with a small index of refraction $n_x \gg 1$. The scattering of a plasma wave into a plasma wave ions of the equilibrium plasma also plays an important role.[10] Despite the high probability of scattering of an extraordinary electromagnetic wave into an ordinary one, this process cannot exert a significant influence on the dynamics of conversion because of the high group velocity of the ordinary wave and the small size of the source along the field. The conversion of a plasma wave into electromagnetic radiation is pulsatory in nature. The period of the pulsations (4-7 msec) in the narrow-band emission of S bursts. The level of electromagnetic radiation corresponding to the extraordinary mode can reach $6\cdot10^{-20}$ W/(m$^2\cdot$Hz) in the model under discussion, which corresponds to the emission level in S bursts observed at the earth.

1)For simplicity, we shall discuss the dynamics of the conversion of plasma waves based on the example of a dynamic spectrum without splitting.

2)We do not consider the zero state of equilibrium $\tilde{W}_p = \tilde{W}_p' = \tilde{W}_x = 0$, which is unstable for any values of the system parameters (3)-(5).

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Translated by Edward U. Oldham

Evolution times for disks of planetesimals

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(Submitted September 4, 1986; resubmitted July 5, 1987)

Astron. Zh. **65**, 1075–1085 (September–October 1988)

We describe the basic design principles behind an algorithm for computer modeling of the evolution of disks consisting of a large ($\gtrsim 10^4$) number of planetesimals, making it possible to investigate the main stages of solid-body accumulation of planets or their cores. The accumulation times for planets are estimated on the basis of computer models of the evolution of disks consisting of hundreds of bodies, and analytical investigations of a series of models for the evolution of disks consisting of a considerably larger number of bodies. As in the work of Safronov, Vityazev, and Pechernikova, the accumulation time for the main mass of the earth is $\sim 10^8$ years, while for Neptune it is $\sim 10^9$ years. In the zone of Neptune certain planetesimals, moving in eccentric and inclined orbits, could be preserved up to the present.

1. INTRODUCTION

Many authorities now maintain[1] that the planets of the terrestrial group and the cores of the giant planets were formed from a disk of solid bodies or planetesimals, initially moving in almost circular orbits. According to Refs. 1-3, the total mass of the initial bodies in the zones of the giant planets exceeded by severalfold the mass of solid material that went into these planets. Having reached $\sim$1-3 $m_\oplus$ ($m_\oplus$ is the mass of the earth) in $3\cdot10^7$ and $3\cdot10^8$ years, the embryos of Jupiter and Saturn, respectively, began to accrete gas.[4] Other models of the formation of the planets also exist.[1,5-7] The process of solid-body accumulation has been investigated both analytically[1-2,8] and by computer modeling.[3,6-7,9-19] The results of computer modeling of the evolution of disks initially consisting of hundreds of bodies have been used to study the final stages of solid-body accumulation of the planets.[3,6,10-12,14-18] The evolution of swarms that initially consisted of a large number ($\sim10^{11}$-10^{12}) of planetesimal bodies has been studied by a computer investigation of the variations of the distribution functions of the bodies with respect to masses and distances from the sun.[9,13,19] Upon the appearance of large bodies (planetary embryos) in the swarm, these calculations were ended, since the "particle in a box" method used in Refs. 9 and 19 became inapplicable.

The actual process of accumulation of the planets depended in a complex way on many factors. The study of relatively simple models, however, makes it possible to draw a number of important conclusions about the process of accumulatioin of the planets. In the present paper we consider the evolution of disks of bodies moving around a massive central body (the sun). Models in which the bodies combine in collisions are predominantly studied, but we also discuss the influence of fragmentation of bodies and gas drag on the accumulation process. The mutual gravitational influence of the bodies is taken into account by the method of spheres (mainly spheres of action),[14,20] i.e., inside a sphere the relative motion of bodies is treated within the framework of the two-body problem, while outside the sphere the bodies move around the sun in unperturbed Keplerian orbits.

In Sec. 2 we consider certain general questions in the construction of an algorithm for computer modeling of the evolution of disks consisting of a large number ($\gtrsim 10^4$) of bodies. As in Refs. 9, 13, and 19, the bodies of the disk are divided into groups (bins), but the interactions of the bodies are taken into account differently. In the limit, when each group consists of one body, this algorithm, not yet implemented on a computer, is the same as the algorithm used to investigate the evolution of disks consisting of hundreds of bodies.[18,20] The formulas that we use to determine the number of encounters between bodies up to the radius r_c of the sphere used and the number of collisions between bodies in a time Δt (Sec. 3) were obtained for a more complex model than the "particle in a box" model considered in Refs. 9 and 19, and they enable us to take into account not only the relative velocity of the bodies but also the orbital elements of proximate bodies. The time of the fall of small bodies into a larger body is determined (Sec. 4) for models differing from the models considered by Safronov et al.[1-2,8] In Sec. 5 we give the evolution time for disks initially consisting of hundreds of bodies. The evolution of such disks was investigated by computer modeling. On the basis of the results given in Secs. 3 and 5, the evolution time for disks consisting of a large number of original bodies is estimated for a series of models of Secs. 6 and 7. Although the evolution of the disks of planetesimals was three-dimensional in nature,[2] a plane model was also considered in the paper along with the three-dimensional model. The investigation of the plane model enables us to estimate the minimum changes in the orbits of gravitationally interacting planetesimals.

2. BASIC DESIGN PRINCIPLES BEHIND AN ALGORITHM FOR COMPUTER MODELING OF THE EVOLUTION OF DISKS CONSISTING OF A LARGE NUMBER OF PLANETESIMALS

The group of bodies examined usually includes bodies for which the masses m and the semimajor axes a (as well as the eccentricities $\bar{e}$ and inclinations i when a sufficiently powerful computer is available) of the orbits lie within certain fairly narrow limits. In the algorithm under consideration, each (k-th) group is characterized by the number $N^{(k)}$ of bodies included in it (instead of $N^{(k)}$ we may consider the total mass $M_{\Sigma}^{(k)}$ of all the bodies of the group) and the mean values of m, a, e, and i designated as $\bar{m}^{(k)}$, $\bar{a}^{(k)}$, $\bar{e}^{(k)}$, and $i^{(k)}$, respectively. Moreover, in the three-dimensional case each group can also be characterized by the mean values $\bar{\Omega}^{(k)}$ and $\bar{\omega}^{(k)}$ of the longitude Ω of the ascending node and the argument of perihelion ω.

As was done in Refs. 9, 13, and 19, we break down the entire process into a series of successive steps Δt in time. The number of encounters $N^{(k,\ \ell)}$ between bodies up to a distance r_c and the number of collisions $N_{col}^{(k,\ \ell)}$ which occur between bodies of the k-th and ℓ-th groups, as well as the changes in the quantities $N^{(k)}$, $\bar{m}^{(k)}$, $\bar{e}^{(k)}$, $\bar{i}^{(k)}$ ($\bar{\Omega}^{(k)}$ and $\bar{\omega}^{(k)}$) in a time Δt can be determined differently for different models. Below we shall consider, in particular, a relatively simple model, which we call the model of group-averaged bodies. The more complex models can be investigated on the basis of this model. In the model of group-averaged bodies we shall assume that in the interaction of bodies of two groups, for all $N^{(k)}$ bodies of one (the k-th) group the masses m and the orbital elements, a, e, i, (Ω and ω) are the same and equal to $m^{(k)}$, $a^{(k)}$, $e^{(k)}$, $i^{(k)}$, (Ω and $\omega^{(k)}$), respectively. Such bodies will be called averaged bodies. To improve conservation of the integrals of motion, it is better to determine the values of $\bar{a}^{(k)}$, $\bar{e}^{(k)}$, $\bar{i}^{(k)}$ and $\bar{\Omega}^{(k)}$ not as the mean values of the corresponding orbital elements (as was done in Refs. 9 and 19) but so that the mechanical energy $h^{(k)}/2$ and the angular momentum $\mathbf{c}^{(k)} = \{c_x^{(k)}, c_y^{(k)}, c_z^{(k)}\}$ of all the averaged bodies of the group are the same as for all the actual (different) bodies of the group. For each averaged body of the k-th group the values of $\mathbf{c}$ and h are $\bar{\mathbf{c}}^{(k)} = \{\bar{c}_x^{(k)}, \bar{c}_y^{(k)}, \bar{c}_z^{(k)}\} = \mathbf{c}^{(k)}$ and $\bar{h}^{(k)} = h^{(k)}/N^{(k)}$ (remember that in the method of spheres, bodies outside the spheres do not interact with each other). Using the laws of conservation of energy and angular momentum and dropping the group (k) we have

$$\bar{a} = -\bar{m}\bar{\mu}/\bar{h}, \quad \bar{e} = \sqrt{1 - [(\bar{c}_x)^2 + (\bar{c}_y)^2 + (\bar{c}_z)^2]/\bar{\mu}\bar{m}\bar{a}},$$
$$\cos\bar{i} = \bar{c}_z/\bar{m}\sqrt{\bar{\mu}\bar{p}}, \quad \sin\bar{i} = \sqrt{(\bar{c}_x)^2 + (\bar{c}_y)^2}/\bar{m}\sqrt{\bar{\mu}\bar{p}},$$
$$\cos\bar{\Omega} = -\bar{c}_y/\bar{m}\sin\bar{i}\sqrt{\bar{\mu}\bar{p}}, \quad \sin\bar{\Omega} = \bar{c}_x/\bar{m}\sin\bar{i}\sqrt{\bar{\mu}\bar{p}}, \tag{1}$$

where $\bar{p} = (1-\bar{e}^2)a$, $\mu = G(M_\odot + \bar{m})$, $M_\odot$ is the mass of the sun, G is the gravitational constant. In accounting for the mutual gravitational influence of the bodies by the method of spheres, h and c do not depend on ω or the true anomaly ν. Therefore, if ω is treated as one of the properties of the group, we can calculate $\bar{\omega}$ as the mean value of ω_j for all the bodies of the k-th group.

Witout dwelling in detail here on modeling the gravitational interactions and collisions of bodies, we note that at each step Δt of the algorithm one must know the values of $N_c^{(k,\ell)}$ and $N_{col}^{(k,\ \ell)}$ for $k = 1, \ldots, N_g$ and $\ell = 1, \ldots, N_g$, where N_g is the number of groups in the disk. The values of Δt are chosen, in particular, so that there is no more than one collision per body of a group, on the average, in time Δt. If the number of bodies in the groups is large, then the results of interactions between bodies in the time Δt can be determined from computer modeling of a relatively small number of characteristic encounters and collisions.

3. NUMBERS OF ENCOUNTERS AND COLLISIONS BETWEEN BODIES IN TIME Δt

For $N_c^{(k,\ \ell)}$ and $N_{col}^{(k,\ \ell)}$ we shall take the expectation values of these quantites. In the stochastic approach under consideration,

$$N_c^{(k,\,l)} = N^{(k)} N_e^{(l,\,k)} \Delta t/\bar{\tau}^{(k,\,l)}, \tag{2}$$

where $N^{(k)}N_e^{(\ell,\ k)}$ is the number of pairs of bodies (body j_1 belongs to the k-th group while body j_2

belongs to the ℓ-th group) for which the values of the aphelion and perihelion distances (r_α and r_π) allow an encounter between the j_1-th and j_2-th bodies (the minimum distance between the segments $[r_\pi(j_1), r_\alpha(j_2)]$ does not exceed r_c); $1/\overline{\tau}^{(k,\ell)}$ is the mean values of the quantities $1/\tau(j_1, j_2)$, where $\tau(j_1, j_2)$ is the time between encounters (up to r_c) of the j_1-th and j_2-th bodies. We can represent $N_e(\ell, k)$ in the form

$$N_e^{(l,k)}=N^{(l)}\overline{\varkappa}^{(l,k)}, \text{where } \overline{\varkappa}^{(l,k)}\leqslant 1. \quad (3)$$

In the general case $\overline{\kappa}^{(\ell,k)}$ can be calculated numerically. In the interaction between bodies of the same group (for $k = \ell$), instead of $n^{(k)}N_e^{(\ell,k)}$ in Eq. (2) we have $N^{(k)}(N^{(k)}-1)\overline{\kappa}^{(k,k)}/2$.

First let us consider the plane model. Let $\Delta\phi$ be the sum of the angles (in radians), with the apex at the sun, within which the distance between the orbits of the j_1-th and j_2-th bodies along the central ray (with apex at the sun), is less than r_c, while $T^{(j)}$ is the period of revolution of the j-th body around the sun. Bodies that lie on the same central ray will, after a time close to (equal to, for circular orbits) the synodic period of revolution $T_S = T(j_1)T(j_2)/|T(j_1)-T(j_2)|$, again lie on the same central ray. If as an approximation we take the direction of this ray to be random, then the ratio $\Delta\phi/2\pi$ is the probability that the distance between bodies lying on this ray is less than r_c. Also allowing for the fact that the mean value of the initial angle (with apex at the sun) between the directions toward the bodies is π for a random location of the bodies, while after an encounter it is 2π, we find that for $\Delta t \leq T_S/2$ the expectation value of the number of encounters (up to r_c) between the j_1-th and j_2-th bodies in the time Δt is $\Delta t/\tau(j_1, j_2) \approx \Delta t\Delta\phi/\pi T_S$, while for $\Delta t > T_S/2$ it is $\Delta t/\tau(j_1, j_2) \approx (0.5 + \Delta t/T_S)\Delta\phi/2\pi$. In the study of the process of solid-body accumulation it is proposed to take $\Delta t \gg T_S$. In this case, therefore, designating the mean value of $\Delta\phi$ for different pairs of bodies of the k-th and ℓ-th groups as $\overline{\Delta\phi}$ and assuming that the values of T_S for these pairs differ only slightly, we obtain

$$\overline{\tau}^{(k,l)}\approx 2\pi\overline{T}_s/\overline{\Delta\varphi}, \quad (4)$$

where $\overline{T}_S$ is the mean value of T_S. For eccentric orbits ($e_m = \max\{\overline{e}^{(k)}, \overline{e}^{(\ell)}\} < r_c/R$) we consider the the quantity $k_\phi = \overline{\Delta\phi}/r_c^*$, where $r_c^* = r_c/R$ and R is the distance of the bodies from the sun. The values of k_ϕ are smaller for larger eccentricities. It is found numerically, for example, that for $m = 0.01\, m_\oplus$ we have $k_\phi \approx 30$ for $e_m = 0.1$, while $k_\phi \approx 10$ for $e_m = 0.3$ (different mutual orientations of the orbits of bodies of the different groups were considered for these estimates of k_ϕ). Using Eqs. (2)-(4), we can determine the value of $N_c^{(k,\ell)}$ for $k \neq \ell$.

In an investigation of interactions between bodies of the same group, the bodies can be divided into two subgroups in a certain way and the same Eqs. (2)-(4) can be applied. Moreover, in that case the values of $N_c^{(k,k)}$ can also be determined from Eq. (2), assuming that

$$\overline{\tau}^{(k,k)}\approx k_T(\overline{a}^{(k)}/a_\oplus)^{3/2}T_\oplus/r_c^\star\approx 2\pi k_T(\overline{a}^{(k)})^{3/2}/r_c^\star\sqrt{GM_\odot}, \quad (5)$$

where $k_T \approx 3.5$, $a_\oplus$ is the semimajor axis of the earth's orbit, and $T_\oplus$ is the period of its revolution around the sun. Equation (5) was obtained on the basis of computer modeling of the evolution of a series of disks consisting of 100 identical particles of mass m_0. The initial eccentricities e_0 of the particle orbits were the same. In such modeling it was assumed that the ratio of the maximum to the minimum values of the semimajor axes of the orbits of bodies of the initial disk was $a_{max}/a_{min} = 2$, $10^{-8}\, m_\oplus \leq m_0 \leq m_\oplus$, and $0.01 \leq e_0 \leq 0.3$, and it was found that $\tau^{(k,k)}$ hardly depended on $\overline{e} \approx e_0$.

Now let us consider the calculation of $N_c^{(k,\ell)}$ for the three-dimensional model. We designate the angle between the orbital planes of two encountering bodies as Δi, while u' is the angle between one of the two rays formed by the line of intersection of the orbital planes and the direction toward one of these bodies, which we call the second. Encounters between bodies up to r_c can occur only for those values of u' for which $h(u') = R_2 \sin u' \sin \Delta i \leq r_c$, where R_2 is the distance from the sun to the second body. We designate the sum of the angles, with apex at the sun, within which this inequality is satisfied as $\Delta u'_\Sigma$. Taking $R_2 \approx R$, $\sin u' \approx u'$, $\sin \Delta i \approx \Delta i$, and $\max\{h(u')\} \geq r_c$ (i.e., $\Delta i \geq r_c^*$), we obtain $\Delta u'_\Sigma \approx 4r_c^*/\Delta i$, and the probability $p(\Delta i)$ of an encounter between two bodies up to r_c in a time Δt is smaller in the three-dimensional case than in the plane case by a factor of $k_i \approx (2\pi/\Delta u'_\Sigma)(r_c/r_c') \approx \pi R\,\Delta i/2r_c'$, where r_c' is the mean value (in the region $\Delta u'_\Sigma$) of the projection of r_c onto the orbital plane of the first particle. Since

$$r_c'=\left[\int\limits_{h(u')\leqslant r_c}\sqrt{(r_c)^2-(h(u'))^2}\,du'\right]\Big/\int\limits_{h(u')\leqslant r_c}du'\approx r_c/2,$$

$p(\Delta i)$ and N_c are smaller by about a factor of $k_i = \pi\Delta i/r_c^*$ for $\Delta i \geq r_c^*$ than for $\Delta i = 0$ (for the same values of m, e, and a). In the case when Δi assumes values from 0 to $\Delta i^*_{max} - r_c^*$ with equal probability, taking $k_i \approx 2$(where $1/\overline{k}_i$ is the mean value of $1/k_i$ for different Δi) for $0 \leq \Delta i \leq r_c^*$, for $0 \leq \Delta i \leq \Delta i^*_{max}$ we obtain

$$1/\overline{k}_i=\left[\int(1/k_i)\,d(\Delta i)\right]\Big/\int d(\Delta i)=r_c^\star\eta_i/\pi\Delta i^\star_{max}, \quad (6)$$

where $\eta_i = 0.5 + \ln(\Delta i^*_{max})-\ln(r_c^*)$. For i uniformly distributed from 0 to $i^{(k)}_{max}$, the probability that $\Delta i \approx \Delta i_{max} = i^{(k)}_{max} + i^{(\ell)}_{max}$ is lower than it is for $\Delta i \approx 0$. Therefore, $\Delta i^*_{max} < \Delta i_{max}$. We shall take approximately $\Delta i^*_{max} \approx \overline{i}^{(k)} + \overline{i}^{(\ell)}$. For example, if r_c is the radius of the sphere of action and $\Delta i^*_{max} = 0.15$, we have $\eta_i \approx 9$ for $m = 10^{-6}\, m_\oplus$ and $\eta_i \approx 4$ for $m = m_\oplus$. Equation (6) was obtained for the case when the bodies are not divided into different groups according to i. But if the values of i for bodies of k-th group range from $i^{(k)}_{min}$ to $i^{(k)}_{max}$, while the minimum distance Δi_{min} between the segments $[i^{(k)}_{min}, i^{(k)}_{max}]$ and $[i^{(\ell)}_{min}, i^{(\ell)}_{max}]$ is greater than r_c^*, then, integrating k_i from Δi_{min} to Δi^*_{max}, by analogy with Eq. (6), we obtain

$$\overline{k}_i\approx\pi(\Delta i^\star_{max}-\Delta i_{min})/r_c^\star\ln(\Delta i^\star_{max}/\Delta i_{min}).$$

Using Eq. (2), we obtain

$$N_{col}^{(k,l)}=N_c^{(k,l)}/\overline{N}_c^{(k,l)}=N^{(k)}\Delta t/k_N T_\varphi^{(k,l)}, \quad (7)$$

where $N_c^{(k,\ell)}$ is the mean number of encounters up to r_c leading to one collision and

$$T_\varphi^{(k,l)}=\overline{\tau}^{(k,l)}\overline{N}_c^{(k,l)}/k_N N_e^{(l,k)} \quad (k_N=1 \text{ for } k\neq l \text{ and } k_N=2 \text{ for } k=l). \quad (8)$$

Usually in the algorithm under consideration $\Delta t < T_\phi^{(k,\ell)}$, but formally we find from (7) that in a time $T_\phi^{(k,\ell)}$ each body of the k-th group takes part in one collision with some body of the ℓ-th group, on the average. We assume below that bodies collide (combine or are destroyed) when the distance between their centers of mass becomes equal to the sum r_Σ of their radii. With allowance for the additional capture or destruction of some of the bodies that have entered the Roche zone,[1,13] it must be borne in mind that the accumulation time is $T \propto (r_3)^{-1}$ for $\Delta i = 0$ and $T \propto (r_3)^{-2}$ for $\Delta i\ r_c^*$, where r_3 is the capture radius. According to Refs. 21 and 23, the additional capture is minor.

Using the concept of the effective radius r_e of a body, in the plane case for averaged bodies we have

$$\overline{N}_{c\,pl}^{(k,l)} \approx r_c/r_e \approx r_c/r_\Sigma\sqrt{1+2\theta'}, \quad (9)$$

where $\theta' = 0.5(v_{par}/v_{rel})^2$, $v_{par} = \sqrt{2G(\overline{m}^{(k)}+\overline{m}^{(\ell)})/r_\Sigma}$, and v_{rel} is the relative velocity of bodies encountering each other at a distance r_c. According to Refs. 1 and 2, the rms velocity of bodies relative to a circular orbit of a planet of mass M and radius r is $v_{rel}^M = \sqrt{GM/\theta r}$, where θ is Safronov's parameter. If $m^{(k)} = M \geq m^{(\ell)} = \gamma M$, then $\theta' = \theta(1+\gamma)/(1+\gamma^{1/3})$, where $3/4 \leq (1+\gamma)/(1+\gamma^{1/3}) \leq 1$. We designate $\xi = v_{rel}/v_{orb} = k_\xi e$, where v_{orb} is the velocity of a body in an orbit of radius $(\overline{a}^{(k)} + \overline{a}^{(\ell)})/2$. On the basis of Eq. (9) and the values of $\overline{N}_c$ obtained in computer modeling of the evolution of plane disk consisting of hundreds of bodies, for a disk consisting of one group we have $k_\xi = 1$. If $\Delta i > r_c^*$, then $\overline{N}_c \approx (\overline{N}_{c\,pl}^*)^2$, where $N_{c\,pl}^*$ is the value of $\overline{N}_c$ for a plane model with the same orbital eccentricities as in the three-dimensional case. For $\overline{i} = \overline{e}/\sqrt{2}$ (Ref. 13) the values of ξ and k_ξ are larger by a factor of about $\sqrt{3}/2$ than for $\overline{i} = 0$ with the same values of $\overline{e}$.

In the course of evolution of a disk of planetesimals, θ varies. In the zone of the giant planes, for example, $\theta \gg 1$ for a gasless model disk of approximately equal bodies.[23] We represent r_c in the form $r_c = R(m/M_\odot)^\alpha(1+\delta)^{\alpha/2}$. When spheres of action are used, $\alpha = 0.4$, while in Ref. 10 $\delta = m_2/m_1$, where m_1 and m_2 are the masses of the bodies encountering each other ($m_2 \leq m_1 = m$). If $N^{(k)} \gg 1$, using Eqs. (5), (6), (8), and (9) and omitting the group numbers, in the interaction between bodies (of density ρ) of the same group and for $i_{max} \approx 2\overline{i}$ we have:

a) for $i + 2\theta' \approx 1$,

$$\overline{N}_{c\,pl} \approx \overline{a}\rho^{1/3}\overline{m}^{\alpha-1/3}(\pi/6)^{1/3}(1+\delta)^{\alpha/2}/(M_\odot)^\alpha,$$

$$T_{\phi\,pl} \approx \overline{a}^{5/2}\rho^{1/3}k_T(\pi/6)^{1/3}T_\oplus/\overline{m}^{1/3}N\overline{\varkappa}(a_\oplus)^{3/2} \propto \overline{m}^{2/3}/M_\Sigma,$$

$$T_{\phi 3d} \approx \overline{a}^{7/2}\rho^{2/3}\overline{i}k_T\pi(\pi/6)^{2/3}T_\oplus/\overline{m}^{2/3}N\overline{\varkappa}\eta_i(a_\oplus)^{3/2} \propto \overline{m}^{1/3}/M_\Sigma, \quad (10)$$

b) for $1 + 2\theta' \approx 2\theta'$ (for $\theta' \gg 1$),

$$\overline{N}_{c\,pl} \approx \overline{a}^{1/2}\rho^{1/6}\overline{m}^{\alpha-2/3}\xi(M_\odot)^{1/2-\alpha}(1+\delta)^{\alpha/2}(\pi/6)^{1/6}/2 \propto k_\xi\overline{e},$$

$$T_{\phi\,pl} \approx \overline{a}^2\rho^{1/6}(\xi/\overline{\varkappa})k_T(M_\odot)^{1/2}(\pi/6)^{1/6}T_\oplus/2\overline{m}^{2/3}N(a_\oplus)^{3/2} \propto \overline{m}^{1/3}/M_\Sigma.$$

$$T_{\phi 3d} \approx \overline{a}^{5/2}\rho^{1/3}\xi^2\overline{i}k_T\pi(\pi/6)^{1/3}M_\odot T_\oplus/4\overline{m}^{4/3}N\overline{\varkappa}\eta_i(a_\oplus)^{3/2} \propto \overline{m}^{-1/3}/M_\Sigma. \quad (11)$$

For the "particle in a box" model,[9,19] $N_{col}^{(k,\ell)} \approx N^{(k)}N^{(\ell)}v_{rel}\pi(r_e)^2\Delta t/V$, where V is the volume of space while v_{rel} is determined through $\overline{e}$ and $\overline{i}$. The values of N_c and N_{col}, determined using Eqs. (2)-(9), also depend on $\Delta\phi$, T_s, and κ, which are determined, in turn, by the orbital elements of the bodies encountering each other. The different values of the semimajor axis of the orbits of the bodies are thereby taken into account, in contrast to Refs. 9 and 19. Usng the equations obtained above, we can analyze, in particular, the cases of i = 0 and e = 0. The algorithm whose basic principles were discussed above enables us to investigate not only the initial stage, as in Refs. 9 and 19, but also subsequent stages of solid-body accumulation of the planets (or of their cores). If each group consists of only one body, then such an algorithm is analogous to the algorithm used for computer modeling of the evolution of disks consisting of hundreds of gravitating bodies.[17-18,20] In studying the interaction between single large bodies, instead of using the stochastic approach one can determine the times until encounters between pairs of bodies and subsequently model (with allowance for encounters that have already occurred) the encounters between bodies occurring in the time interval Δt under consideration.

4. TIMES OF FALL OF SMALL BODIES INTO A LARGER BODY (PLANET)

When the k-th group consists of one large body (planet) of mass M and the ℓ-th group consists of N bodies of mass m, in a time Δt about $N\Delta t/T_\phi^{\ell,k}$ bodies fall onto body M. Using the results of Sec. 3, in the case of $\overline{e} = \overline{e}^{(\ell)} \gg \overline{e}^{(k)}$ and $\overline{i} = \overline{i}^{(\ell)} \gg \overline{i}^{(k)}$ we have

$$T_{\phi\,pl}^{(l,k)} \propto M^{-1/3}/\overline{\varkappa}, \quad T_{\phi 3d}^{(l,k)} \propto \overline{i}M^{-2/3}/\overline{\varkappa} \quad \text{for} \quad 1+2\theta' \approx 1 \quad (12)$$

and

$$T_{\phi pl} \propto \overline{e}M^{-2/3}/\overline{\varkappa}, \quad T_{\phi 3d} \propto \overline{i}\overline{e}^2M^{-4/3}/\overline{\varkappa} \text{ for } 1+2\theta' \approx 2\theta'. \quad (13)$$

From Eqs. (12) and (13) it is seen, in particular, that, other conditions being equal, larger bodies grow faster. If the number of bodies m that can encounter the body M is proportional to e, then $\kappa \propto \overline{e}$ in Eqs. (12) and (13). In the case when all the bodies m can encounter body M (for $N_e^{(\ell,k)} = 1$), the mean time for a body m to fall onto body M is $T = T_\phi^{(\ell,k)}N_c$. If the inclinations of the orbits of the bodies m to the orbital plane of body M are the same and equal to Δi, then, with allowance or the results of Sec. 3, for eccentric orbits of the bodies m we find that in the plane case $\overline{T}_{pl} \approx \overline{N}_{c\,pl}2\pi T_s/k_\phi r_c^* \approx 2\pi T_s R/r_\Sigma k_\phi\sqrt{1+2\theta'}$, while for $\Delta i > r_c^*$, $\overline{T}_{3d} \approx \overline{N}_{c\,3d}2\pi^2T_s\Delta i/k_\phi(r_c^*)^2 \approx 2\pi^2T_s\Delta i(R/r_\Sigma)^2/k_\phi(1+2\theta')$.

Analytic formulas for the time of growth of a planet as it exhibits the bodies from its feeding zone are given in Refs. 2 and 8. Here the thickness $H = 2R \sin i$ of the zone and the eccentricities $\overline{e}$ of the orbits (the same for all the bodies) are determined by the mass M of the planet, which varies greatly in the course of evolution. In obtaining these formulas, it is assumed that θ does not vary with time, and that the surface density of material near the planet's orbit is the same as in the entire feeding zone (continuous mixing of bodies in the zone occurs). The formulas given in Sec. 4 were obtained for the case when M varies little over the investigated time interval, while e and $\sin i$ (or Δi) cannot depend on each other or on M.

Let us consider the fall of small bodies m_0 into a large body, a planet, of mass M ($M \gg m_0$) and radius r in the case when the feeding zone of body M has the form of a torus, formed by rotation about the Oz axis of part of a circular ring, bounded by two arcs of circles of radii $R_1 = R(1 - \delta_1)$ and $R_2 = R(1 + \delta_2)$ and by two segments of rays with the apex at the sun. We take the angle between these rays to be 2i and we assume that the bisector of this of this angle is the Oy axis, perpendicular to Oz. The volume of this zone is $V_T \approx 4\pi[(R_2)^3 - (R_1)^3]\cdot \sin i/3$. We designate the total mass of the bodies m_0 lying inside such a torus as m^0_Σ. We shall assume that the ratio λ of the mass of the bodies thrown into hyperbolic orbits or falling into other planetary embryos to the mass m'_Σ of bodies falling into body M is constant in the course of evolution. We assume that near the orbit of body M the spatial density of bodies of the swarm is $\rho_0 = \kappa' \rho_{cs}$, where $\rho_{cs} = [m^0_\Sigma - m'_\Sigma(1 + \lambda)]/V_T$ and $\kappa' =$ const. Then, using the well-known[2,8] relation $dM/dt = \pi(r_e)^2\rho_0 v^M_{rel}$, for $i > r_e/R$ and nearly constant values of M, $\sin i$, and v^M_{rel} (and hence of $\bar{e}$ and ξ) in the course of evolution, we find that the time in which bodies m_0 with a total mass $k_s m^0_\Sigma(1 + \lambda)$ fall into body M is $k_s m_\Sigma^0/(1+\lambda)$, $T(k_s) \approx -2(\delta_1+\delta_2)\sin i (R/r)^2 P \ln|1-(1+\lambda)k_s|/\pi(1+2\theta')\xi\kappa$, where P is the period of revolution of body M around the sun.

The formulas given in this section can be used to determine the total mass of bodies m that fell into body M (not necessarily a planetary embryo) in a time Δt, as well as to determine the time of existence of bodies in the vicinity of a nearly formed planet. While bodies are falling into a planet $\theta' \approx \theta = (4\pi/3)^{1/3}\rho^{1/3}M^{2/3}a/\xi^2 M_\odot$ (Ref. 23) and $\delta_1 \approx \delta_2 \approx e$, where ρ is the density of the planet and a is the semimajor axis of its orbit. If $\bar{e} = \sqrt{2}\xi$ (Ref. 23), $\lambda = 0$, and $k_s = 0.97$, then for the earth $T(k_s) \approx 3\cdot 10^9 \sin i/(1 + 2\theta)$ yr, while for Neptune $T(k_c) \approx 3\cdot 10^3 \sin i(1 + 2\theta)$ yr. In this case, in particular, for $\bar{e} = 0.2$ in the terrestrial zone $T(k_s) = 5\cdot 10^8 \sin i$ yr, while for $\bar{e} = 0.4$ in the zone of Neptune $T(k_s) = 2\cdot 10^{11} \sin i$ yr. If $T_s = 5P$, then for these planets and values of $\bar{e}$ we have $\bar{T}_{3d} = 4\cdot 10^8 \Delta i$ (for $k_\phi = 20$) and $\bar{T}_{3d} = 2\cdot 10^{11}\Delta i$ (for $k_\phi = 10$), resectively. The estimates given above indicate that in the zone of Neptune individual planetesimals may still exist, moving in inclined and eccentric orbits. The extended gaseous envelopes possessed earlier by the giant planets[12] could contribute to greater efficiency of capture of planetesimals (smaller values of $\bar{T}$ and $T(k_s)$).

5. TIMES OF EVOLUTION OF DISKS CONSISTING OF HUNDREDS OF BODIES

The distribution of bodies of an evolving disk as a function of their masses and orbital elements was investigated earlier[3,14-18] by computer modeling of the evolution of disks initially consisting of hundreds of bodies. The mutual gravitational influence of the bodies was taken into account mainly by the method of spheres of action, and it was assumed that colliding bodies combine. In this case the bodies were not divided up into groups, pair-wise interactions of all the bodies were taken into account, and the algorithm described in Refs. 14 and 20 was used. In this section we give the time T of evolution of these disks (up to the last collision between bodies). The value of T can vary by an order of magnitude with variation of the pseudorandom numbers used to determine the positions in their orbits of the bodies encountering each other.[20] Besides T, therefore, we also consider the values of $T^{(25)}$, the time of evolution until 25 bodies remain in the disk.

For the three-dimensional model (with $\bar{i} = \bar{e}/2$) in the terrestrial zone, with a mass M_Σ of solid material in the feeding zone of the planet equal to $m_\oplus$, we have $T^{(25)}_{3d} \approx 10^7$ yr and $T_{3d} \approx 5\cdot 10^7$-$5\cdot 10^8$ yr, while in the zone of Uranus and Neptune with $M_\Sigma = 200\ m_\oplus$, we have $T^{(25)}_{3d} \approx 5\cdot 10^7$ yr and $T_{3d} \approx 3\cdot 10^8$-$10^{11}$ yr. Here, as below, the numerical estimates are made for the case when the density of bodies in the terrestrial zone is ~5.6 g/cm³, while in the zone of Neptune it is ~2 g/cm³. The results obtained, like the data of Ref. 11, indicate that the time it takes for the individual bodies from its feeding zone to fall into the earth may reach $5\cdot 10^8$ years. In a computer-aided investigation of the evolution of three-dimensional disks initially consiting of nearly formed planets and several hundred bodies in the zone of Uranus and Neptune,[3] the time it took for most of the bodies to fall into these planets was $T_{UN} \approx 10^9$ yr. Although the evolution of the disk of planetesimals had a three-dimensional character,[2] for comparison we note that $T_{UN} \approx 10^7$ yr in the plane case. When the initial mass of the plane disk of approximately equal bodies wss close to the mass of solid material in the corresponding planets, for the terrestrial zone $T^{(25)}_{3d} \approx 10^4$ yr and $T_{pl} \approx 2\cdot 10^4$-$10^5$ yr (here not one but several small planets are formed[15]), while for the zone of Neptune $T^{(25)}_{3d} \approx 10^6$ yr and $T_{pl} \approx 3\cdot 10^6$-$3\cdot 10^7$ yr.

6. TIME OF EVOLUTION OF A DISK CONSISTING OF APPROXIMATELY EQUAL BODIES

Using the formulas given in Sec. 3, we can make certain analytic estimates of the time of evolution of disks for a number of the simplestmodels. In Sec. 6 we consider an auxiliary model in which the masses m of the bodies of the evolving disk are always approximately equal to each other (although they change in time), and in which the bodies combine in collisions. In that case we can treat the entire feeding zone of a planet as one group and use Eqs. (10) and (11), taking $\kappa \propto e$. Such a model does not occur in reality, but we can use it to investigate more complex models, treating the approximately equal bodies of the disk as one of the groups. For a model of equal bodies, in a time T_ϕ the bodies combine in pairs and the masses of the bodies double.

As can be seen from Eqs. (10) and (11), $T_{\phi\,pl}$ grows with an increase in m. At the end of evolution $\theta > 1$ (Refs. 2 and 23) and $T_{\phi\,pl}$ does not depend on e for $\bar{\kappa} \propto eT_{\phi pl}$. Therefore, most of the time of evolution of a plane disk is spent in the end stages of evolution. In the terrestrial zone this statement is also valid for a three-dimensional model, since in this case $T_{\phi\,3d}$ is larger for larger m. For a disk of different bodes it follows from this, in particular, that in the terrestrial zone the majority of the initial planetesimals were not preserved up to the final stage of formation of this planet. In the case when the influence of gravitational interactions of the bodies on the variation of $\bar{e}$ and $\bar{i}$ is greater than that of the gas, for the three-dimensional model in the zone of the giant planets e and i can reach a maximum rather rapidly, while, apart from the initial stage of evolution, $\theta > 1$ (Refs. 1 and 25). Therefore (see Eq. (11)), in this case $T_{\phi 3d}$ can decrease in the couse of evolution. For example, if in the zone of Neptune the mass of the disk is $M_\Sigma = 100\ m_\oplus$, $\bar{e} = 0.2$, $\bar{i} \approx \bar{e}/\sqrt{2}$, and $\rho = 2$ g/cm^3, the maximum of $T_{\phi 3d} \approx 4 \cdot 10^9$ yr is reached for $m \approx 10^{-4}$-0.01 $m_\oplus$, while for $m = m_\oplus$ and $m = 10^{-6}\ m_\oplus$ the values of $T_{\phi 3d}$ are lower than this maximum by a factor of two to three. For $\bar{e} = 0.4$, $T_{\phi 3d}$ is about four times larger than for $\bar{e} = 0.2$. Therefore, in the evolution of a disk of different bodies combining in collisions, bodies with masses of ~10^{-4}-0.01 $m_\oplus$ could exist in the final stages of the accumulation of Neptune.

Using Eqs. (7), (10), and (11), we can compare the rate of growth of bodies in the feeding zones of different planets. If the ratio of the total masses of the planetesimals in the zones of Jupiter and Earth is $M_\Sigma{}^R \geq 5^{5/2}$, while $\theta \gg 1$, then in the case of a thre-dimensional gasless model with equal values of m and e, the values of $T_{\phi 3d}$ are smaller for the zone of Jupiter, and hence the rate of growth is greater, than in the terrestrial zone. Since $T_{\phi 3d} \propto m^{-1/3}/M_\Sigma$ for $\theta \gg 1$, in this case allowance for the larger bodies in the zone of Jupiter than in the terrestrial zone only strengthens this statement. If $1 + 2\theta \approx 1$, then with $M_\Sigma{}^R \leq 5^{7/2}$ for the three-dimensional model and with $M_\Sigma{}^R \leq 5^{5/2}$ for the plane model, the values of T_ϕ in the zone of Jupiter are larger (for the same m and $\bar{e}$). The drag of the gas decreases $\bar{e}$ and i (Ref. 7). According to Ref. 13, $\bar{e} \approx \bar{i}\sqrt{2}$. Therefore, with allowance for Eqs. (19) and (11), we find that (except for the plane case with $1 + 2\theta \approx 1$) gas drag decreases T_ϕ and hence the time of evolution. Gas probably disappeared earlier from the terrestrial zone than from the zone of Jupiter. Therefore, allowance for the influence of gas only accelerates the relatively faster (in comparison with the terrestrial zone) growth of bodies in the zone of Jupiter. This also results from allowance for the fact that θ may be larger in the zone of Jupiter than in the terrestrial zone. The results obtained indicate that the embryo of Jupiter with a mass of ~2-3 $m_\oplus$ (capable of accreting gas) could have formed before the accumulation of the earth ended. The maximum value of e for bodies moving in a gas are estimated in Refs. 18 and 23.

7. TIMES OF EVOLUTION OF DISKS CONSISTING OF DIFFERENT BODIES

Let us consider a gasless model of the evolution of a disk consisting of large bodies m and smaller bodies m'. We designate their total masses as m_Σ and m_Σ', respectively, and their mean eccentricities as e(m) and e(m'). We shall assume that for each time under consideration, the masses of the bodies m are approximately equal to each other. Let $T_{m,m'}$ be the mean time between collisions of two bodies m and m', while $k' = T_{m,m'}/T_{m,m}$. Then for a model in which bodies combine in collisions, the rate of growth of the bodies m is proportional to $[m_\Sigma 2\bar{e}(m) + m_\Sigma'(\bar{e}(m) + \bar{e}(m'))/k']/T_{m,m}$. If the mean eccentricities and inclinations of the orbits of bodies in the variants being compared are about the same for bodies m and m' (this condition can be satisfied for $m_\Sigma' < m_\Sigma$), as well as for different variants, then $k' > 1$ and the time in which the masses of the bodies m double is $T_* = T_\phi(m_\Sigma, m)k' m_\Sigma/(k'm_\Sigma + m_\Sigma') \approx T_\phi(M_\Sigma, m)c_k$, where $c_k = k'/[1 + (k'-1)m_\Sigma/M_\Sigma]$, $M_\Sigma = m_\Sigma + m_\Sigma$, while $T_\phi(m_\Sigma, m)$ is the value of T_ϕ for the case when the disk consists of bodies m whose total mass is m_Σ. For such a model $k' \leq 2$ in the plane case and $k' \leq 4$ in the three-dimensional case and for M_Σ = const, T_* is of the same order as for a model disk of bodies that are always approximately equal.

Let us consider the model of evolution of a disk, consisting of bodies m and m', in which bodies m colliding with each other are converted into bodies m', and bodies m' colliding with bodies m always combine with them. Then, under the assumption that the mean eccentricities and inclinations of the orbits of bodies m and m' are approximately the same, and in the absence of the ejection of bodies into hyperbolic orbits, we can show that the disk evolves to a state in which $m_\Sigma' = k'm_\Sigma$. If this relation is satisfied, then for the same values of M_Σ, the bodies m grow more slowly by a factor of $M_\Sigma/M_\Sigma = k' + 1$ than in the case when the disk consists only of equal bodies m.

The results given above indicate that in the evolution of a disk of different bodies, for a model allowing for fragmentation of the bodies and gas drag, the times of growth of the largest bodies of the disk do not exceed (by more than a factor of five, in any case) the times of growth of bodies for the model of approximately equal bodies if the masses of the disks are the same in both cases. Bodies with larger masses grow faster, Therefore, allowance for the differentiation of the bodies by mass only strengthens the statement made above. Although planetary embryos may be destroyed in collisions with large bodies, breaking up and partially vaporizing, in the opinion of Cox and Lewis[10] a considerable part of the fragments can be collected into one body again under the influence of gravity. Even while losing some mass in collisions with large bodies, however, planetary embryos could grow, on the whole, due to the accumulation of small bodies.

8. CONCLUSION

An algorithm for computer modeling of the evolution of a disk initially consisting of a large ($\geq 10^4$) number of bodies, the basic design of which was discussed in the present paper, makes it possible to study only the initial stage, as in Refs. 9, 13, and 19, but also subsequent stages of solid-body accumulation of planets (or their cores).

We obtained the analytic dependence of the time of evolution of a disk on the number, masses, and mean eccentricities and inclinations of the orbits of the bodies comprising. On the basis of these functions, as well as computer modeling of the evolution of disks of hundreds of bodies, the characteristic times of evolution of disks consisting of a considerably larger number of bodies were studied for a number of models. The times for the falling of small bodies into large bodies and planetary embryos were studied. The estimates obtained for the time of accumulation of the main mass of the planets are close to the results of Safronov, Vityazev, and Pechernikova[1,2,8] (~10^8 years for Earth and ~10^8 years for Neptune). In the zone of Neptune certain planetesimals, moving in inclined and eccentric orbits, might be preserved up to the present, and some bodies with masses of ~10^{-4}-0.01 $m_\oplus$ could survive up to the final stages of formation of the planet.

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Translated by Edward U. Oldham