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CONSTRAINTS FOR TRANSONIC BLACK HOLE ACCRETION

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ABSTRACT

Regularity conditions and global topological constraints leave some forbidden regions in the parameter space of the transonic isothermal, rotating matter onto black holes. Unstable flows occupy regions touching the boundaries of the forbidden regions. We discuss the astrophysical consequences of these results.
Subject headings: black holes

I. INTRODUCTION

Many of the X-ray sources believed to be compact objects accreting rotating matter show transient behavior, switching themselves between high and low states. There is growing evidence that this may be due to particular properties of transonic accretion onto black holes and neutron stars, in particular to regularity conditions at the critical points and the global topological constraints.

Thin, stationary, axially symmetric accretion flows are often described by a one-dimensional model in which all the physical quantities are vertically integrated and the laws of mass, momentum, angular momentum, and energy are expressed by a set of ordinary differential equations (Muchotrzeb and Paczyński 1982). The necessary integration constants are provided by the boundary conditions, which depend on the details of a particular astrophysical situation. For example, it is often assumed that the flow at some radius $R_{\rm out}$ far away from the central accreting body is identical with that given by the Shakura-Sunyaev solution. We shall call this particular possibility *Keplerian boundary conditions.*

Accretion onto black holes is always transonic, and the same is true for most situations involving neutron stars. Differential equations describing transonic dissipative flows have subcritical points in addition to the well-known critical points (Flammang 1982). Any locally acceptable solution must pass through these points regularly. The regularity conditions for critical points are in the form of algebraic constraints, while those for subcritical points are given by integral conditions. In the isothermal case the energy conservation equation is trivially fulfilled and the complications with the subcritical points do not arise (Flammang 1982). This, together with the fact that the sound speed c_S is constant, enormously simplifies the problem and offers the possibility of studying the transonic flows analytically.

An acceptable local transonic solution must be globally correct, going all the way from radial infinity to the center. For a regular, globally acceptable solution it is necessary, although not sufficient, for the critical point to be of saddle or nodal type; spiral- or center-type solutions are excluded (see Ferrari *et al.* 1985 for an excellent discussion on classification of types of critical points).

The sonic point/local regularity conditions, together with the global topological demands, cannot be met for all the choices of boundary conditions and parameters describing the material properties (equation of state, opacity, viscosity). This creates *forbidden regions* in the parameter space of the problem and means that not all astrophysically acceptable boundary conditions can lead to regular stationary flows.

Abramowicz and Zurek (1981) found that in the case of dissipation-free black hole accretion the *forbidden region* separates two physically different regimes. The first, *Bondi accretion*, contains solutions with the sonic point far away from the hole. They resemble spherical accretion of nonrotating matter. The second regime, *disk accretion*, contains solutions with the sonic point close to the hole. They are similar to the innermost parts of thick accretion disks and have no Newtonian analogy. A speculation by Abramowicz and Zurek that when an astrophysical situation locates a stationary model of the flow in the forbidden region, then the nonstationary response of the flow will be *bistable*, with the flow oscillating between the Bondi and disk regimes, was later supported, although not proved, by a rough time-dependent model of Abramowicz, Livio, and Lu (1986). They suggested that the *bistability* can be connected with the observed high state–low state transitions of some X-ray sources. None of the nonstationary numerical models of Hawley (1986) show the bistability. They were connected with shocks and covered only a limited region of the parameter space, however, since they always started from supersonic flow at the outer boundary.

Muchotrzeb (1983) continued the studies of dissipative accretion flows started by Paczyński and Bisnovatyi-Kogan (1981) and by herself and Paczyński (1982). She worked in a particular subset of the parameter space consistent with the Keplerian boundary conditions. She found a part of the boundary of the region where globally acceptable transonic solutions of the saddle type exist. She interpreted the existence of such a boundary as an upper limit, $\alpha_* \approx 10^{-2}$, to the Shakura-Sunyaev viscosity parameter α and suggested that when α is greater than α_* no stationary solution is possible. Matsumoto *et al.* (1984) cleared up this point by showing that *unstable* nodal-type transonic solutions exist beyond the boundary found by Muchotrzeb. They demonstrated that the boundary of the forbidden region corresponds to the boundary between regions of spiral- and nodal-type solutions.

Instability of the transonic solutions was suggested by Muchotrzeb-Czerny (1986) and studied in great detail analytically by Kato, Honma, and Matsumoto (1988) and numerically by Matsumoto, Kato, and Honma (1988). They found local

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instabilities for isothermal disks when

$$\alpha\Omega(R_c) > \left|\frac{dv}{dR}\right|_c . \tag{1.1}$$

Here Ω is the angular velocity of rotation, v is the radial component of the accretion velocity, R is the radial distance, and the subscript c refers to the critical point. The instability arises when oscillations of the azimuthal component of the viscous force are in phase with the variations of azimuthal velocity, so the viscous force does positive work on oscillations. They showed that these local oscillations may develop to global trapped ones in the transonic region because the epicyclic frequency,

$$\chi^2 = \frac{2\Omega}{R}\frac{dl}{dR} , \tag{1.2}$$

is small there. In addition to these, there is a nonpropagating mode located at $R = R_c$. When inequality (1.1) predicts instability, this mode also becomes unstable. For isothermal flows with the Shakura-Sunyaev viscosity prescriptions the instability criterion (1.1) says that nodal sonic points are unstable, while the saddle-type points are stable. This is not, however, as we show later in this paper, a general property.

Recently Paczyński (1987) proposed an explanation for the complex phenomenon of the quasi-periodic oscillations (QPOs) observed (see, e.g., Stella 1986 for references) in the several bright low-mass X-ray binaries. The explanation utilizes the existence of the upper limit for the viscosity parameter α for stable flows, suggested by Muchotzreb.

We shall also discuss the location of the two other boundaries in the parameter space. The first is connected with the condition that the solution must be *complete*, i.e., it must pass from infinity to the center. In the $\alpha = 0$ case it was found by Lu (1986). The second is connected with the stability condition (1.1).

The most important conclusion of our paper is that the unstable region is located close to the forbidden region in the parameter space. As we explain later, this may indicate that the bistable flows oscillating between the Bondi and disk states should be in addition affected by instability, i.e., be intrinsically noisy, or show time-dependent pulsations with frequencies much higher than those characteristic for their high state–low state transitions.

II. ISOTHERMAL TRANSONIC ACCRETION

From now on we assume that the accreted gas is isothermal, i.e., that the sound speed c_S is constant. Another simplification adopted here is the Paczyński and Wiita (1980) potential to describe, in the Newtonian theory, the external gravitational field at distance R from a black hole or a neutron star with mass M:

$$\psi = -\frac{GM}{R - R_G} , \quad R_G \equiv \frac{2GM}{c^2} . \tag{2.1}$$

In this potential the free particles on circular orbit have the *Keplerian* angular velocity Ω_K and specific angular momentum l_K given by

$$\Omega_K^2 \equiv \frac{GM}{(R - R_G)^2 R} , \quad l_K^2 = \frac{GMR^3}{(R - R_G)^2} . \tag{2.2}$$

Still another important simplification follows from the assumption that the vertical thickness H of the flow is very small:

$$H = c_S/\Omega_K \ll R . \tag{2.3}$$

This allows us to use vertically integrated or averaged physical quantities: density Σ, pressure W, specific angular momentum l, accretion velocity v, and the viscous stress g. The basic equations describing the mass, momentum, and angular momentum conservation, together with the equation of state and the Shakura-Sunyaev viscosity prescription, are (cf. Matsumoto *et al.* 1984)

$$\frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma v) = 0 , \tag{2.4a}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial R} - \frac{l^2 - l_K^2}{R^3} + \frac{1}{\Sigma}\frac{\partial W}{\partial R} + \frac{W}{\Sigma}\frac{\partial \ln \Omega_K}{\partial R} = 0 , \tag{2.4b}$$

$$\frac{\partial l}{\partial t} + v\frac{\partial l}{\partial R} - \frac{1}{2\pi R}\frac{1}{\Sigma}\frac{\partial}{\partial R}(2\pi R^2 g) = 0 , \tag{2.4c}$$

$$W = c_S^2\Sigma , \tag{2.4d}$$

$$g = -\alpha W . \tag{2.4e}$$

In our convention $v < 0$ corresponds to accretion and $v > 0$ to a wind solution. We are not interested in the second possibility, so we assume $v < 0$.

a) Stationary Case

In the stationary case the equation of mass conservation (2.4a) can be trivially integrated, giving $2\pi R\Sigma v = -\dot{M}_0 =$ constant, with $\dot{M}_0$ the accretion rate. To integrate the equation of angular momentum conservation (2.4c), one assumes that there is no viscous stress across the horizon of the black hole: $g(R_G) = 0$. The result of the integration is $(l - l_{in})\dot{M}_0 = 2\pi R^2\alpha W$. It will be convenient to introduce dimensionless variables by the following scaling:

$$\begin{aligned}
&\text{Radius:} && r = R/R_G \\
&\text{Velocity:} && \vartheta = v/c , \\
&\text{Sound velocity:} && K = c_S/c , \\
&\text{Angular momentum:} && \lambda = l/(R_G c) , \\
&\text{Angular velocity:} && \omega = \Omega/(c/R_G) .
\end{aligned} \tag{2.5}$$

With this scaling the *stationary* model of the flow is described by

$$\left(\vartheta - \frac{K^2}{\vartheta}\right)\frac{d\vartheta}{dr} = (\lambda^2 - \lambda_K^2)r^{-3} - K^2\frac{d\ln(\omega_K/r)}{dr} \equiv \mathcal{N}(r, \lambda) , \tag{2.6}$$

$$\lambda - \lambda_{in} = -\alpha r K^2\vartheta^{-1} . \tag{2.7}$$

The functions $\lambda_K(r)$ and $\omega_K(r)$ are dimensionless versions of $l_K(r)$, $\Omega_K(r)$, given explicitly by (2.2) and (2.5). In particular, we have

$$\frac{d\ln(\omega_K/r)}{dr} = -\frac{5r - 3}{2r(r - 1)} ,$$
$$\frac{d^2\ln(\omega_K/r)}{dr^2} = \frac{5r^2 - 6r + 3}{2r^2(r - 1)^2} . \tag{2.8}$$

Note that equations (2.6) and (2.7) were obtained by performing *two* integrations. However, only *one* integration constant, λ_{in}, appears in these equations. The second integration constant, $\dot{M}_0$, is lost because of the assumption c_S = constant (we shall discuss this later).

b) Sonic Point Local Regularity Conditions

At the critical point *both* the left-hand and right-hand sides of equation (2.6) are equal to zero, and

$$\vartheta = \vartheta_c = -K \,, \tag{2.9a}$$

$$\mathcal{N}_c \equiv \frac{\lambda_c^2 - \lambda_K^2(r_c)}{r_c^3} + K^2 \frac{5r_c - 3}{2r_c(r_c - 1)} = 0 \,. \tag{2.9b}$$

The critical points are located outside the horizon of the black hole, $r_c > 1$. Thus, because $5r_c > 3$ and $K^2 > 0$, equation (2.9b) can be fulfilled only when

$$\lambda_c^2 \le \lambda_K^2(r_c) \,, \tag{2.10}$$

a condition found previously by Abramowicz and Zurek (1981). Here equality corresponds to $K = 0$, and it is consistent with a boundary of a forbidden region in the parameter space.

In general topological type of critical points for a linear first-order differential equation $dy/dx = f(x, y)$ can be seen from an expression for the derivative $\eta \equiv (dy/dx)_c$ at the sonic point. It follows from the quadratic equation $\eta^2 - 2a\eta + b = 0$, and may be written as

$$\left(\frac{dy}{dx}\right)_c = a \pm (a^2 - b)^{1/2} \,. \tag{2.11}$$

The spiral-type critical points exist for $a^2 - b < 0$, while the saddle and nodal points exist for $a^2 - b > 0$. The saddle points correspond to $b < 0$ and the nodal ones to $b > 0$. When $a = 0$, there are no nodal points. In this case the spiral points (which are reduced to circular ones) correspond to $b > 0$ and the saddle points to $b < 0$. (See, e.g., Ferrari *et al.* 1985 for more details.)

To apply these ideas to equations (2.6) and (2.7), we introduce three new variables, $x = r - r_c$, $y = \vartheta - \vartheta_c$, $z = \lambda - \lambda_c$. With accuracy in quadratic terms in x, y, z, one has

$$\left(\frac{d\vartheta}{dr}\right)_c = \left(\frac{dy}{dx}\right)_c = \left(\frac{y}{x}\right)_c \,, \quad \left(\frac{d\lambda}{dr}\right)_c = \left(\frac{dz}{dx}\right)_c = \left(\frac{z}{x}\right)_c \,. \tag{2.12}$$

Expansion of equation (2.6) near the critical point gives

$$y \frac{dy}{dx} = \lambda_c \frac{\partial \mathcal{N}}{\partial \lambda^2} z + \frac{1}{2} \frac{\partial \mathcal{N}}{\partial r} x \,. \tag{2.13}$$

The derivatives $\partial \mathcal{N}/\partial \lambda^2$ and $\partial \mathcal{N}/\partial r$ can be found from equation (2.6). From equation (2.7) one has, expanding around $r = r_c$,

$$z = \alpha x K + \alpha r_c y \,, \tag{2.14}$$

and from the last three equations one derives the solution for the derivative $(d\vartheta/dr)_c$:

$$\left(\frac{d\vartheta}{dr}\right)_c = \left(\frac{dy}{dx}\right)_c = \frac{\alpha\omega_c}{2} \pm \left(\frac{\alpha^2\omega_c^2}{4} - X\right)^{1/2} , \tag{2.15}$$

$$X \equiv \frac{3}{2} \lambda_c^2 r_c^{-4} - \alpha\lambda_c K r_c^{-3} - \frac{1}{2}(r_c - 1)^{-3} + \frac{5r_c - 6r_c + 3}{4r_c^2(r_c - 1)^2} K^2 \,. \tag{2.16}$$

Comparing this with equation (2.11), we conclude that the spiral, nodal, and saddle regions are consistent with

$$\text{Spiral:} \quad \frac{\alpha^2\omega_c^2}{4} \le X \,, \tag{2.17}$$

$$\text{Nodal:} \quad 0 \le X \le \frac{\alpha^2\omega_c^2}{4} \,, \tag{2.18}$$

$$\text{Saddle:} \quad 0 \ge X \,. \tag{2.19}$$

A necessary, but not sufficient, condition for an acceptable solution is that the critical point should be of either saddle or nodal type. Therefore equation (2.17) with *equality* defines the boundary of a forbidden region. When X changes between 0 and $\alpha^2\omega_c^2/4$, i.e., the critical point is of the nodal type, the velocity gradient changes between

$$0 < \left(\frac{d\vartheta}{dr}\right)_c < \frac{\alpha\omega_c}{2} \,. \tag{2.20}$$

On the other hand, the criterion (1.1) demands for an unstable flow

$$0 < \left(\frac{d\vartheta}{dr}\right)_c < \alpha\omega_c \,. \tag{2.21}$$

However, owing to discontinuity of $(d\vartheta/dr)$ across the critical point (see Fig. 1), the regions defined by inequalities (2.20) and (2.21) are, in fact, equivalent: the nodal-type flows are unstable, while those of the saddle type are stable.

c) Global Topological Constraints

The conditions $X \le \alpha^2\omega_c^2/4$ and $\lambda_c^2 \le \lambda_K^2(r_K)$ demand that the critical point have the correct topological type, saddle or nodal, and correspond to an acceptable equation of state, with positive sound-velocity square. These conditions are *local*; they cannot guarantee that a solution crossing a locally correct critical point reaches the central accreting object: there is a possibility that the solution turns back to a subsonic region through a nonregular sonic point with $\mathcal{N}(r, \lambda) \ne 0$. Figure 2 illustrates this possibility and defines what we understand by a *final* critical point. The question whether a critical point is final can be always answered by numerical integration. Here we give a simple analytical argument, sufficiently accurate for our purposes.

Integration of equation (2.6) from r_c to some value $r < r_c$ gives

$$\frac{1}{2}(\vartheta^2 - K^2) - K^2 \ln \frac{|\vartheta|}{K} = F(r, \lambda) \,, \tag{2.22}$$

$$F(r, \lambda) = \int_{r_c}^{r} \lambda^2(r) r^{-3}\, dr + \frac{1}{2}\left(\frac{1}{r - 1} - \frac{1}{r_c - 1}\right) - K^2 \ln \left[\frac{\omega_K(r)}{\omega_K(r_c)} \frac{r_c}{r}\right] . \tag{2.23}$$

The function of ϑ and K on the left-hand side of equation (2.22) is zero for $\vartheta = -K$, i.e., at a critical point, and positive for the supersonic part $|\vartheta| > K$. Hence the condition that the critical point is final reads $F(r, \lambda) > 0$ for all $r < r_c$. On the other hand, from $\lambda = \lambda_{in} + \alpha r K^2/|\vartheta|$ and $|\vartheta| > K$ we have $\lambda_{in} < \lambda < \lambda_{in} + \alpha r K$, and from equation (2.23) we can write

$$F(r, \lambda_{in} + \alpha r K) < F(r, \lambda) < F(r, \lambda_{in}) \,. \tag{2.24}$$

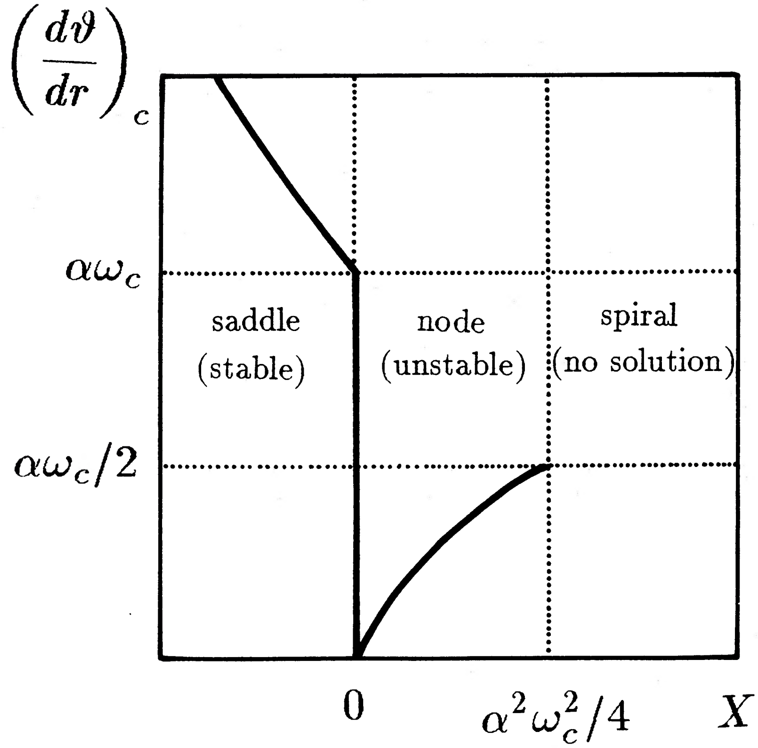

FIG. 1.—Gradient of the accretion velocity at the critical point as a function of X

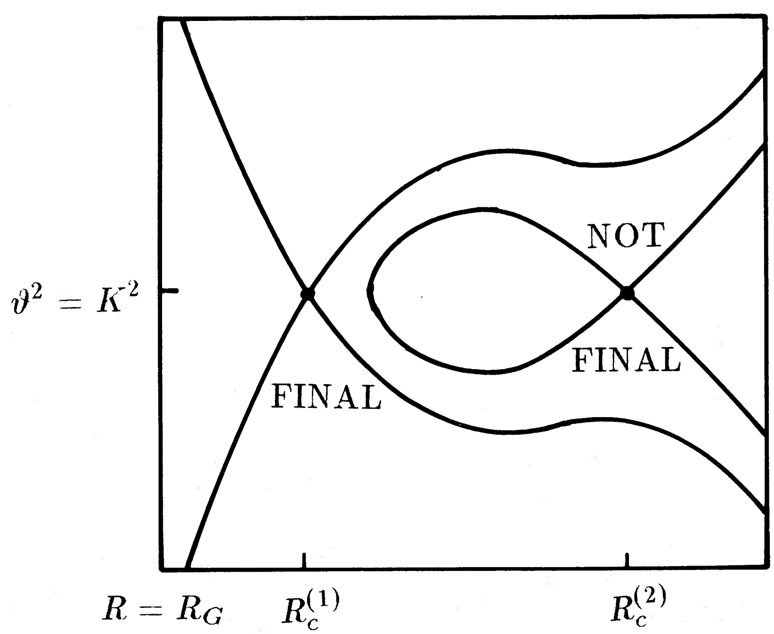

FIG. 2.—Final critical point

If $F(r, \lambda_{in} + \alpha r K) > 0$ for all $r < r_c$, then also $F(r, \lambda) > 0$ for all $r < r_c$. Hence

$$F(r, \lambda_{in} + \alpha r K) > 0 \quad \text{for all} \quad r < r_c \tag{2.25}$$

gives a *sufficient* condition for the critical point $r = r_c$ to be final. Similarly, if $F(r, \lambda_{in}) < 0$ for some $r < r_c$, then it would be also $F(r, \lambda) < 0$ for some $r < r_c$. Hence

$$F(r, \lambda_{in}) > 0 \quad \text{for all} \quad r < r_c \tag{2.26}$$

is a *necessary* condition for the critical point at $r = r_c$ to be final. Both of these conditions can be explicitly expressed as algebraic conditions, and for fixed K and α they give two lines on the (λ_c, r_c)-plane. The boundary between the final and non-final critical points,

$$F(r, \lambda) = 0\,, \tag{2.27}$$

lies between them. When $\alpha = 0$ the two conditions (2.25) and (2.26) coincide, and the location of the boundary (2.27) is explicitly known.

III. PHYSICAL REGIONS IN THE PARAMETER SPACE

a) Parameter Space

There are *three* first-order derivatives, $(d\Sigma/dR)$, (dv/dR), (dl/dR), in the stationary problem and therefore three constants of integration connected with them. Material properties of the fluid are characterized by *two* parameters, α and $K = c_S/c$. The location of the sonic point R_c gives *one* additional parameter to the problem. The value of the central mass M is not a parameter, because the scaling (2.5) erases it completely.

The six constants which characterize the problem parameters are

$$\dot{M}_0,\ \lambda_{in},\ \lambda_c,\ R_c,\ \alpha,\ K\,. \tag{3.1}$$

We shall see that only three of them are independent. The *two* regularity conditions at the sonic point reduce the number of independent parameters from *six* to *four*. We have already seen that the integration constant $\dot{M}_0$ (accretion rate) does not appear in the regularity conditions at the sonic point. This is a particular property of the isothermal flow which arises from simplification of the pressure gradient force term $\Sigma^{-1}(\partial W/\partial R) \equiv P'$ in equation (2.4b).

In general, for a one-parameter equation of state, $W = W(\Sigma)$ and $c_S^2 = (dW/d\Sigma) \neq$ constant. Therefore,

$$P' \equiv \frac{1}{\Sigma}\frac{\partial W}{\partial R} = c_S^2(\Sigma)\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial R}\,. \tag{3.2}$$

One then uses equation (2.4a) and its integrated version to eliminate both $\partial\Sigma/\partial R$ and Σ from equation (3.2), replacing them by $\partial v/\partial R$, v, and known functions of R. Because $\Sigma = -\dot{M}_0/2\pi R v$, one obtains in general $P' = P'(v, dv/dR, R, \dot{M}_0)$. However, since P' depends on $\dot{M}_0$ only through c_S, when c_S is constant the dependence of $\dot{M}_0$ is lost. Therefore, only three of the original six parameters are independent.

One might worry that the absence of the accretion rate $\dot{M}_0$ in the set of parameters describing the flow due to the assumption $c_S =$ constant implies that this very assumption is quite unacceptable from the physical point of view: after all, $\dot{M}_0$ contains the most important physical information about the flow. However, in a sense, exactly the same information is carried out by c_S. For example, physically the vertical thickness H of the flow is regulated by the accretion rate, but because $H = c_S/\Omega_K$ one can regulate the thickness by tuning c_S. The thickness is zero either when $\dot{M}_0 = 0$ or when $c_S = 0$. In Figure 3 the location of the critical points is shown in the $\alpha = 0$ case. Figure 3*a* corresponds to the isothermal situation considered here, while Figure 3*b*, taken from Abramowicz and Zurek (1981), corresponds to the general one-parameter equation of state $W = W(\Sigma)$. The topological equivalence of the $c_S =$ constant curves in the isothermal case with the $\dot{M}_0 =$ constant curves in the general case is quite visible in this figure.

The angular momentum conservation equation and the regularity condition $\mathscr{N}_c = 0$ help to make the best choice of

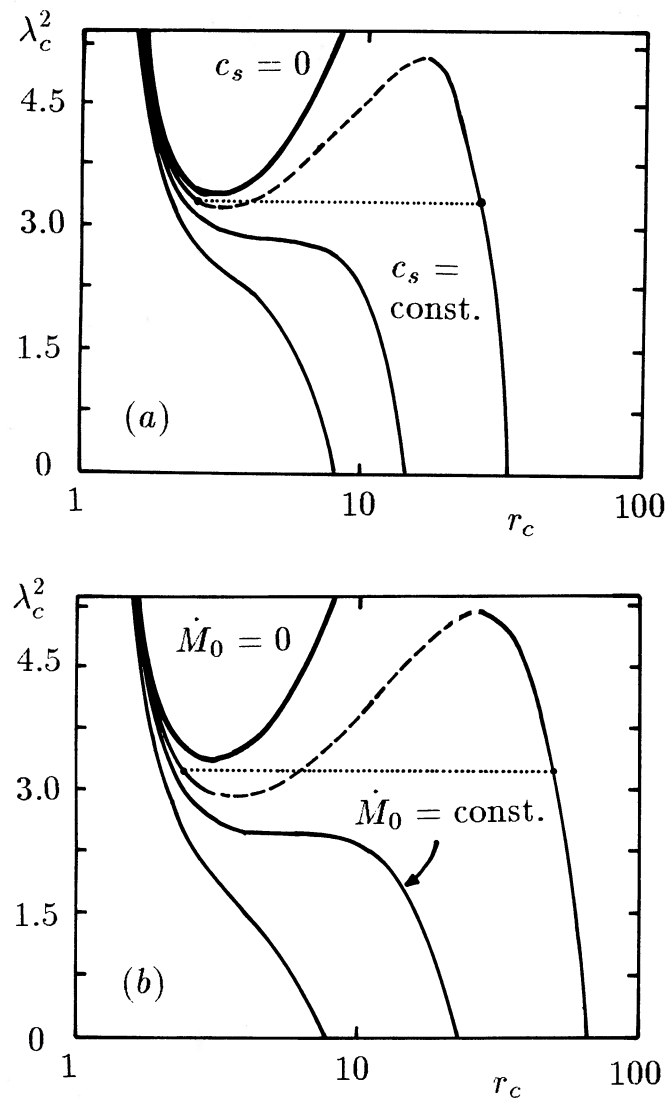

FIG. 3.—Location of the critical points in the $\alpha = 0$ case for (*a*) isothermal and (*b*) general one-parameter equation of state cases. The regularity condition $\mathscr{N} = 0$ gives $c_S =$ constant curves for the isothermal case or $\dot{M} =$ constant curves for the general case. It is clear from these figures that the properties of these curves are very similar: in both cases the negative slope of the curves indicates the saddle-type sonic point (*solid lines*), and the positive slope indicates the circle or spiral type (*dashed lines*). For the same set of parameters two solutions with the acceptable, saddle-type, critical point topology are sometimes possible, as shown by horizontal dotted lines. If such a situation occurs, it is always possible to reach infinity from the outside critical point and the accreting center from the inside critical point. In this paper we are considering the outside critical points, and therefore the question is whether the critical curve passing a critical point can reach the center.

the three independent parameters. These equations at the critical point have the form

$$\lambda_{\rm in} = \lambda_c - \alpha r_c K\,, \quad K^2 = \frac{2[\lambda_K^2(r_c) - \lambda_c^2](r_c - 1)}{r_c^2(5r_c - 3)}\,. \quad (3.3)$$

It is obvious that one must choose two of the three parameters K, r_c, λ_c and one of the two parameters $\lambda_{\rm in}$, α. The most convenient choice of the three independent constant parameters, completely describing the three-dimensional parameter space of the problem, is

$$r_c,\ \lambda_c,\ \alpha\,. \quad (3.4)$$

b) *Boundaries of the Physical Regions*

The different boundaries discussed in the previous section form two-dimensional surfaces in the three-dimensional parameter space. We shall describe them in the two-dimensional plane (λ_c^2, r_c), treating α as a parameter.

From formulas (2.17), (2.18), and (2.19) it follows that when $\alpha = 0$ the nodal region shrinks to a single line $X = 0$, and that the spiral (*circle* in the limit $\alpha = 0$) and the saddle regions are divided by this line. The line $X = 0$ for $\alpha = 0$ corresponds to the loci of the extrema of the $\mathcal{N} = 0$ (with K = constant) curves. In Figure 4 the curve $X = 0$ for $\alpha = 0$ is shown by a thick solid line marked $B_{\rm SS}$, while the $\mathcal{N} = 0$ curves for four given K's are shown by thin lines, marked by the corresponding values of K. Their solid parts indicate saddle-type points, and the dashed parts denote the circle-type critical points. The $\mathcal{N} = 0$ curve for $K = 0$ (*solid line marked* λ_K^2) gives the boundary of the forbidden region connected with inequality (2.10). Inside this region $K^2 < 0$, and in the region above the line $B_{\rm SS}$ the physically acceptable solution is impossible when $\alpha = 0$.

When $\alpha \neq 0$ the line $B_{\rm SS}$ shifts upward and splits to form a nodal region, sandwiched between the spiral and the saddle regions. This is shown in Figure 4 by two pairs of thick solid lines marked $\alpha = 0.1$ and $\alpha = 0.3$. In the case $\alpha = 0.1$ the two lines almost coincide, so it is not possible to show them as separate ones. For the Shakura-Sunyaev viscosity prescription (2.4e) the nodal region coincides with the region of the unstable solutions; instability criterion (1.1) is fulfilled there.

Figures 5*a* and 5*b* show in detail how the boundaries of the forbidden regions change with increasing α. As α increases, the saddle region on $\mathcal{N} = 0$ (with K = constant) curves, indicated by solid lines, invades beyond the extrema of the curves. In addition, the nodal region, indicated by dotted lines, emerges between the saddle and the spiral regions. For sufficiently large K the minimum and maximum first come very close together, then join, and finally, for still larger K, disappear (see Figs. 3–5). Because of this, for sufficiently large K and α, the spiral region, indicated by dashed lines, disappears on the $\mathcal{N} = 0$ (with K = constant) lines, leaving two saddle regions sandwiching a nodal one (see Fig. 5*b*). The $\mathcal{N} = 0$ (with K = constant) lines with no extremal points belong to the saddle region.

The boundary of the regions where the sufficient and necessary conditions for a final critical point are satisfied are shown in Figures 5*a* and 5*b* by dash-dotted lines marked $B_{\rm FS}$ and $B_{\rm FN}$, respectively. Critical points below $B_{\rm FS}$ are guaranteed to be final, while those above the curve $B_{\rm FN}$ are certainly not final. The region of final critical points extends beyond the local minima of the $\mathcal{N} = 0$ (with K = constant) curves and enters slightly into the unstable nodal region. Between these curves our criteria cannot judge whether or not a critical point is final, but this is not a problem, because for our present purpose it is

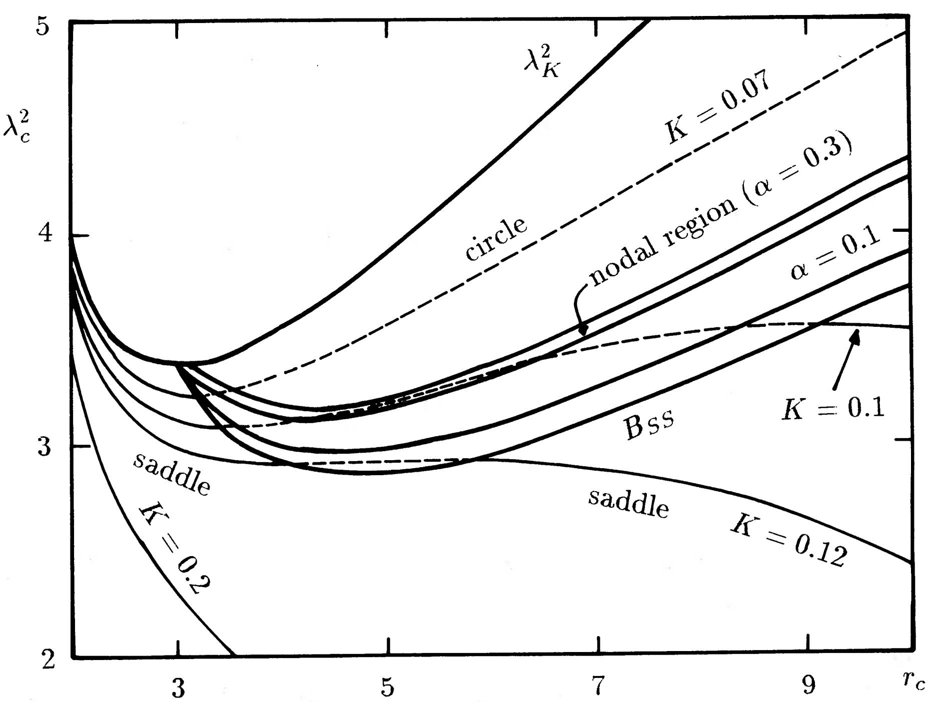

FIG. 4.—Diagram showing boundaries of various physical regions in the (λ_c^2, r_c)-plane. The curve marked λ_K^2 is the Keplerian one. The curve marked B_{SS} is the boundary between the regions of saddle critical point and circular critical point, in the case of $\alpha = 0$. When $\alpha = 0$, the whole regions above the Keplerian curve as well as above the $B_{\rm SS}$ curve are forbidden ones. The region of nodal critical point which appears when $\alpha \neq 0$ is also shown for $\alpha = 0.1$ and $\alpha = 0.3$. The region in the case of $\alpha = 0.1$ is so narrow that it is shown by a single curve. The region below the strip is the saddle region, while the region above is the spiral region. The $\mathcal{N} = 0$ curves with K = constant (*thin lines*) are shown in order to demonstrate that the $B_{\rm SS}$ curve is the locus of the extrema of these curves.

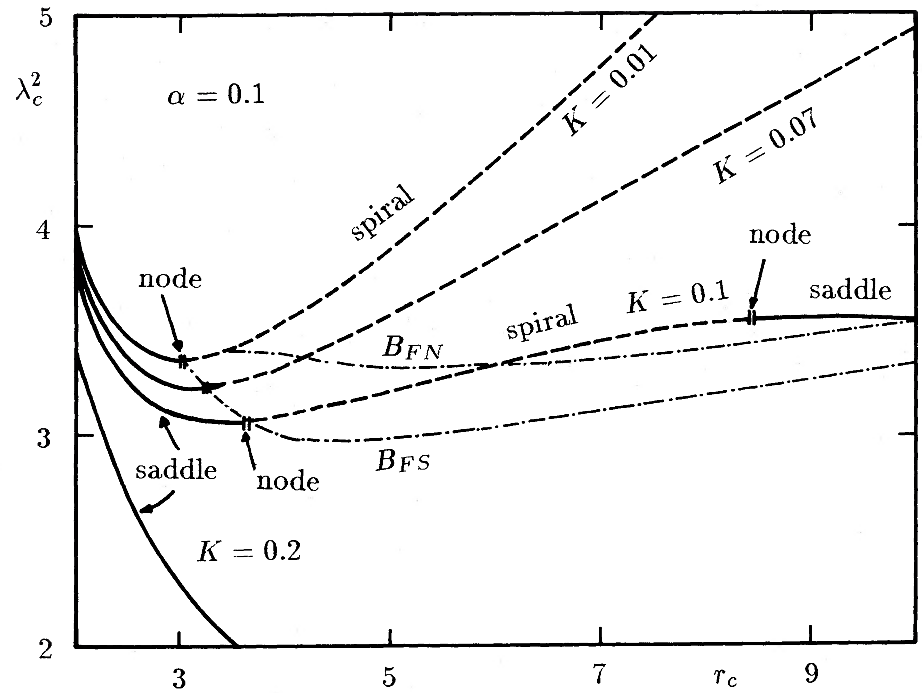

FIG. 5a

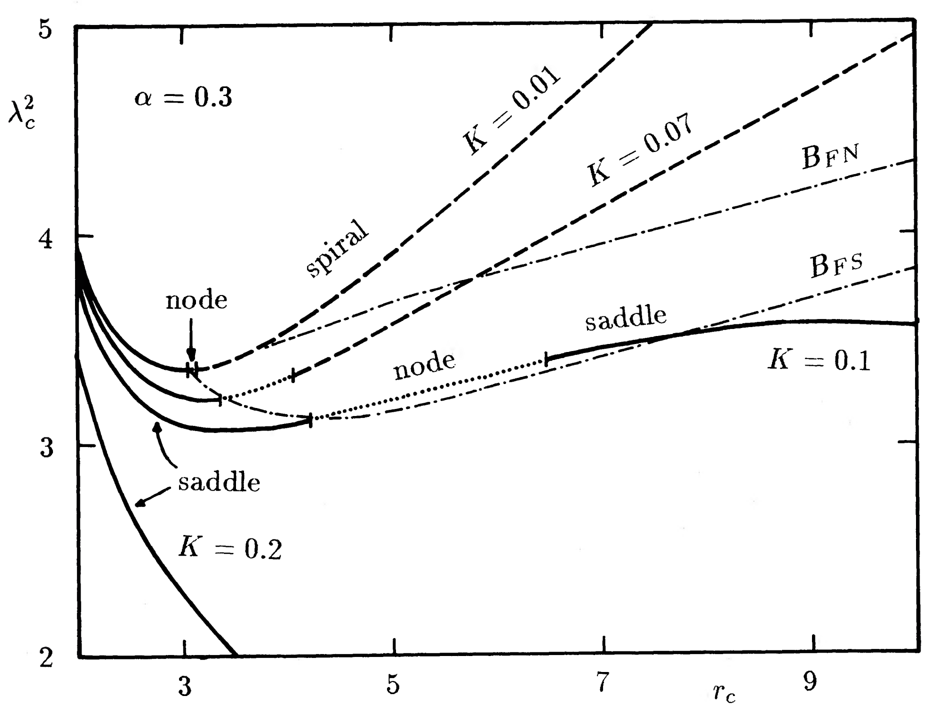

FIG. 5b

FIG. 5.—(a) Diagram showing how the $\mathcal{N} = 0$ curves (with K = constant) are divided into regions of saddle, nodal, and spiral critical points, in the case of $\alpha = 0.1$. The $\mathcal{N} = 0$ curves are shown for four values of K: 0.01, 0.07, 0.1, and 0.2. The curve for $K = 0.01$ is almost equal to the Keplerian one in the range of r under consideration. The saddle-type critical point region is shown by solid curves, and the spiral-type region by dashed curves. The nodal-type region is so narrow that only the boundaries are shown by vertical lines. The boundary of a sufficient condition for the final critical point is shown by curves marked B_{FS}. Below this curve, the critical point is definitely the final one. The boundary of a necessary condition for the final critical point is also shown by curves marked B_{FN}. Above the curve, the critical point cannot be the final one. (b) Same as (a), but for $\alpha = 0.3$. The nodal critical point region on the $\mathcal{N} = 0$ curves is shown by dotted curves. Notable differences from the case of $\alpha = 0.1$ are that the saddle region as well as the nodal region are wide, and the spiral region disappears in the case of $K = 0.1$.

enough to notice that a physically acceptable region, in which *all* the local and global conditions are fulfilled, exists for *any* value of α. There is *no* critical value for viscosity beyond which steady solutions are impossible.

IV. DISCUSSION

Our assumptions about viscosity and the equation of state are, of course, quite unrealistic. We shall now discuss how the results depend on these assumptions.

a) Viscosity Law

The Shakura-Sunyaev viscosity law has the unphysical property of giving nonzero viscous torque in the shear-free, $d\Omega/dR = 0$, case. For this reason many authors adopt today another phenomenological viscosity law, assuming that the kinematic viscosity ν is given by

$$\nu = \alpha c_S H , \tag{4.1}$$

where α is a constant viscosity parameter. In this case the viscous stress is $g = \alpha W \Omega_K^{-1} R(d\Omega/dR)$, and the dimensionless equations describing the steady state are

$$\left(\vartheta - \frac{K^2}{\vartheta}\right)\frac{d\vartheta}{dr} = \mathcal{N}(r, \lambda) , \tag{4.2}$$

$$\lambda - \lambda_{\rm in} = \alpha r^2 K^2 \vartheta^{-1} \omega_K^{-1} \frac{d\omega}{dr} . \tag{4.3}$$

Compare this with equations (2.6) and (2.7) for the Shakura-Sunyaev viscosity. At the critical point $\mathcal{N}(\lambda, r) = 0$, which is the same regularity condition as in the Shakura-Sunyaev case. This should be expected anyway, since the regularity condition does not depend on α.

We shall use the same method to study the topology of the critical points that was used in § II*b*. The expression for $y(dy/dx)$ is the same as that given by equation (2.13), while the expression for z, obtained from expansion of expression (4.3) around the critical point, now reads

$$z = \left(\frac{d\lambda}{dr}\right)_c x = \frac{2\alpha\lambda_c K - (\lambda_c - \lambda_{\rm in})\vartheta_K(r_c)}{\alpha r_c K} x , \tag{4.4}$$

where $\vartheta_K(r_c)$ is the Keplerian rotation speed at the critical point r_c. Therefore, the square of the velocity gradient at the sonic point equals

$$\left(\frac{d\vartheta}{dr}\right)_c = \pm Y^{1/2} , \tag{4.5}$$

with $Y = Y(r_c, \lambda_c, \lambda_{\rm in})$ given by

$$Y = \frac{2\alpha K \lambda_c^2 - (\lambda_c - \lambda_{\rm in})\lambda_c \vartheta_K(r_c)}{\alpha K r_c^4} + \frac{1}{2}\left(\frac{\partial \mathcal{N}}{\partial r}\right)_c , \tag{4.6}$$

$$\left(\frac{\partial \mathcal{N}}{\partial r}\right)_c = -3\lambda_c^2 r_c^{-4} + (r_c - 1)^{-3} - K^2 \frac{5r_c^2 - 6r_c + 3}{2r_c^2(r_c - 1)^2} . \tag{4.7}$$

From this we conclude that only spiral- and saddle-type critical points exist in the present case:

$$\text{Spiral:} \quad Y < 0 , \tag{4.8}$$

$$\text{Saddle:} \quad Y \geq 0 , \tag{4.9}$$

$$\text{Nodal:} \quad \text{does not exist} . \tag{4.10}$$

When $\alpha \ll 1$, one has

$$(\lambda_c - \lambda_{\rm in})\vartheta_K(r_c) = 2\alpha K \lambda_c + O(\alpha^2) . \tag{4.11}$$

This proves that for $\alpha = 0$ the boundary between the saddle and circle (spiral) regions in the Shakura-Sunyaev case, given by $X = 0$, and the boundary between these regions in the present case, given by $Y = 0$, coincide, as they obviously should. When $\alpha \neq 0$, these two boundaries not only locate in different places; in addition, they have different topology, one being a nodal *region*, the other one being a line.

The absence of the nodal region in the present case might make one think that there is no unstable region either. This is, however, not true. Let us linearize equations (2.4a)–(2.4d), with $g = \alpha W \Omega_K^{-1} R(d\Omega/dR)$, assuming that every physical quantity f differs slightly from its equilibrium, steady state, value f_0, $f = f_0 + \delta f$. We assume that

$$\delta f \sim \exp i(\sigma t - \mathcal{K} R) , \tag{4.12}$$

where σ is the frequency of the perturbation, $\mathcal{K} = 2\pi/\Lambda$ its wavenumber, and Λ its wavelength. We consider only short-wave perturbations, $\Lambda \ll R_0$, with R_0 being the length scale of change for f_0. Keeping terms linear in δf and dropping some terms according to $\Lambda \ll R_0$, we arrived at the following dispersion relation:

$$n^3 + \alpha \mathcal{K}^2 c_S^2 \Omega_K^{-1} n^2 + (\chi^2 + \mathcal{K}^2 c_S^2) n + \alpha \mathcal{K}^2 c_S^2 \Omega_K^{-1}(2\Omega\Omega_K h_0 + c_S^2 \mathcal{K}^2) = 0 , \tag{4.13}$$

where $n \equiv i(\sigma - v_0 \mathcal{K})$, $h_0 \equiv -(R/\Omega_K)(d\Omega/dR)$, and χ is the epicyclic frequency defined by equation (1.2).

The full analytical discussion of stability of the transonic, isothermal accretion flow with the Shakura-Sunyaev viscosity prescription was done by Kato, Honma, and Matsumoto (1988). Because our dispersion relation (4.13) becomes identical with theirs when our quadratic term is dropped and $h_0 = 1$, some of the results concerning the Shakura-Sunyaev case can also be recovered from our present discussion.

First of all, let us note that because $d\Omega/dR$ is less than zero for accretion flows, all the coefficients a_i in the dispersion relation $a_i n^i = 0$ are positive. The condition for *instability* is Im $(\sigma) < 0$ or Re $(n) > 0$. For $\alpha = 0$ the dispersion relation has the three solutions

$$n = 0 , \quad n = \pm i(\chi^2 + \mathcal{K}^2 c_S^2)^{1/2} . \tag{4.14}$$

These solutions represent a trivial neutral mode and two sound waves with Doppler-shifted frequency propagating through rotating and moving medium. Let us now consider the case of nonzero but very small viscosity, $\alpha \ll 1$, by expanding the frequency according to $n = n_0 + \alpha\delta n$. The solution connected with $n_0 = 0$ is

$$\alpha\delta n = \frac{-\alpha\mathcal{K}^2 c_S^2(2\Omega\Omega_K h_0 + c_S \mathcal{K}^2)}{\Omega_K(\chi^2 + \mathcal{K}^2 c_S^2)} < 0 , \tag{4.15}$$

and this corresponds to a *stable* viscous mode. On the other hand, the sound waves become overstable for $\alpha \ll 1$:

$$\alpha\delta n = \alpha\Omega \frac{\mathcal{K}^2 c_S^2}{\chi^2 + \mathcal{K}^2 c_S^2} \frac{2\Omega^2 - \chi^2}{\Omega\Omega_K} > 0 . \tag{4.16}$$

Note also that the growth rate of the unstable sound modes n is of the order of $\alpha\Omega$, i.e., the instability occurs by the *thermal* time scale. It is the same instability as that found by Kato, Honma, and Matsumoto (1988). It is also for the $\nu = \alpha c_S H$ viscosity law, despite the fact that there are no nodal-type critical points there. This makes us believe that the presence of the unstable region in the parameter space is a general property, not dependent on a particular viscosity description.

b) *Equation of State; Energy Equation*

In this paper we have shown how to analyze local and global constraints connected with the occurrence of the critical points in the flow in the simple case of the isothermal accretion onto a black hole. Such an analysis is complicated in the general case, when the equation of state is assumed to be that of the gas and radiation mixture, various heating and cooling processes are allowed, and radiation transfer is considered. In particular, no stability analysis is available yet. For this reason it would be premature to apply the exact results of our paper to numerical models with realistic equation of state and realistic treatment of dissipative processes. Such realistic numerical models for a wide range of accretion rates, from $\dot{M}_0 = 10^{-3}\dot{M}_E$ to $\dot{M}_0 = 10^2\dot{M}_E$, have been recently computed by Abramowicz *et al.* (1988). The authors used the Shakura-Sunyaev viscosity law, included heat transport by radiation, conduction, and advection in both vertical and horizontal directions, allowed the angular momentum distribution to be different from the Keplerian one (although boundary conditions are Keplerian), and explicitly solved the transonic part of the flow.

V. CONCLUSIONS

We have shown that the local regularity conditions and the global topological demands do not prevent steady state, transonic, isothermal accretion flows to exist for any value of the Shakura-Sunyaev parameter α. However, these conditions cut off some forbidden regions from the parameter space of the problem. This means that if the astrophysical situation locates the flow in these forbidden regions, the flow cannot be stationary. This can be manifested as switching between high and low states. As the boundaries of the forbidden regions are warped by unstable regions, the unsteady flows should in addition show some activity on time scales shorter than that of the high state–low state switching.

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