

TURNING-POINT METHOD FOR AXISYMMETRIC STABILITY OF ROTATING RELATIVISTIC STARS

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ABSTRACT

A turning-point method is used to obtain a simple criterion governing the secular stability of uniformly rotating relativistic stars to axisymmetric perturbations. In particular, along a sequence of rotating neutron stars with fixed angular momentum and increasing central density, the configuration with maximum mass marks the onset of instability.

Subject headings: instabilities — relativity — stars: neutron — stars: rotation

I. INTRODUCTION

The turning-point method is a simple way to locate points of secular instability along a sequence of equilibrium configurations. An early version was devised by Poincaré (1885), and recent treatments are given by Thompson (1979), Katz (1978), and Sorkin (1981, 1982). The last of these references proves a quite general theorem governing the stability of many-parameter equilibrium families, without reliance upon existence or completeness of normal modes, and, as we will see, it is immediately applicable to uniformly rotating stellar models based on one-parameter equations of state. In particular, along a sequence of rotating neutron stars with constant angular momentum, instability to collapse (i.e., instability of the quasi-radial mode) sets in at the configuration where the mass is an extremum as a function of the sequence parameter—central density, radius, or injection energy.

To locate points of *dynamical* stability to axisymmetric perturbations along sequences of rotating relativistic stars, one must evaluate a complicated functional (the second-order change in energy) for a set of trial functions (Schutz 1972; Chandrasekhar and Friedman 1972), and no explicit calculations have as yet been attempted. For slowly rotating stars one need only use a single trial function (Hartle, Thorne, and Chitre 1972; Chandrasekhar and Friedman 1972), but this approach and a related static stability criterion (Hartle and Thorne 1969) rely on nondegenerate perturbation theory, and for isentropic stars at the instability point the radial mode is degenerate with the set of zero-frequency convective modes (Bisnovatyi-Kogan and Blinnikov 1974). A revised static stability criterion (Hartle 1975) overcomes the difficulty, but it requires the iterative construction of a “comparison” sequence of differentially rotating equilibria, and calculations have been done only for $n = 3/2$ polytropes (Hartle and Munn 1975).

In addition to its simplicity, the present criterion has the virtue that secular rather than dynamical instability can be expected to limit the class of observed neutron stars: we show at the end of § II that a sequence of uniformly rotating neutron stars becomes dynamically unstable *after* it has become secu-

larly unstable. That is, for fixed angular momentum, the point of secular instability occurs at lower central density.¹

II. SECULAR INSTABILITY CRITERIA

Consider a family of stellar models based on a one-parameter equation of state $\epsilon = \epsilon(p)$. Suppose that the mass, plotted as a function of angular momentum J and some other parameter, say the central density ϵ_c , has the shape of a ridge (or valley): along curves of constant J , the mass M reaches a maximum (minimum) as a function of ϵ_c (see Fig. 1). We will show that lines of extremal mass, where $\partial[M(\epsilon_c, J)]/\partial\epsilon_c = 0$ separate the regions of stable and unstable stars.

For spherical models in which the pulsations are governed by the same one-parameter equation of state as the equilibrium models, a turning-point marks the location of a zero-frequency mode, the point along the equilibrium sequence at which the frequency of the radial mode changes from real to imaginary (see, e.g., Harrison *et al.*, 1965; Bardeen 1965; Bardeen, Thorne, and Meltzer 1966; Zel'dovich and Novikov 1971). In neutron stars, however, the oscillations are adiabatic, effectively governed by an equation of state that differs from that of the equilibrium star, and the corresponding instability is secular—it proceeds on a time scale slow enough to accommodate the nuclear reactions and energy transfer that accompany the change to a nearby equilibrium. In the case of rotating stars, the turning point similarly marks a zero-frequency “mode,” in the sense of a time-independent perturbation of the star. The perturbation is again not dynamical, and in this case the time scale must also be long enough to accommodate the transfer of angular momentum needed to keep the angular velocity uniform. That is, the growth rate of the instability is limited by

¹ For nonrotating and slowly rotating neutron stars, the stable configuration with lowest density is similarly the model with minimum mass. For rapidly rotating models, however, as one decreases the central density with angular momentum fixed, the equilibrium sequence terminates by mass shedding (the equator rotates at the frequency of a Keplerian orbit) before an extremal value of the mass is reached.

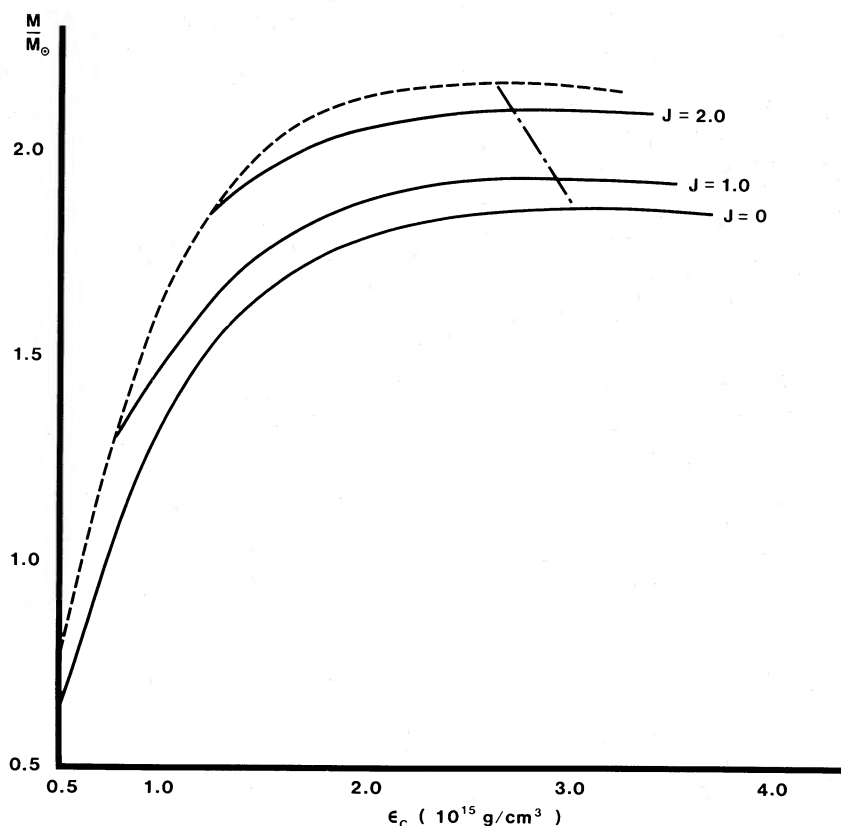


FIG. 1.—Solid curves show neutron star mass as a function of central density for three values of angular momentum (measured in units of 10^{49} g cm²); the curves are based on models constructed from the Bethe-Johnson (I) equation of state. Points to the right of the dotted line represent secularly unstable models. The dashed line represents configurations whose equator rotates at the frequency of a particle in circular orbit: above this line there are no equilibria.

the time required for viscosity to redistribute the star's angular momentum.

We first state the principal result in a way that does not restrict the parameterization. A stellar model will mean a stationary, axisymmetric solution to the Einstein field equations with perfect fluid source.

LEMMA. Consider a two-parameter family of uniformly rotating stellar models having an equation of state of the form $p = p(\epsilon)$. Suppose that along a continuous sequence of models labeled by a parameter λ , there is a point λ_0 at which both $\dot{N}$ ($=dN/d\lambda$) and $\dot{J}$ vanish and where $(\dot{\Omega}\dot{J} + \dot{\mu}\dot{N}) \neq 0$. Then the part of the sequence for which $(\dot{\Omega}\dot{J} + \dot{\mu}\dot{N}) > 0$ is unstable for λ near λ_0 .

Proof. The result follows from Theorem I of Sorkin (1982), with his function S replaced by $-M$, with E^2 replaced by N and J , and with his β_x replaced by μ and Ω . The conditions of the theorem are satisfied because stellar models are configurations for which M is minimized at fixed N and J .

One may also phrase the criterion in terms of an extremum of the mass and choose a sequence of stars along which the total angular momentum is constant:

THEOREM. Consider a continuous sequence of uniformly rotating stellar models based on an equation of state of the form $p = p(\epsilon)$. Suppose that the total angular momentum is constant along the sequence, and that there is a point λ_0 where $\dot{M} = dM/d\lambda = 0$ [and where $\mu > 0$, $(\dot{\mu}\dot{M}) \neq 0$]. Then the part of the sequence for which $\dot{\mu}\dot{M} > 0$ is unstable for λ near λ_0 .

Proof. The theorem follows immediately from the Lemma when one uses the relation (Bardeen 1972)

$$dM = \Omega dJ + \mu dN$$

to express $\dot{M}$ in terms of $\dot{N}$.

As noted earlier, the theorem has immediate application to rotating neutron stars (Friedman, Ipser, and Parker 1986; Friedman and Ipser 1986), implying that, as in the spherical case, configurations with maximum mass (at fixed angular momentum) bound the region of stable stars.

In order to make this claim, one must prove that stars on the low-density side of the ridge of maximum-mass configurations are in fact stable. In the following paragraph we argue that this result is implied by the assumption that uniformly rotating equilibria will never become differentially rotating as a result of adding viscosity: differentially rotating models in the neighborhood of a uniformly rotating model will settle down to uniform rotation. This assumption—that viscosity leads to uniform rotation—seems intuitively obvious, and is not known to fail for any dynamically stable model. However, near the end of the Maclaurin sequence (well past the dynamical instability point for the bar mode), there are uniformly rotating models for which nearby differentially rotating configurations have lower energy (Bardeen 1971).

Axisymmetric instability sets in through a zero-frequency mode in relativity, whether or not viscosity is present. That is, along a sequence of rotating stars, a change of stability occurs

only when there is a time-independent solution to the perturbation equations; and for such a solution the energy of the perturbation vanishes (the difference in mass between the perturbed and unperturbed configurations is zero to second order). Prior to the instability point, all perturbations have positive energy. Now if the time-independent perturbation that marks the onset of instability maintained uniform rotation, then the instability point would simply be the turning point. If, on the other hand, the perturbed configuration is differentially rotating, then by our assumption there is a uniformly rotating

model with the same total angular momentum and baryon number but with smaller mass. This means that there was a turning point at an earlier point of the sequence and that the zero-frequency mode occurred in a model that was already secularly unstable.

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