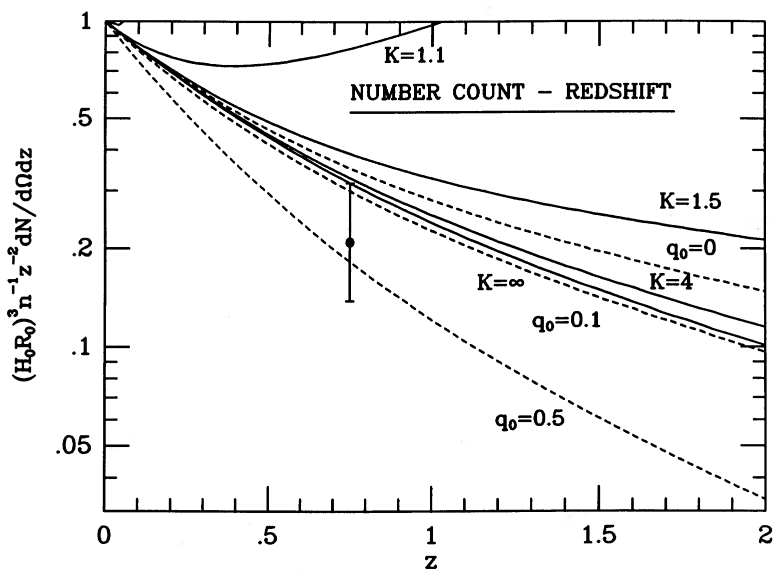

FIG. 2*b*

FIG. 2.—The number count–redshift relation for a matter-dominated universe (*dashed curves*) and a K-dominated universe (*solid curves*). Results for the matter-dominated model are given for $q_0 = 0$, 0.5, and 1.0. (*a*) The K-dominated results for a closed model with $K = 1.01$, 1.1, 1.5, and ∞. (*b*) The K-dominated results an open model with $K = 1.1$, 1.5, 4, and ∞. The data point and error bar are from Loh and Spillar (1986).

objects converge due to the geometry. (6) The distance to the horizon diverges. (7) The age of the universe is H_0^{-1}.

Models are parameterized by a dimensionless constant, $K \equiv 8\pi G\rho_K R^2/3$. Kinematic tests limit K (see Table 1). Since for the $k = +1$ case the limits imply $K \gtrsim 1.2$, the radius of the universe is today greater than $R_0 \geq H_0^{-1}(K-1)^{1/2} \gtrsim 1300h^{-1}$ Mpc.

The fraction of the critical density contributed by K-matter is $\Omega_K = K/(K-k)$, ($K \geq 1$ for $k = +1$). For the closed model, $K \gtrsim 1.2$ implies $\Omega_K \lesssim 6$, while for the open model $K \gtrsim 4$ implies $\Omega_K \lesssim 0.8$. The K energy density scales as $(1+z)^2$ while the matter energy density scales as $(1+z)^3$. If the fraction of the critical density contributed by matter is Ω_M, the universe was matter-dominated a $z \gtrsim z_*$, given by

$$1 + z_* = \frac{K}{K-k}\frac{1}{\Omega_M} \lesssim \begin{cases} 6\Omega_M^{-1} & k = +1 \\ 0.8\Omega_M^{-1} & k = -1 \, . \end{cases} \tag{43}$$

Even for Ω_M as small as 0.01, the very early history of the universe is unchanged (primordial nucleosynthesis, recombination, and so on).

If the universe is spatially flat ($k = 0$), $\Omega_K \equiv 1$ for any value of K, and if $k \neq 0$, $\Omega_K \to 1$ as $K \to \infty$. It is interesting to note that a spatially flat K-dominated model gives different results for the kinematic tests than does the spatially flat matter-dominated model.

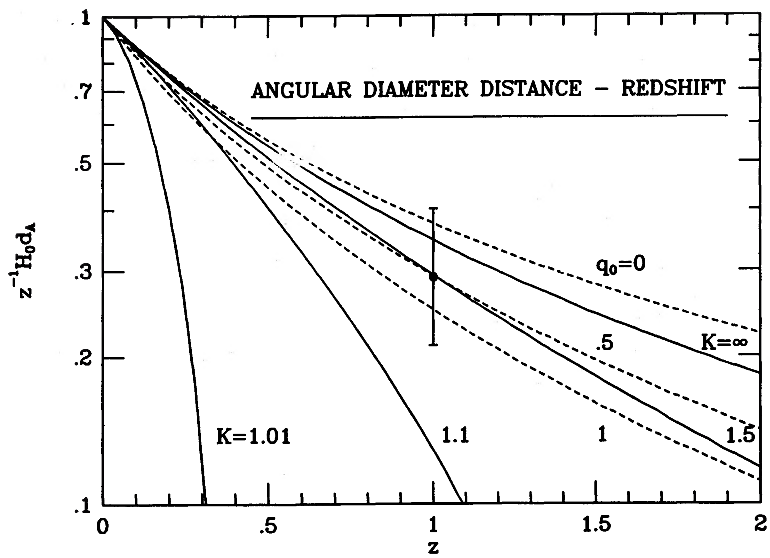

FIG. 3*a*

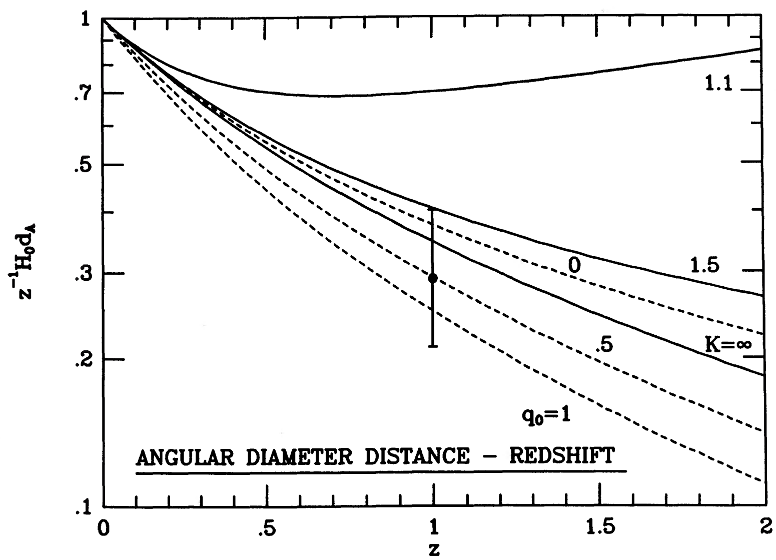

FIG. 3*b*

FIG. 3.—The angular diameter distance–redshift relation for a matter-dominated universe (*dashed curves*) and a K-dominated universe (*solid curves*). Results for the matter-dominated model are given for $q_0 = 0$, 0.5, and 1.0. (*a*) The K-dominated results for a closed model with $K = 1.01$, 1.1, 1.5, and ∞. (*b*) The K-dominated results an open model with $K = 1.1$, 1.5, and ∞. The data point and error bar are from Bruzual and Spinrad (1978).

Is there any reason to motivate the consideration of the existence of K-matter? I mention the possibility of a universe dominated by cosmic strings (Vilenkin 1984). In Vilenkin's paper, K is determined by the scale of spontaneous symmetry breaking of the symmetry responsible for the cosmic strings.

A dynamical question not discussed here is the growth of perturbations in the K-dominated model. If there are no perturbations in K-matter and the universe is K-dominated, then growth in matter perturbations is slowed (Turner 1985). However, if there is the possibility of perturbations in K-matter, then structure formation might proceed faster than in the matter-dominated model. The growth of perturbations in the energy density, δ, on scales inside the horizon well above the Jeans length in a spatially flat FRW model in the linear regime is described by the equation (Weinberg 1972)

$$\ddot{\delta} + 2\,\frac{\dot{R}}{R}\,\dot{\delta} - 4\pi G\rho\delta = 0\ . \tag{44}$$

In the K-dominated model, $\dot{R}/R = t^{-1}$, and $\rho_K = 3K/8\pi G R^2 = 3K/8\pi G(K - k)t^2$, and equation (44) becomes for $k = 0$

$$\ddot{\delta} + 2t^{-1}\dot{\delta} - \tfrac{3}{2}t^{-2} = 0\ , \tag{45}$$

which has growing-mode solution $\delta \propto t^{(7-1)^{1/2}/2} = t^{0.8229}$. This is to be compared with the growing-mode solution $\delta \propto t^{2/3}$ for the matter-dominated case.

Note added in manuscript.—Gott and Rees (1987) have pointed out that K-matter would lead to gravitationally lensed quasars. Their limit is $K > 1.2$.

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REFERENCES

Bruzual G., and Spinrad, H. 1978, *Ap. J.*, **220**, 1.
Gott, J. R., and Rees, M. J. 1987, *M.N.R.A.S.*, **227**, 453.
Kristian, J., Sandage, A., and Westphal, J. 1978, *Ap. J.*, **221**, 383.
Loh, E., and Spillar, E. 1986, *Ap. J.* (*Letters*), **307**, L1.
Rowan-Robinson, M. 1985, *The Cosmological Distance Ladder* (San Francisco: Freeman).
Turner, M. S. 1985, *Phys. Rev. Letters*, **54**, 252.
Vilenkin, A. 1984, *Phys. Rev. Letters*, **53**, 1016.
Weinberg, S. 1972, *Gravitation and Cosmology* (New York: Wiley).

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