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A STUDY OF THE WHITE DWARF LUMINOSITY FUNCTION[1]

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ABSTRACT

We examine the dependence of white dwarf luminosity functions on the assumed age of the Galactic disk, taking into account (1) the relationship between white dwarf mass and progenitor mass, (2) the progenitor lifetime, (3) the mass dependence of cooling curves, and (4) possible variations in the birth rate function in the past. At low luminosities, the resultant distribution in number versus luminosity depends very strongly not only on the shape of the cooling curves for different masses, but also on the spectrum of white dwarf masses that contribute to the distribution. Although the disk age which we estimate by comparing with observations is near 9 Gyr, as found earlier by Winget and colleagues in 1987, we predict for this age that at luminosities only a quarter of a magnitude fainter than the dimmest white dwarfs yet found, the space density of white dwarfs is larger by a factor of 10 than the space density which the Winget *et al.* luminosity function suggests.

Subject headings: luminosity function — stars: evolution — stars: stellar statistics — stars: white dwarfs

I. INTRODUCTION

The theoretical white dwarf luminosity function has been the focus of several previous investigations. In the decades following the classical development of the theory of white dwarf evolution by Kaplan (1950), Mestel (1952), and Schatzman (1953), much of the attention in the field was devoted to comparing observationally based white dwarf luminosity functions with theoretical cooling curves, with the major aim of testing the theory. Reviews by Weidemann (1968, 1979), Shaviv (1979), and Liebert (1980) describe some of this work.

The use of cooling curves to estimate something about the history of star formation was introduced by Schmidt (1959). That the finite age of the Galactic disk should lead to a deficiency of white dwarfs at low luminosity (an abrupt falloff in the observed distribution at a point where the time required by a white dwarf to reach this point is equal to the age of the disk) was demonstrated explicitly by D'Antona and Mazzitelli (1978).

D'Antona and Mazzitelli developed a basic formalism for constructing white dwarf luminosity functions, assuming the availability of a number of inputs: white dwarf cooling curves as a function of the white dwarf mass m; a time-dependent stellar creation function $\phi(t)$; a stellar mass function dN/dM, where M is the mass of a progenitor star and N is the space density of progenitors; main-sequence evolutionary time scales for progenitors $t_{\rm evo}(m)$; and a progenitor mass–white dwarf mass relationship dm/dM. With these ingredients, one can in principle estimate an age for the Galactic disk by constructing luminosity functions for a variety of assumed disk ages and then comparing the theoretical curves with observed white dwarf luminosity functions, *if* the observed function exhibits an abrupt falloff at low luminosity. They then prepared theoretical luminosity functions for assumed Galactic ages of 10 and 11 Gyr, showing that, at these ages, a falloff should occur at luminosities less than $\sim 10^{-4}\ L_\odot$.

Liebert *et al.* (1979) (see also Liebert 1979, 1980) established the existence of an abrupt falloff in the observed luminosity function and some attention was given to the question of whether this falloff could be due to incompleteness in the observations (Liebert 1979, 1980) or to some defect in the theory (e.g., Iben and Tutukov 1984). Winget *et al.* (1987) accepted the falloff as real and, by fitting the observed luminosity function with theoretical luminosity functions, estimated the age of the disk to be $T_d = 9.3 \pm 2$ Gyr.

In constructing their theoretical luminosity functions, Winget *et al.* assumed that, at a luminosity of $10^{-2}\ L_\odot$, the white dwarf number-mass distribution is the same as the overall (luminosity-independent) number-mass distribution given by the observations (Weidemann and Koester 1984). They then evolved this distribution, using the cooling theory as described by Lamb (1974) and Lamb and Van Horn (1975). This approach does not take into account the fact that, if there is a single-valued relationship between white dwarf mass and mass of the main-sequence progenitor, then mass-dependent main-sequence lifetimes will play a role in the structure of derived luminosity functions. It is also possible that, due to observational scatter, the observed number-mass distribution is broader than the actual distribution, and this will affect the structure of the derived luminosity function as well.

In this paper, we adopt the spirit of the D'Antona-Mazzitelli approach. We choose a progenitor mass–white dwarf mass relationship, take explicitly into account the lifetimes of the main-sequence progenitors, and adopt a Salpeter-like initial mass function. We also examine the effects of a time-dependent star formation rate, showing that, although the shape of the white dwarf luminosity function at luminosities below where the maximum space density occurs depends sensitively on the luminosity and mass dependence of the cooling curves, it is virtually independent of any variations in the star formation rate after the very earliest times.

II. THEORETICAL PRELIMINARIES

Our objective is to construct a luminosity function for white dwarfs in the solar neighborhood that has the general form shown in Figure 1, where l is the white dwarf luminosity and $(dn/dl)dl$ is the space density of white dwarfs with luminosities in the range l to $l + dl$. We assume that we have at our disposal a set of mass-dependent cooling curves of the sort illustrated in Figure 2, where m_1

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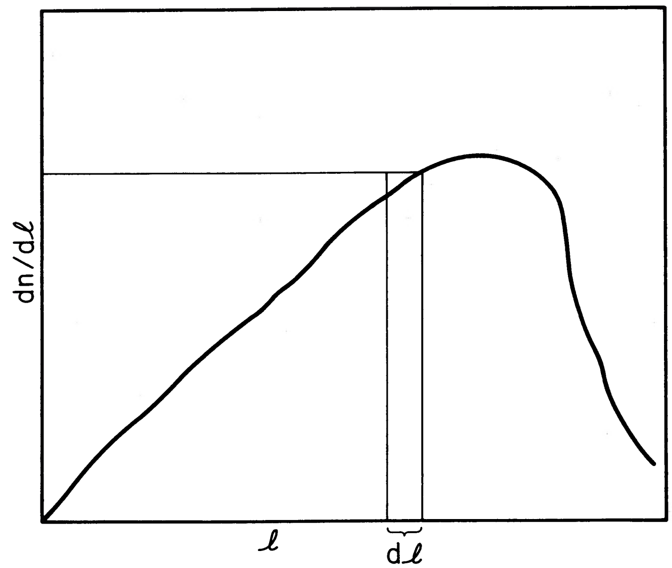

FIG. 1.—General form for luminosity function

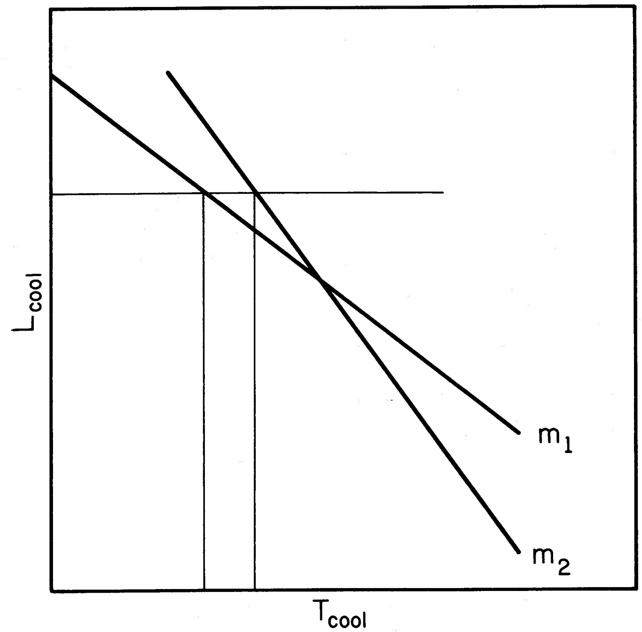

FIG. 2.—Idealized cooling curves

and m_2 are the masses of two model white dwarfs and $t_{\rm cool}$ is the time for a model to cool down to the luminosity $l_{\rm cool}$. In this paper, we shall ignore the complication that the $t_{\rm cool}$-$l_{\rm cool}$ relationship is composition-dependent and assume that

$$t_{\rm cool} = t_c(l, m) . \tag{1}$$

The next step is to adopt some approximation to the relationship between the mass m of a white dwarf and the mass M of its main-sequence progenitor. The true relationship is a function of composition (as well as of the environment, if the white dwarf is formed in a close binary), but we shall ignore these complications and adopt

$$m = m(M) . \tag{2}$$

The third necessary ingredient is the nuclear burning lifetime of the progenitor. Once again we ignore the dependence on composition and suppose that

$$t_{\rm nuclear\ burning} = t_{\rm evo}(M) \tag{3}$$

Now, focusing our attention on a fixed luminosity interval l to $l + dl$ in Figure 1, we wish to determine the range of progenitor masses among the stars formed at time t after the start of disk star formation which can contribute to the observed (dn/dl) at the present time T_d after the start of star formation in the disk. The equation we must solve in conjunction with Equation (1) is

$$t + t_{\rm evo}(M) + t_c(l, m) = T_d . \tag{4}$$

For every choice of l and t, this transcendental equation can be solved for m and then M as illustrated in Figure 3.

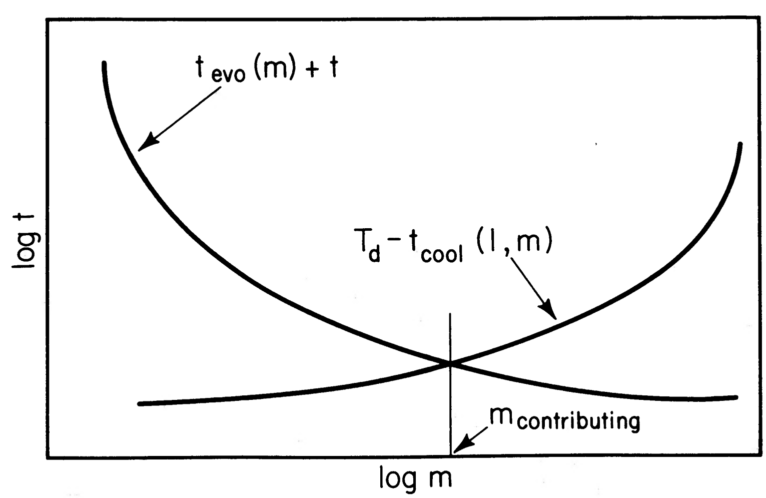

FIG. 3.—Graphical solution of eq. (4) in text. The argument of the function $t_{\rm evo}$ should be $M(m)$.

Differentiating equations (2) and (4) we have

$$dM = -\frac{(\partial t_c/\partial l)_m}{(dt_{\rm evo}/dM) + (\partial t_c/\partial m)_l(dm/dM)}\, dl \,, \tag{5}$$

where dM is the mass range for stars formed at time t which can populate the white dwarf luminosity function in the interval dl about l at time T_d.

The actual contribution to the luminosity function by stars formed at time t depends on the rate of star formation and on the mass function at that time. We shall assume that the number density $d^2n(t)$ of stars of mass M to $M + dM$ that are formed in the interval t to $t + dt$ is given by

$$d^2n(t) = \phi(t)\left(\frac{\partial N}{\partial M}\right)_t dM\, dt \,, \tag{6}$$

where $\phi(t)$ is a birth rate or creation function and $(\partial N/\partial M)_t$ is a mass function which defines how the space density N of progenitors varies with progenitor mass at time t.

In constructing a numerical luminosity function for white dwarfs it is convenient to think of star formation as occurring in delta function bursts. We write

$$\phi(t) = \sum \phi_i \delta(t - t_i) \,, \tag{7}$$

where ϕ_i is the strength of a burst at time t_i and $\delta(t - t_i) = 0$ when $t \neq t_i$, but $\int_{-\infty}^{\infty} \delta(t - t_i)dt = 1$. The number of stars formed in the ith burst that will become white dwarfs of luminosity l at time T_d is

$$dn_i = \phi_i\left(\frac{\partial N}{\partial M}\right)_i dM = \phi_i\, d\bar{n}_i \,, \tag{8}$$

where dM is given by equation (5).

The full luminosity function can now be written as

$$\frac{dn}{dl} = \sum \phi_i \frac{d\bar{n}_i}{dl} \,, \tag{9}$$

where

$$\frac{d\bar{n}_i}{dl} = -\left(\frac{\partial N}{\partial M}\right)_i \frac{(\partial t_c/\partial l)_m}{(dt_{\rm evo}/dM) + (\partial t_c/\partial m)_l(dm/dM)} \,, \tag{10}$$

and all the derivatives at time t_i must be evaluated at the mass $M = M_i$ given by a solution of equation (4) with $t = t_i$.

Luminosity functions derived from the observations are usually written in the form $dn/dM_{\rm bol}(M_{\rm bol} = -2.5 \log L + 4.77)$, and for computational purposes, the relationships between t_c, $t_{\rm evo}$, m, and M are more conveniently written in logarithmic form. In order to make more rapid headway, we also choose the mass function to be independent of time: $(\partial N/\partial M)_t \to dN/dM$. So, for each delta function contribution we have

$$\frac{d\bar{n}_i}{dM_{\rm bol}} = \frac{\ln_e 10}{2.5} \frac{(dN/dM)t_c(\partial \log t_c/\partial \log l)_m}{(dt_{\rm evo}/dM) + (t_c/m)(\partial \log t_c/\partial \log m)_l(dm/dM)} \,. \tag{11}$$

When the creation function is time dependent, a numerical solution requires the construction of libraries of $d\bar{n}_i/dl$ for a set of t_i's and l's. For constant ϕ, however, it is far easier and quicker to use the continuum equivalent of equation (9). We set $\phi_i = \phi(t)dt$ and replace the sum by an integral. Using equations (2)–(4), we have that, for fixed l and T_d,

$$-\frac{dt}{dM} = \frac{dt_{\rm evo}}{dM} + \left(\frac{\partial t_c}{\partial m}\right)_l \frac{dm}{dM} \,, \tag{12}$$

so that

$$\frac{d\bar{n}_i}{dl} \to -\frac{dN}{dM}\left(\frac{\partial t_c}{dl}\right)_m \frac{dM}{dt} \,, \tag{13}$$

and equation (9) becomes

$$\frac{dn}{dl} = -\int_{M_1}^{M_2} \phi(t)\left(\frac{dN}{dM}\right)\left(\frac{\partial t_c}{\partial l}\right)_m dM \,, \tag{14}$$

where the limits on M are M_1 = the mass given by equation (4) when $t = 0$ and M_2 = the maximum mass of a progenitor which can produce a white dwarf.

Finally, if $\phi(t) = \text{constant} = \phi_0$, we have

$$\frac{dn}{dl} = -\phi_0 \int_{M_1}^{M_2} \left(\frac{dN}{dM}\right)\left(\frac{\partial t_c}{\partial l}\right)_m dM \,, \tag{15}$$

or

$$\frac{dn}{dM_{\rm bol}} = l\, \frac{\ln_e 10}{2.5}\, \phi_0 \int_{M_1}^{M_2} \frac{dN}{dM} \left(\frac{\partial t_c}{\partial l}\right)_m dM \,. \tag{15'}$$

III. A SIMPLE EXAMPLE

To illustrate the general trend of solutions, let us suppose that

$$\frac{dN}{dM} = \frac{1.3}{M_0}\left(\frac{M_0}{M}\right)^{2.3} , \tag{16}$$

which is essentially the Salpeter (1955) mass function. The parameter M_0 has been chosen in such a way that the integral of this expression from M_0 to ∞ is 1. Next, we adopt the simple relation given by Iben and Tutukov (1987) for the nuclear burning lifetimes of the progenitor stars,

$$t_{\rm evo} = \frac{10^{10}\ \rm yr}{M^{3.5}} \,. \tag{17}$$

Third, we take,

$$t_c = 10^{10}\ {\rm yr}\left(\frac{l_0}{l}\right)^{5/7} , \tag{18}$$

which is the Mestel (1952) cooling function normalized to the Iben and Tutukov (1984) cooling curves. In equation (18), $l_0 = 10^{-4.5}\ L_\odot$, and in equations (16)–(17), M is in solar units.

With these choices, equation (14) gives

$$l\frac{dn}{dl} = -(\phi_0 \times 10^{10}\ {\rm yr})M_0^{\,1.3}\, \frac{5}{7}\left(\frac{l_0}{l}\right)^{5/7}\left\{\left[\frac{T_d}{10^{10}\ \rm yr} - \left(\frac{l_0}{l}\right)^{5/7}\right]^{1.3/3.5} - \left[\frac{1}{8}\right]^{1.3}\right\}, \tag{19}$$

where we have chosen $M_2 = 8$ (e.g., Becker and Iben 1979).

For comparison with the observations we rewrite this equation as

$$\frac{dn}{dM_{\rm bol}} = c\left(\frac{l_0}{l}\right)^{5/7}\left\{\left[T_{10} - \left(\frac{l_0}{l}\right)^{5/7}\right]^{0.37} - 0.067\right\}, \tag{19'}$$

where $T_{10} = T_d/10$ Gyr and

$$c = \left(\frac{5}{7}\right)\left(\frac{\log_e 10}{2.5}\right)M_0^{\,1.3}(\phi_0\, 10^{10}\ {\rm yr})\ {\rm pc}^{-3} \,. \tag{19''}$$

In Figure 4, we plot the theoretical white dwarf luminosity function given by this equation for assumed disk ages of 9, 12, 15, and 18 Gyr. The constant c has been set equal to $10^{-2.2}\ {\rm pc}^{-3}$.

The observational estimates plotted along with our theoretical luminosity functions have been taken from Winget *et al.* (1987), who base their estimates on data reviewed in Liebert, Dahn and Monet (1988). The total number density of white dwarfs implied by the observational distribution is about $3 \times 10^{-3}\ {\rm pc}^{-3}$. By any reckoning, these observational data points have significant uncertainties due to factors such as the uncertainty in the "discovery function" and the choice of luminosity intervals into which the distribution of white dwarfs has been binned. Particularly uncertain is the lowest luminosity point in the observed distribution, because the overall level of the number density in this region is extremely low. The last point actually represents only *two* white dwarfs (J. Liebert 1988, personal communication), and it is certainly difficult to apply standard statistical methods to such a small sample. It is perhaps quite possible that if the southern sky were searched as thoroughly as the northern sky, the value of the last point could be raised significantly.

In any case, our basic objective is not to quarrel with the legitimacy of the observationally based estimate of the luminosity function, but rather to examine the degree of correspondence with theoretical estimates. We consider a good theoretical fit to be one which has the same general slope as that defined by the observational data at luminosities less than the maximum and which shows a steep dropoff at a luminosity which is coincident with the luminosity of the dimmest white dwarfs observed. We also note that, by normalizing the general luminosity level of a theoretical curve to the observational data, we determine the value of the birthrate function ϕ_0.

With these criteria in mind, we judge that the best fit to the data using the simple theory is obtained by assuming a disk age somewhere between 12 and 15 Gyr. The age estimated in this way hinges almost entirely on the last data point, and, because of this, we can use equation (19) in conjunction with the luminosity dropoff defined by the last data point to show analytically how the

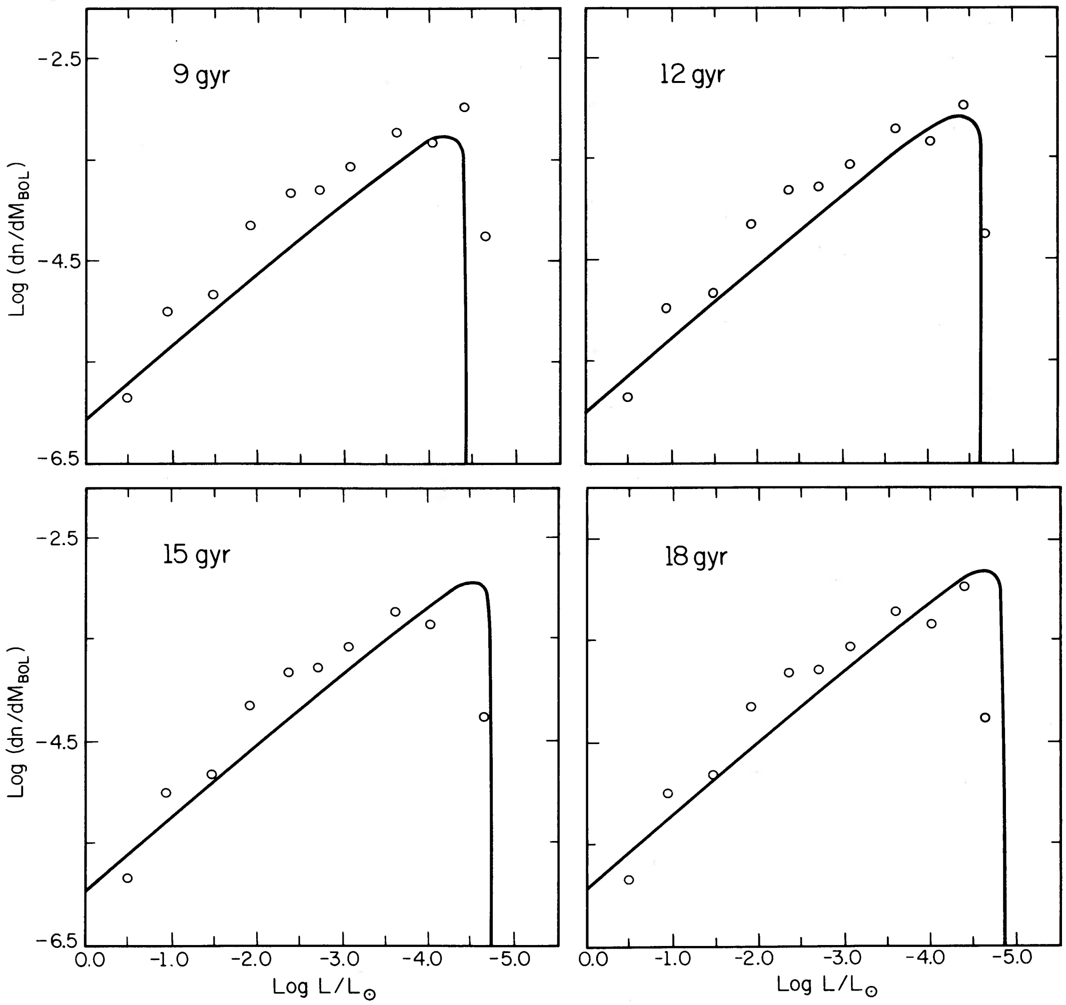

FIG. 4.—Luminosity functions (*solid lines*) for a simple power-law cooling theory (Mestel) compared with the observed luminosity function (*open circles*).

derived age depends on dropoff luminosity. That is, setting $dn/dM_{\rm bol} = 0$ at $l = l_{\rm dropoff}$ in equation (19′), we have to a good approximation that

$$T_{10} \approx \left(\frac{l_0}{l_{\rm dropoff}}\right)^{5/7} . \tag{20}$$

Table 1 gives the dropoff luminosities for a number of different assumed galactic disk ages.

By taking both the plots of the entire luminosity functions and Table 1 into account, we estimate a disk age of 14 ± 1 Gyr, an age which is considerably longer than that which we will obtain with mass-dependent cooling curves in § V. The formal uncertainty in this estimate is obtained by assuming an uncertainty of ±0.1 in log $l_{\rm dropoff}$.

The Winget *et al.* luminosity function gives a total number density of 3×10^{-3} white dwarfs per cubic parsec, and, with $c \approx 10^{-2.2}$ pc^{-3}, the theoretical curves fit the observed one. From equation (19″) we have that

$$\phi_0 \approx 1.52 \times 10^{-10} \times 10^{-2.2} M_0^{1.3} \text{ yr}^{-1} \text{ pc}^{-3} \approx 9.59 \times 10^{-13} M_0^{-1.3} \text{ yr}^{-1} \text{ pc}^{-3} .$$

Choosing the parameter $M_0 \sim 0.9$, so that ϕ_0 approximates the birth rate of stars more massive than M_0, we have $\phi_0 = 1.10 \times 10^{-12}$ yr^{-1} pc^{-3}. Approximating the volume of the disk by $V_d \approx 3 \times 10^{11}$ pc^3, we finally obtain 0.33 yr^{-1} as the rate at which stars more massive than 0.9 $M_\odot$ are being formed.

Another aspect of interest is the relationship between white dwarf mass and luminosity. While the heaviest white dwarfs can be present anywhere where there are stars to be found in the luminosity function, there is a certain minimum progenitor mass, and hence a minimum mass for a white dwarf which can contribute at a given luminosity. This minimum contributing mass is obtained by considering the earliest period of star formation, $t = 0$, and then using equations (17) and (18) in equation (4) to find an expression for $M_{\rm min}$:

$$M_{\rm min} = \left[t_{10} - \left(\frac{l}{l_0}\right)^{5/7}\right]^{-1/3.5} . \tag{21}$$

TABLE 1

DISK AGE VERSUS DROPOFF LUMINOSITY

τ_{disk}	$\log l_{dropoff}$
8	−4.36
9	−4.44
10	−4.50
11	−4.56
12	−4.61
13	−4.65
14	−4.70
15	−4.74
16	−4.78
17	−4.82
18	−4.85
19	−4.89
20	−4.92

As is evident from Figure 5, the simple relations which we consider here lead to a very sharp rise near the end of the luminosity function, and the minimum mass restriction only becomes important at the very lowest luminosities. In some sense, this is why the dropoff in the luminosity function is so sharp. When we consider more complex (e.g., Winget *et al.* 1987) cooling curves, we will find that the increase in the minimum mass occurs at a more modest rate over a much larger luminosity interval, leading to interesting structure in the luminosity function in the region beyond the maximum.

IV. PHYSICS AND MODEL INPUTS

Before seeking more realistic solutions, we need more carefully chosen approximations to a number of relations. Of primary interest is a good analytic relationship between the initial stellar mass and the total evolution time to the planetary nebula stage. Evolutionary time scales are of course dependent not only on mass, but also on chemical composition. We have therefore plotted in Figure 6 the theoretical evolutionary time scales for a variety of initial masses and compositions. These data were taken from a number of sources, including the work of Vandenberg and Bell (1985), Iben (1986), Mengel *et al.* (1979), and Sweigart (1978). Although there is some dispersion (mainly due to compositional differences) it is negligible to the accuracy with which we need t_{evo}. A least-squares routine can be used to obtain a good third-order polynomial fit,

$$\log t_{evo} = 9.921 - 3.6648 \log M + 1.9697(\log M)^2 - 0.9369(\log M)^3 , \quad (22)$$

which can be differentiated to obtain an equation for dt_{evo}/dM.

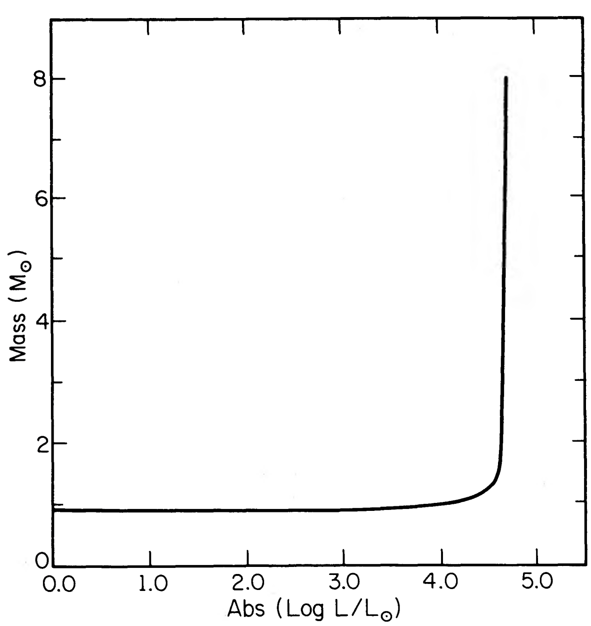

FIG. 5

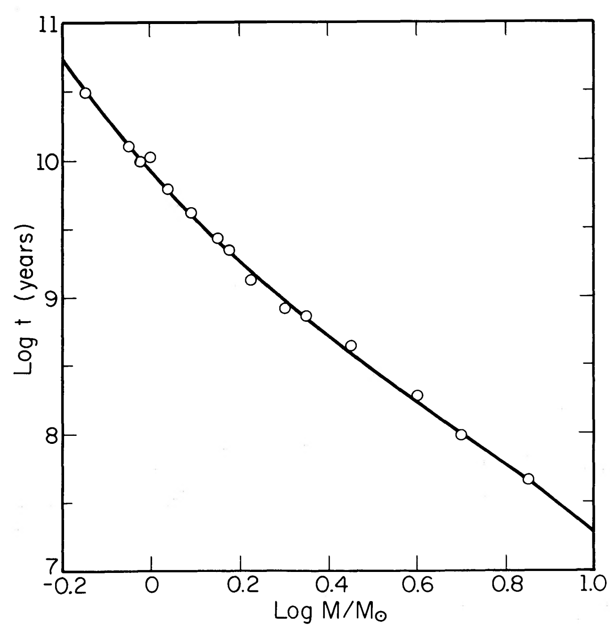

FIG. 6

FIG. 5.—Minimum contributing mass as a function of luminosity for a 14 Gyr luminosity function based on the Mestel cooling theory

FIG. 6.—Evolutionary time scales for progenitor stars as a function of progenitor mass. The curve is a polynomial fit to the data points.

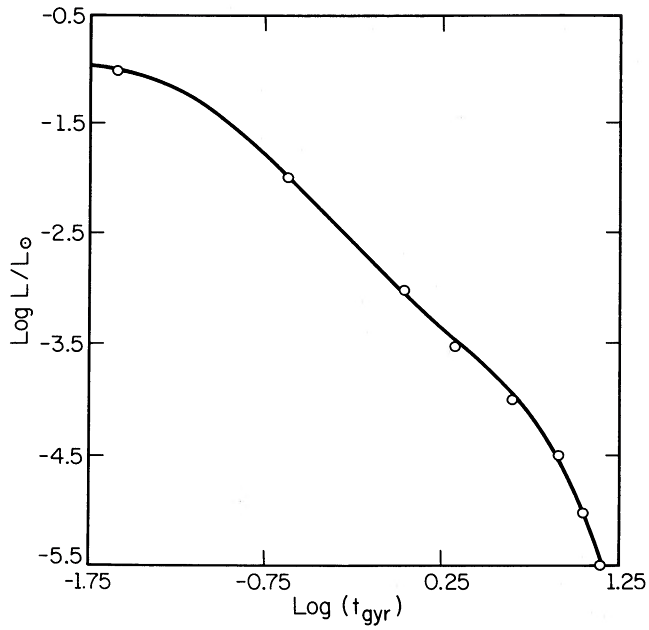

FIG. 7.—Iben and MacDonald cooling curve for a 0.6 $M_{\odot}$ white dwarf

The white dwarf cooling curves are also crucial in the construction of the luminosity function. We have already made use of the simple cooling curve of Mestel (1952). A more recent cooling curve, which requires a fifth-order polynomial to yield an accurate analytical form, is given by Iben and MacDonald (1986) for a model of mass 0.6 $M_{\odot}$. Our approximation to their result,

$$\log t_{\rm gyr} = -2.5503 - 0.6947(\log l) + 0.47657(\log l)^2 + 0.26616(\log l)^3 + 0.05374(\log l)^4 + 0.003864(\log l)^5 \,, \quad (23)$$

was obtained by performing a least-squares fit with log (t) as the independent variable to the sample of theoretical points shown in Figure 7.

We have also prepared polynomial fits to the cooling curves for pure carbon white dwarfs of mass 0.6, 0.8, and 1.0 $M_{\odot}$ given in the paper of Winget *et al.* (1987). For these masses our approximations are, respectively,

$$\log t_{\rm gyr} = -3.515 - 3.131(\log l) - 1.657(\log l)^2 - 0.5274(\log l)^3 - 0.0769(\log l)^4 - 0.00401(\log l)^5 \,, \quad (24)$$

$$\log t_{\rm gyr} = -4.991 - 6.337(\log l) - 4.210(\log l)^2 - 1.491(\log l)^3 - 0.2481(\log l)^4 - 0.01545(\log l)^5 \,, \quad (25)$$

and

$$\log t_{\rm gyr} = -3.969 - 4.868(\log l) - 3.325(\log l)^2 - 1.257(\log l)^3 - 0.2231(\log l)^4 - 0.1469(\log l)^5 \,, \quad (26)$$

and they are compared with the Winget *et al.* curves in Figure 8. After fitting second-order polynomials in mass to the coefficients of these expressions we are in a position to construct analytic expressions for $(\partial t_c/\partial l)_m$ and for $(\partial t_c/\partial m)_l$.

Another important relation gives the mass m of a white dwarf that a progenitor star of given mass M will yield. This is not a quantity which is known from first principles, but estimates can be made by comparing observations with theoretical AGB models and theoretical isochrones (e.g., Weidemann 1987). We shall assume that all stars smaller than two solar masses will eventually evolve into a 0.6 $M_{\odot}$ white dwarf and that stars with masses greater than 2 $M_{\odot}$ yield white dwarfs of mass

$$m = 0.43333 + 0.08333M \quad (27)$$

(see Iben and Tutukov 1985, Fig. 30). Thus, $dm/dM = 0$ if $M < 2$ and $dm/dM = 0.08333$ for $M > 2$.

We also assume that the mass function conforms to the Salpeter mass function and is uniform over time. That is, during any era of star formation, the distribution of newly created stars is proportional to $1/M^{2.3}$. The mass distribution for our purposes is understood to end at 8 $M_{\odot}$. Although heavier stars are certainly present during any era, white dwarfs are not the end product of their evolution.

V. NUMERICAL SOLUTIONS

We next construct a luminosity function using the Iben and MacDonald (IM) t_c-l relationship approximated by equation (23), assuming a constant rate of star formation and performing a numerical integration of equation (15). The necessary expressions for the minimum contributing mass at a given luminosity and the cooling time to a given luminosity are more complicated than in our previous example, forcing us to generate each value for $dn/dM_{\rm bol}$ numerically.

By varying the assumed age of the disk, we generate a variety of luminosity functions in this fashion and find that the best fit

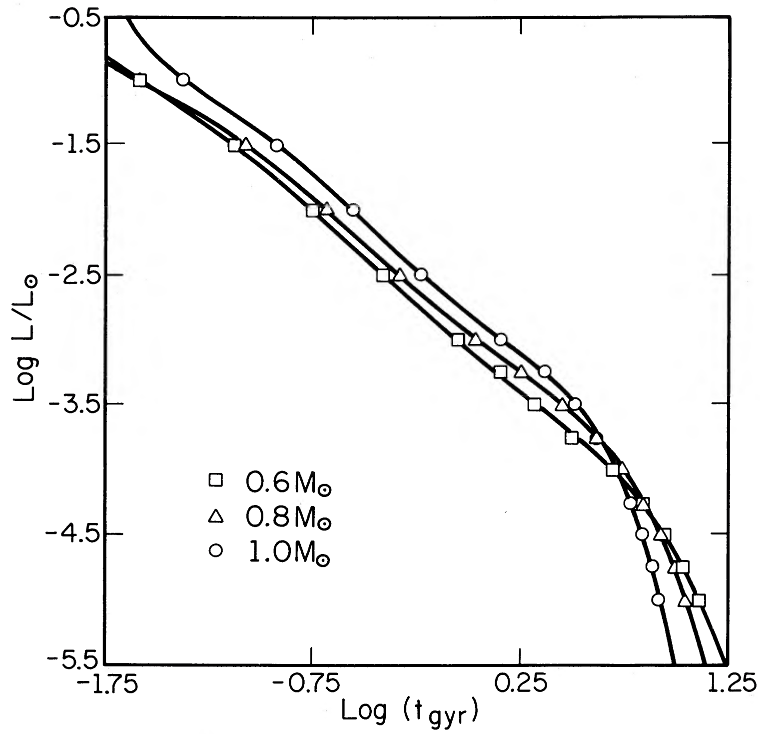

FIG. 8.—Winget *et al.* (1987) cooling curves for carbon white dwarfs of masses 0.6 $M_\odot$ (*squares*), 0.8 $M_\odot$ (*diamonds*), and 1.0 $M_\odot$ (*open circles*). The curves are polynomial fits.

occurs for a disk age of 9 ± 0.5 Gyr (Fig. 9). The formal uncertainty has been estimated by using a linear approximation to the slope at the very low luminosity end of the IM cooling curve (Fig. 7). We find that

$$T_d \approx 0.9\left(\frac{10^{-4.7}}{l_{\rm dropoff}}\right)^{0.28}, \qquad (20')$$

so that $\Delta T_{10}/T_{10} \approx \pm 0.280 |\Delta l_{\rm dropoff}/l_{\rm dropoff}| \approx 0.06$, and $\Delta T_{10} \approx \pm 0.5$ for $\Delta \log l \approx \pm 0.1$.

As was also the case with the simple Mestel (1952) mass-independent cooling curve, the decline in the luminosity function following its maximum is very abrupt, and the galactic disk age is very well defined (to the extent that the observational cutoff is to be trusted). However, the IM luminosity function displays a curvature in the lower luminosity regions preceding the maximum in the distribution that fits better with the observational luminosity function than does the Mestel luminosity function, a result common to many previous studies (see Shaviv 1979, Fig. 2 for a summary).

The integrated $\phi(t)$ = constant luminosity function can be considered to be a superposition of delta function bursts occurring with equal strength at equal time intervals. The contribution of one such burst, lasting for 10^8 yr and starting at $t = 0$ is shown in Figure 10. This single burst luminosity function is characterized by a much steeper slope at the higher luminosities than the integrated function. It should be noted that the very singular nature of the delta burst creation function requires that, for full display, the vertical scale must be considerably expanded over that required by the integrated luminosity functions.

Figure 11 shows the minimum contributing progenitor mass at each luminosity. Comparing this figure with Figure 5, it is evident that the increase in minimum progenitor mass is not quite as abrupt as in the simple theory, but it is still rapid enough to ensure that the minimum mass restriction is not important over a very wide luminosity range.

Putting the analytical machinery that we have developed to its full use, we now investigate luminosity functions which incorporate the mass-dependent Winget *et al.* cooling curves. As in the previous cases, we first assume a flat creation function. Using equation (15) with $\phi(t)$ = constant = 1.25×10^{-12} ($\approx$0.38 yr^{-1} in the entire Galaxy), we construct luminosity functions for galactic disk ages of 6, 9, and 12 Gyr, with the results shown in Figure 12. The minimum progenitor mass is given in Figure 13*a* as a function of luminosity for each of the three chosen ages, while Figure 13*b* shows the average contributing progenitor mass and Figure 13*c* shows the average white dwarf mass as functions of luminosity for these ages.

We have obtained these averages by using a delta burst library to be described later. At every point i along a delta burst luminosity function, the range in white dwarf mass (or in progenitor mass) about the mean contributing mass m_i is quite small relative to m_i. Since the complete luminosity function is $dn/dM_{\rm bol} = \sum_i (dn/dM_{\rm bol})_i$, the average contributing mass is just $\bar{m} = \sum_i m_i(dn/dM_{\rm bol})_i$.

Several interesting conclusions may be drawn from these results. First, the luminosity function corresponding to a disk age of 9 Gyr is clearly a much better fit to the observationally based luminosity function than are the other two. The 6 Gyr luminosity function begins to fall short at luminosities where the observational number densities are most definitely still increasing with decreasing luminosity, and it is difficult to see how enough white dwarfs could be found at low luminosities to conform to the shape

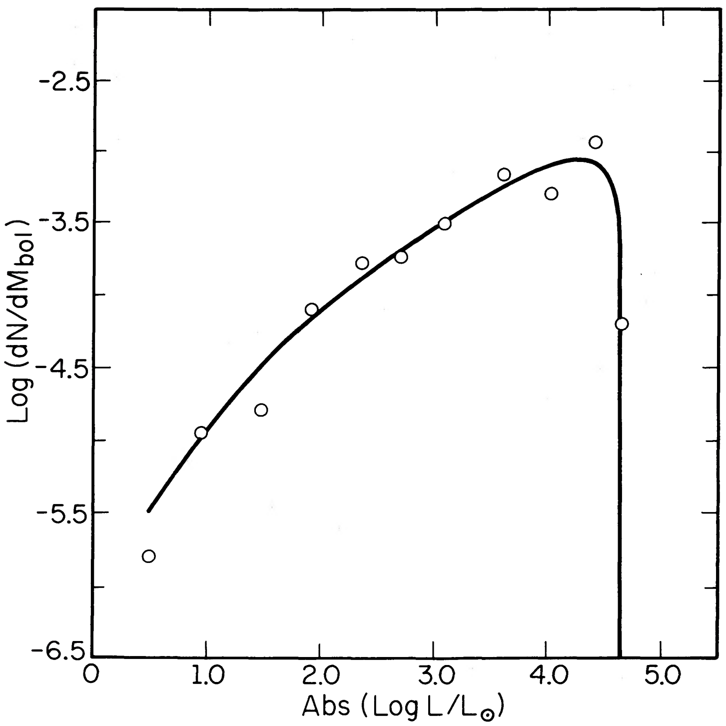

FIG. 9.—Luminosity function for the Iben-MacDonald cooling curve with a 9 Gyr disk

of the theoretical 12 Gyr luminosity function. The formal error in our age estimate would appear to be less than ± 1 Gyr. Our result agrees with the age of 9.3 ± 2 Gyr estimated by Winget *et al.* and shows that the luminosity-independent white dwarf mass weighting algorithm which these authors used does not, with the data as presented by them, introduce serious errors in an age estimate. However, at luminosities below log $l \approx -4.4$, the shape of our theoretical luminosity function differs considerably in quantitative detail from the one produced by Winget *et al.*

This brings us to the second interesting feature in the luminosity functions shown in Figure 12: a temporary "leveling off" in the luminosity function past the maximum. This feature is a result of the fact that the heavier white dwarfs begin to cool more quickly than do the lighter ones in the low-luminosity region (see Fig. 8). Because the luminosity function becomes progressively dominated

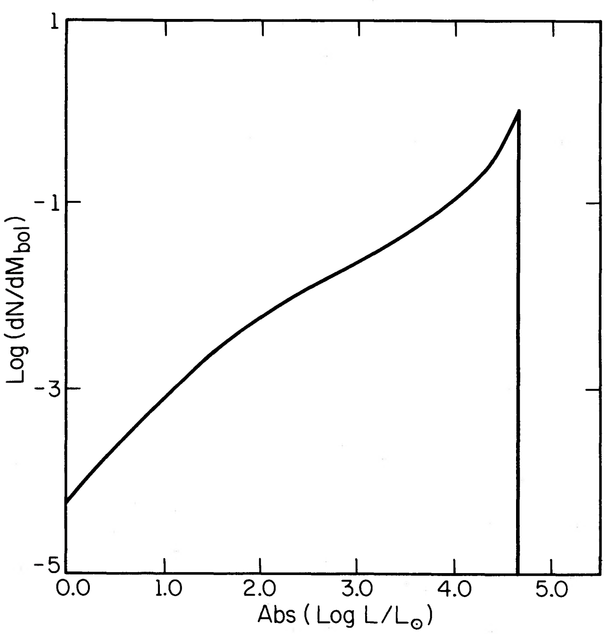

FIG. 10

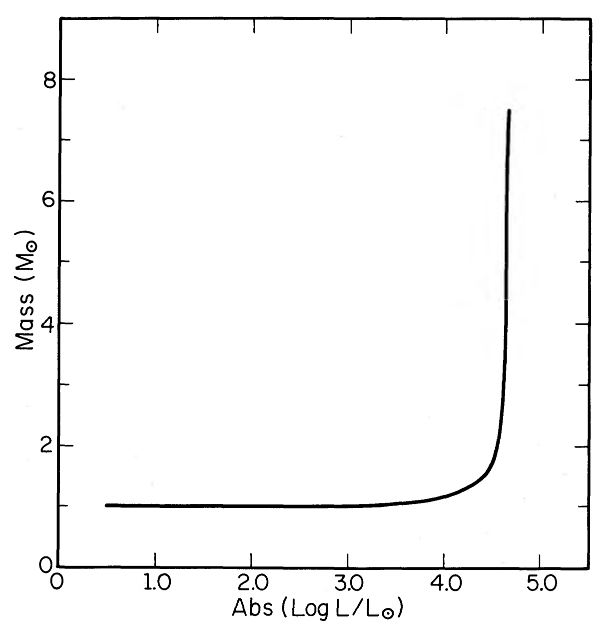

FIG. 11

FIG. 10.—Luminosity function due to delta burst for Iben-MacDonald theory
FIG. 11.—Minimum contributing mass for the Iben-MacDonald luminosity function

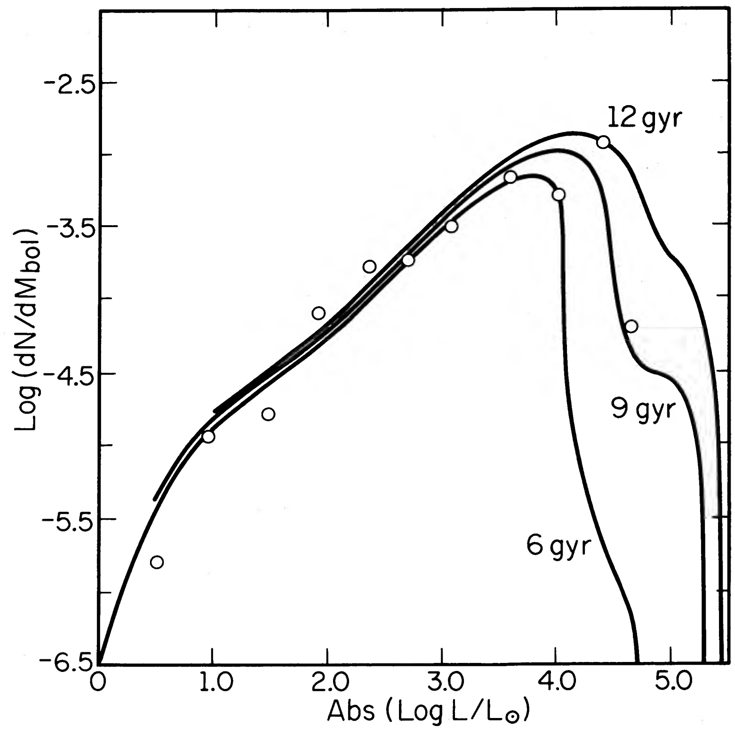

FIG. 12.—Luminosity functions for the Winget *et al.* cooling curves: $T_{\text{disk}} = 6, 9, 12$, Gyr

by the heavier white dwarfs, the 1 $M_\odot$ cooling curve becomes very much more important than either the 0.8 or the 0.6 $M_\odot$ curves in the regions with luminosities less than $\log (L/L_\odot) = -4.5$. This rapid cooling tends to spread the white dwarfs out into a "plateau" at a relatively low number density and effectively postpones the final extinction of the luminosity function due to the finite disk age. Indeed, if the slope of the cooling curve were to become infinite, the plateau would extend off to infinity at an infinitely low number density. What we see in Figure 12 (especially for $T_d = 9$ Gyr) is a mild example of this effect.

It is clear from Figures 13*b* and 13*c* that adoption of a time-independent and luminosity-independent number-mass distribution for white dwarfs is inappropriate at luminosities near the observed turnoff luminosity, where the expected white dwarf mass is significantly larger than the typical mass of white dwarfs in the solar vicinity.

In particular, the 9 Gyr luminosity function predicts that the white dwarfs which define the adopted turnoff luminosity of $l_{\text{turnoff}} \approx 10^{-4.7} L_\odot$ should be of mass near 0.9 $M_\odot$. This prediction is presumably testable.

In Figure 14*a* we compare the Winget *et al.* theoretical luminosity function for 9.3 Gyr with ours for 9 Gyr. We have also introduced a revised observational luminosity function given by Liebert, Dahn, and Monet (1988). This new luminosity function shows blackbody/model bolometric corrections and also features an altered binning scheme and refined estimates of the visual magnitudes for certain stars. We note first that the theoretical luminosity function, which fitted the observational luminosity function as presented in Figure 12, no longer fits the data as presented in Figure 14*a*, particularly at low luminosities.

It is evident that the theoretical curves differ considerably from each other in the dropoff region, precisely where the observational data are most critical for an age estimate. This difference is almost entirely a consequence of our use of a realistic single-valued initial-final mass relation. It is also clear that an extension of the observed luminosity function to lower luminosities by only one-half a magnitude would be very revealing. Unfortunately, such an extension may be decades in coming, but the motivation for searching is clear. With the present data, 9 Gyr would appear to be an upper limit to the age of the disk stars in the solar vicinity.

The statistical treatment of the data at low luminosities can have an important influence on age estimates as well as on the perception of the adequacy of the theoretical luminosity functions. Certainly the Winget *et al.* luminosity function with its steep slope cannot be said to match well with the alternative interpretation of the observed distribution constructed by Liebert, Dahn, and Monet (1988). Nor does our 9 Gyr function provide a particularly good fit. In Figure 14*b* we show that a considerably better fit is given by an 8.0 Gyr luminosity function.

We next examine the effect of adopting a time-dependent rate of star formation, using equations (9) and (11) since it is no longer practical to employ equation (14) in its simple form. We have constructed a library of 180 individual "burst luminosity functions" spanning 18 Gyr in step sizes of 1×10^8 yr. The luminosity function due to any arbitrary creation function and disk age of less than 18 Gyr can be synthesized by superposing an appropriate selection from the library and using an appropriate weighting factor ϕ_i for each individual burst.

The luminosity function which would result from a single burst of star formation 9 Gyr ago of width 1×10^8 yr is detailed in Figure 15*a* (and should be compared with Fig. 10). This luminosity function is a good representative of the 180 which make up our library. A characteristic feature of the burst luminosity functions is a very sharp maximum, which leads to some irregularities in representation by a finite number of equally spaced mesh points. That is, the peak of the delta function is only "caught" by accident

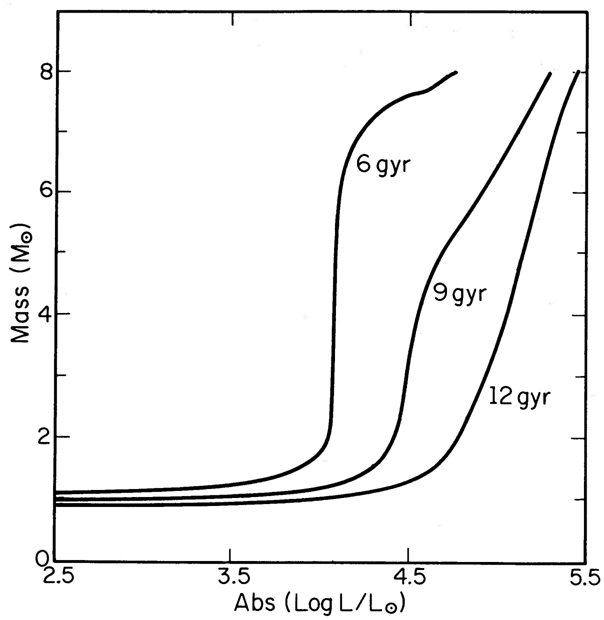

FIG. 13*a*

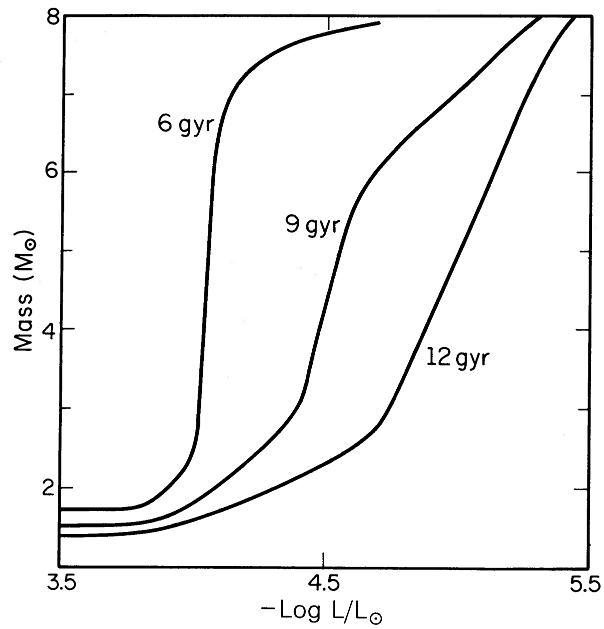

FIG. 13*b*

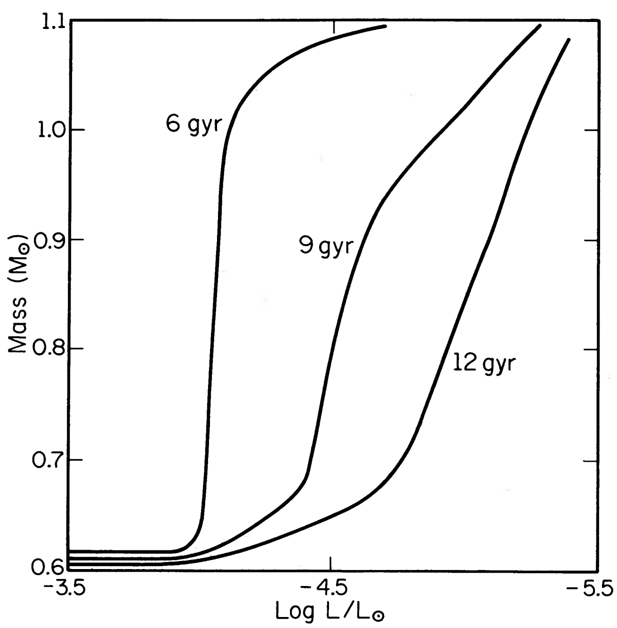

FIG. 13*c*

FIG. 13.—(*a*) Minimum progenitor masses contributing to the luminosity functions of Fig. 12. (*b*) Average progenitor masses contributing to the luminosity functions of Fig. 12. (*c*) Average white dwarf masses contributing to the luminosity functions of Fig. 12.

in the finite grid representation, as demonstrated in Figure 15*b*, where we show (with a solid line) the result of constructing a 6 Gyr delta function at a resolution 10 times finer than that used in constructing the 6 Gyr delta function (*open squares*) which is a part of our library. The linear superpositions of our library luminosity functions therefore contain a certain amount of quantization noise, as is evident from a comparison of Figure 16, which is a plot of the luminosity function corresponding to a constant rate of star formtion ($c = 1$), with Figure 12, which contains a plot of the same luminosity function obtained in the more direct manner suggested by equation (15). The two 9 Gyr luminosity functions should be identical, yet it is easily seen that in the regions where the sharp spikes are modeled, there is a certain amount of noise in the final product. The apparent slight paucity of white dwarfs in certain areas of the luminosity function is due to the fact that the peaks of individual spikes were not completely caught by the mesh of representation points. It is thus important to note that minor oscillations *in the regions preceding the maximum* of the composite luminosity function for varible $\phi(t)$ should be disregarded.

As a specific application of this technique, we construct a luminosity function based on a $\phi(t)$ which resembles the time-dependent rate of star formation suggested by Scalo (1987) in a recent paper on galactic starbursts. This model features starbursts occurring quite recently, and also at a time about 6–8 Gyr in the past superposed upon a relatively constant base rate. This star formation rate

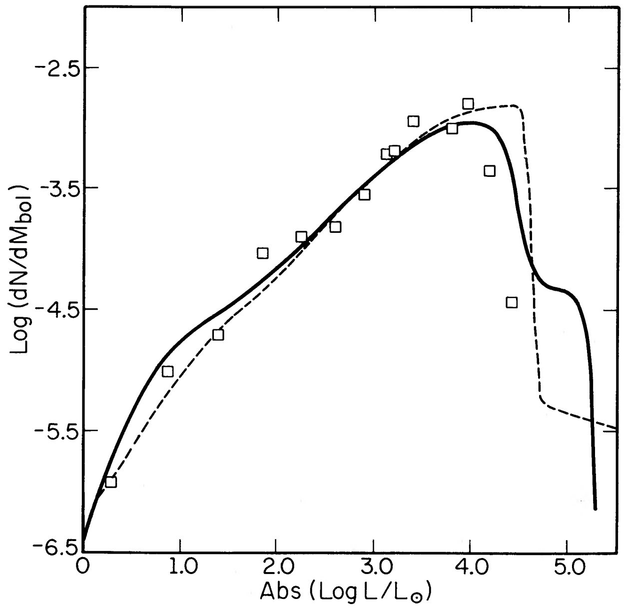

FIG. 14a

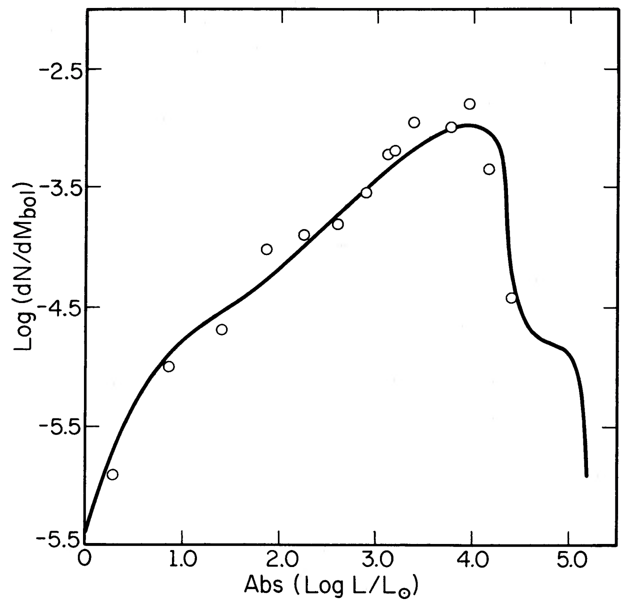

FIG. 14b

FIG. 14.—(*a*) Comparison of the Winget *et al.* theoretical LF with our theoretical LF for a disk age of 9 Gyr. Observational data are binned according to Liebert, Dahn, and Monet (1988). (*b*) Theoretical LF for disk age of 8 Gyr. Observational data are binned according to Liebert, Dahn, and Monet (1988).

(SFR) model was obtained through a study of the chromospheric ages of nearby F and G stars, and our approximation to this model is given in Figure 17. The resultant luminosity function is shown in Figure 18. As is evident, the fit with the observational luminosity function is quite good, and a starburst model of this type, although certainly not confirmed, is not ruled out by the observational luminosity functon. Perhaps the most important conclusion to be drawn from the Scalo luminosity function is the fact that rather large fluctuations in the creation function do not have a massive effect on the luminosity function. This is especially true for the regions past the maximum. These areas are populated almost exclusively by the remnants of heavier stars formed early in the galactic history, and are immune to later fluctuations in star formation.

It should also be stressed that the noisy character of the theoretical luminosity function between $\log(L/L_{\odot})$ of -1.5 and -2.5 is almost entirely due to the discrete nature of the delta burst luminosity functions which are tied to the most recent burst of star formation. In the limit of a continuous summation of luminosity functions, the noisy peaks would merge to form a smooth, relatively insignificant bump similar to that seen at low luminosities for the old burst.

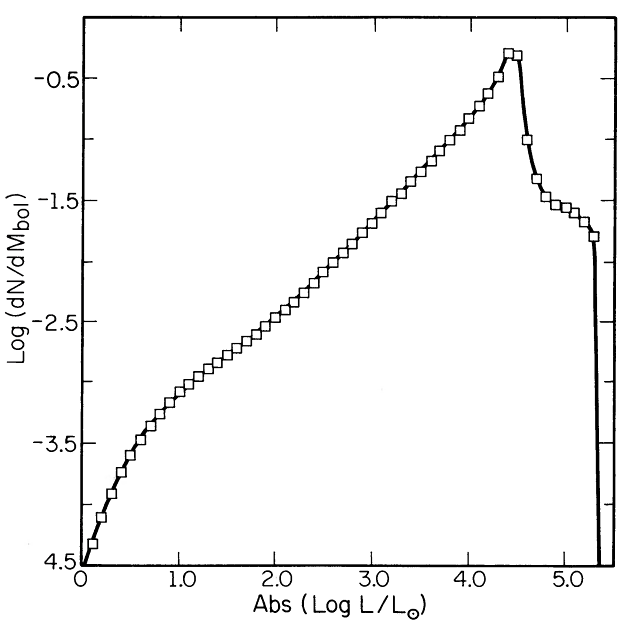

FIG. 15a

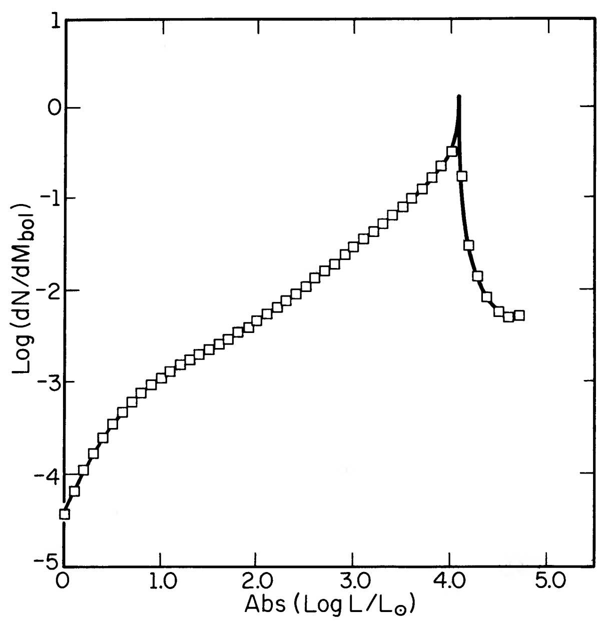

FIG. 15b

FIG. 15.—(*a*) Luminosity function for a delta burst of star formation occurring 9 Gyr ago (employs the Winget *et al.* cooling curves.) (*b*) High-resolution luminosity function for delta burst of star formation occurring 6 Gyr ago.

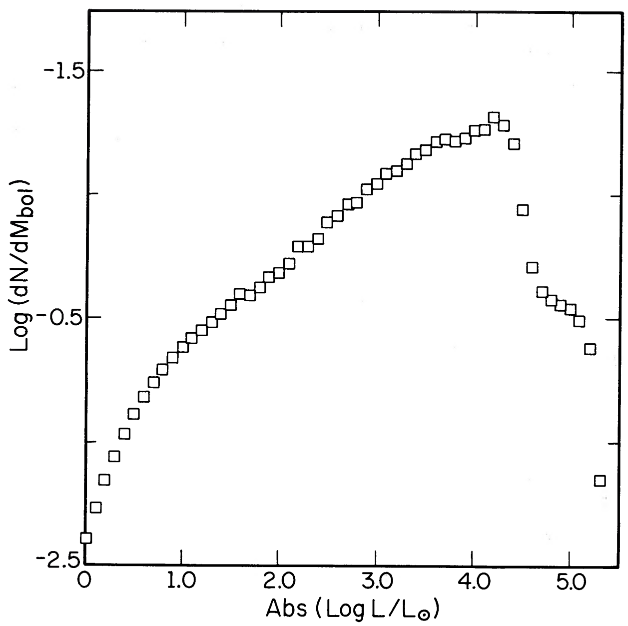

FIG. 16

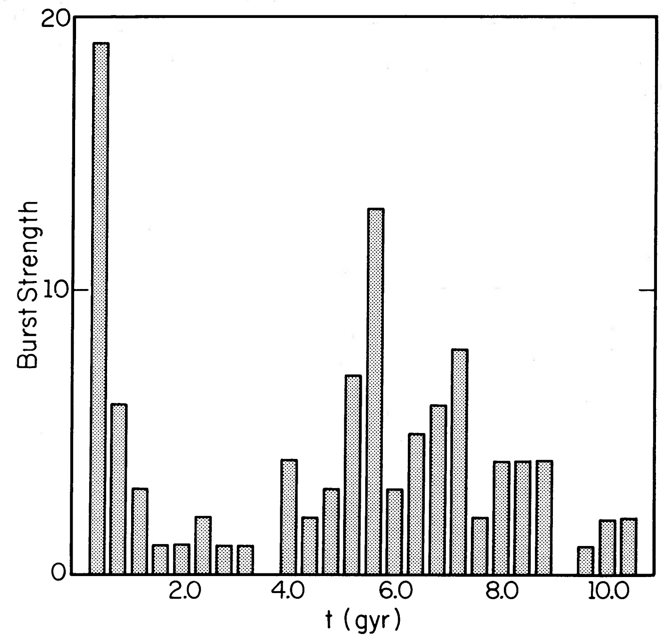

FIG. 17

FIG. 16.—Luminosity function resulting from the summation of equally weighted luminosity functions constructed for 90 successive 1.0×10^8 yr bursts of star formation.

FIG. 17.—Star formation rate suggested by a study of chromospheric ages of nearby F and G stars (Scalo 1987)

VI. DISCUSSION

It is very important to recognize that the theoretical luminosity functions in this study depend very strongly upon the $t_c(l, m)$ cooling curves. The shapes of these curves, especially in the low-luminosity regions, essentially dictate the finer details in the theoretical luminosity function. This sensitive dependence is compounded by the fact that the cooling models have only been explicitly calculated to a luminosity of $\log (L/L_\odot) = -5.00$. We have extended these curves by use of least-squares polynomial fits down to luminosities in the neighborhood of $\log (L/L_\odot) = -5.50$. Although the temporary leveling off which we have observed using the Winget *et al.* cooling curves begins before the extrapolated zone is entered, the final extinction of the 9.0 Gyr and especially the 12.0 Gyr models are somewhat dependent on the choice of extrapolation.

Cooling curves also depend on the choice of composition, on the opacity and equation of state (especially in the outer layers of the white dwarf), and on whether or not diffusion is taken into account. Some of this dependence is described by Winget and Van Horn

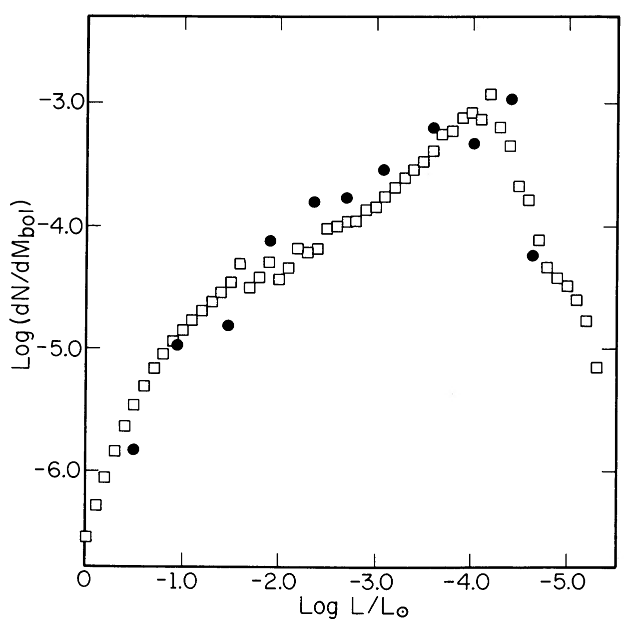

FIG. 18.—Luminosity function (*open squares*) obtained by using Winget *et al.* cooling curves, the Scalo SFR, and a 9 Gyr disk age. The observational points (*solid circles*) are from Winget *et al.* (1987).

(1988). It is also evident from a comparison between the 0.6 $M_\odot$ cooling curve of Winget *et al.* in Figure 8 (pure carbon, no diffusion) and the 0.6 $M_\odot$ cooling curve of Iben and MacDonald in Figure 7 (carbon-oxygen core, hydrogen and helium in surface layers, diffusion).

Another variation in the input physics which might be expected to have an effect upon an estimate of the galactic disk age, would be to assume that all stars heavier than 6 $M_\odot$ go on to form something other than a white dwarf (e.g., Bertelli, Bressan, and Chiosi 1985; Castellani *et al.* 1985*a*, *b*; Renzini *et al.* 1985). This would of course cut the minimum mass limit to 6 $M_\odot$. To examine the effect of such a change, we have constructed a 9 Gyr luminosity function using exactly the same input physics as before, except that the coefficient of M in equation (27) is chosen as 0.1250 and the cutoff progenitor mass is chosen as 6 $M_\odot$. The resultant luminosity function is almost identical with the 9 Gyr luminosity function in Figure 12. Changing the location of the departure from $dm/dM = 0$ in the progenitor → remnant mass relation from 2.0 $M_\odot$ to 1.5 $M_\odot$ (e.g., Bertelli, Bressan, and Chiosi 1985) also has essentially no effect on the resultant theoretical luminosity functon.

Another consideration which could lengthen the estimate of the disk age is the possibility that the galactic disk may become inflated in the vertical direction as time progresses (Spitzer and Schwarzschild 1951, 1953). In a study of disk dynamical evolution, Fuchs and Weilen (1987) have found that the dispersion of stellar velocities in the direction perpendicular to the disk increases with stellar age by approximately a factor of 7 in going from $\sim 10^8$ yr to $\sim 10^{10}$ yr. (See also Gilmore and Wyse 1987 and Freeman 1987.) The resultant increase in scale height for the oldest (and therefore heaviest white dwarfs) implies a correction to the observed white dwarf space densities or to the theoretical luminosity functions in a direction which will increase the estimate of disk age. That is, at any given luminosity [corresponding to a total time since birth of $t_{\rm tot}(m) = t_{\rm evo}(M) + t_c(l, m)$], either the theoretical values of $dn/dM_{\rm bol}$ should be reduced by a factor proportional to the approximate scale height for that age, or, equivalently, the points along the observational distribution should be increased by the appropriate factor.

We have chosen to adjust the observational points by amounts corresponding to the average total age of the white dwarfs populating each particular luminosity region. The effects of adopting two velocity dispersion–scale height relationships have been examined. In one scheme, we assume that the disk expansion factor f is related to the velocity dispersion by $f \propto h_{\rm scale} \propto v^2_{\rm dispersion}$. This leads to a larger effect than does our other assumption that $f \propto h_{\rm scale} \propto v_{\rm dispersion}$. The disk expansion modifications are illustrated in Figure 19, where the $f \propto v^2$ results are shown as open circles, the $f \propto v$ results are shown as squares, and the original observations are shown as open triangles at each luminosity. The lowest two luminosity points experience virtually the same expansion factor, contrary to what one might expect. This is due to the fact that while the average white dwarf mass at the lowest luminosity point is approximately 0.95 $M_\odot$, it has been reduced to nearly 0.65 $M_\odot$ at the second lowest luminosity point (which represents the peak in the distribution). A close examination of the Winget *et al.* cooling curves in Figure 8 shows that the cooling time of a 0.65 $M_\odot$ dwarf to $\log L/L_\odot = -4.4$ is almost exactly the same as that of a 0.95 $M_\odot$ dwarf to $\log L/L_\odot = -4.65$. Since the progenitor time scales $t_{\rm evo}(M)$ for the contributing stars at both of these low luminosities are negligible in comparison with the white dwarf cooling time, the white dwarfs populating the two lowest luminosity points have essentially the same age and have thus suffered equal amounts of dispersion in the disk.

We have attempted theoretical fits to the expanded-disk luminosity functions by keeping ϕ constant and adjusting the assumed age, but no amount of adjusting can produce a satisfactory fit. The 9 Gyr luminosity function from Figure 12 is repeated as curve 1 in Figure 19, and it is evident that no change in the vertical normalization of the theoretical curves can produce a fit to observational points simultaneously at low and high luminosities.

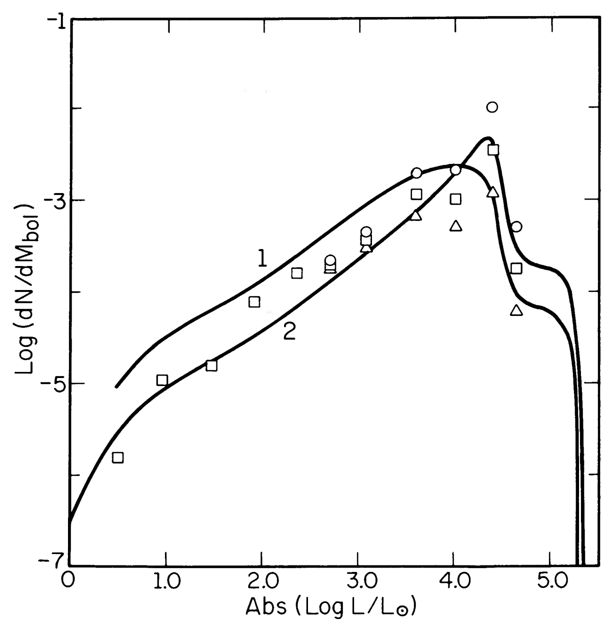

FIG. 19.—Effect of expanded disk on the observational luminosity function. The squares are appropriate if the disk expansion factor is proportional to the velocity dispersion, $f \propto v$. The open triangles represent unaltered data ($f = 1$), and the open circles are appropriate if $f \propto v^2$. The theoretical curves are for a constant SFR, and one which decays exponentially with a time constant of 1 Gyr.

However, by adopting an exponential form for the birth rate function, $\phi = \alpha e^{-t/\tau}$, where τ and α are adjustable parameters, we can achieve a tolerably good fit. This is illustrated by curve 2 in Figure 19 for which $\alpha = 10^{-2.026}$ and $\tau = 1$ Gyr. Thus, for the particular choice of mass-dependent cooling curves which we have employed, we conclude that it is the star formation rate, and not the disk age, which must be readjusted when the expansion of the disk is taken into account. The SFR which best fits the expanded disk data declines at a very rapid rate and is therefore consistent with Butcher's (1987) speculation that most of the elements in the disk may have been created in a single, early burst of nucleosynthesis.

In conclusion, we note that our estimate for a disk age somewhere close to 9–10 Gyr is supportive of other age estimates from different methods. The oldest known galactic cluster, NGC 188, has been assigned an age of around 10^{10} yr by comparing the distribution of stars in the color-magnitude diagram with theoretical models of evolving stars (Vandenberg 1985; Iben 1967). In a recent study using radioactive dating (Butcher 1987), measurements of the line strength ratios of thorium and the stable element neodymium show little evolution with age, suggesting a disk age of less than 9 Gyr. Estimates of the age of the elements in the disk (Fowler and Meisl 1986; Fowler 1987) also give small values of order 10–12 Gyr.

It is important to point out that the disk age and the galactic age are by no means the same thing. It is possible that the disk could have been formed later, via an infall of matter from a previously existing massive halo. Indeed, observations of the ages of the oldest globular clusters indicate an age of 15 ± 3 Gyr (e.g., Iben and Renzini 1984). One may argue that the several independent, but concordant, estimates of the age of the disk demonstrate one or more errors in the assumptions and/or observational data that go into estimating globular cluster ages (thus insisting on a present understanding of how and when the various components of our galaxy formed), or that we have learned something new about the formation and evolution of the halo and disk components that make up our Galaxy.

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