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On the Nongravitational Motion of Comet P/Halley

by

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ABSTRACT

A new method, based on the moments of perihelion passages as observational data to the orbit improvement, was used to investigate the nongravitational motion of the comet over two millennia. The nongravitational effects were examined as a secular change $\dot{a}$ of the semi-major axis of the comet's orbit. By linking of four or five consecutive perihelion times of the comet from the interval 1986 AD – 87 BC, discrete values of $\dot{a}$ were computed for 24 moments in the considered interval of time. It was found that the simplest reasonable function of time for approximation of the nongravitational effects was a parabolic one: $\dot{a}(t) = a_0(1+a_1 t+a_2 t^2)$. Numerical values of the parameters a_0, a_1, a_2 were determined by the least squares method basing on the 24 discrete values of $\dot{a}$. Problem of determination of a_0, a_1, a_2 together with corrections to the six orbital elements directly from observational equations is considered.

Using the 300 best selected positional observations from 1987-1835 and 25 perihelion times from 1835 AD – 87 BC, the orbit of Comet Halley was improved including the nongravitational effects in the form of the parabolic $\dot{a}(t)$. Thus it was possible to link successfully all the observed apparitions of the comet by one system of orbital elements. The equations of motion were integrated by the recurrent power series method backwards till the fifteenth century BC. Evolution of orbital elements during 44 returns of the comet and comparison with results by other authors are presented.

1. Introduction

The first attempt to link four apparitions of Comet Halley has been undertaken by Brady and Carpenter (1971) who applying the method of trial and error found a set of orbital elements representing well the observations of the comet made in 1682, 1759, 1835-36 and 1909-11. However, to obtain a satisfactory result of the linkage, the authors had to introduce an additional parameter ε into equations of the comet's motion:

$$\ddot{\mathbf{r}}+k^2[1-\varepsilon(t-t_0)]\frac{\mathbf{r}}{r^3}=\frac{\partial R}{\partial \mathbf{r}};$$

r is the radius vector, k the Gaussian gravitation constant, R the planetary disturbing function, and t_0 some initial epoch of motion of the comet. The value $\varepsilon = 1.635 \times 10^{-9}$ found by Brady and Carpenter corresponds to the secular deceleration of the comet's motion by 4.1 days per revolution. Kiang (1972), using the method of variation of elements, integrated the equations of motion of the comet back till 240 BC to determine the moments of perihelion passages of the comet basing on ancient historical records on cometary apparitions. He also found a nongravitational effect which amounted to an average lenghtening of the period by 4.1 days after each return.

Modern investigations of the nongravitational motion of Comet Halley, based on Marsden's method of including nongravitational effects (Marsden *et al.* 1973), have been undertaken by Yeomans (1977). Using the observations from 1607 through 1911 Yeomans made the least-squares differential orbit correction, confirmed the value of the nongravitational parameter ε obtained by Brady and Carpenter, and determined the Marsden's nongravitational parameters A_1 and A_2 for Comet Halley. In the known work, Yeomans and Kiang (1981) used the nongravitational parameters A_1 and A_2 as constant values with time to find the best orbital elements satisfactorily representing all the observed perihelion times of the comet till 240 BC. A further step in investigations of Comet Halley's motion was made by Landgraf (1984) who considered the nongravitational parameters A_1 and A_2 as some linear functions of time.

In both works, by Yeomans and Kiang and by Landgraf, the orbit was improved by the least-squares method using the positional observations from few last apparitions of the comet. The equations of motion were then integrated backwards over a time-span of some thousands of years. However, to obtain good agreement of computed perihelion times with the observed ones, the authors had to make some subjective changes in orbital elements for 837 AD when the comet closely approached the Earth to within 0.03 a.u. Although the procedures of computations in both works and the fit to the observed perihelion times were similar, the results of integration were quite different if comparing the perihelion times computed far from the observational interval; *e.g.* for the two predictions of perihelion time in the fifteenth century BC the value of the difference amounted to 68 years!

In order to find a source of the above mentioned discordance, we applied a new approach to the observational material for improvement of orbit of Comet Halley: we used the instants of perihelion passages of the comet as observational data for composing the observational equations and joining them to the equations resulting from the positional observations (Sitarski and Ziołkowski 1986). Thus one could use in a uniform way both modern positional observations as well as inaccurate ancient observations known from the historical records, from which the dates of perihelion passages of the comet have been deduced (Kiang 1972, Hasegawa 1979, Yeomans and Kiang 1981).

According to the results of our investigations, nongravitational forces play a crucial role in the motion of Comet Halley. The secular change in the nongravitational parameter A_2, included in the process of orbit improvement, resulted in different values of improved orbital elements than in case of the solution with constant A_2, and it caused considerably different results of prediction of the past perihelion times of the comet.

The method of using the perihelion times as observational data to the orbit improvement also allows to search a variability of nongravitational parameters of motion in the long intervals of time. It was successfully shown when investigating the nongravitational motion of Comet Encke (Sitarski 1987). Then the nongravitational effects were examined in two ways: either as a secular change $\dot{a} = \mathrm{d}a/\mathrm{d}t$ of the semi-major axis a of the comet's orbit or as the Marsden's nongravitational parameters A_1 and A_2; it was found that $\dot{a}(t)$ as well as $A_2(t)$ could be approximated by similar functions of time, hence both methods were equivalent for searching a variability of nongravitational effects with time.

Is this work we applied the method of $\dot{a}$ to investigate the nongravitational motion of Comet Halley over two millennia and to link all the apparitions of the comet by one system of dynamical parameters of motion.

2. Method of Computations

Let us assume that nongravitational effects in the comet's motion are described by $\dot{a}(t)$ being some function of time t and also of n constant parameters $a_1, \ldots, a_n$ values of which should be determined. The equations of motion of the comet have then the following vectorial form:

$$\ddot{\mathbf{r}} + k^2 \left[1 + \frac{3}{c^2} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - 2\dot{r}^2) \right] \frac{\mathbf{r}}{r^3} = \frac{\partial R}{\partial \mathbf{r}} + \frac{\dot{a}(t)}{2a} \dot{\mathbf{r}} \tag{1}$$

where: $\mathbf{r}$ — radius vector of the comet, $\dot{\mathbf{r}}$ — vector of its velocity, $\dot{r} = (\mathbf{r} \cdot \dot{\mathbf{r}})/r$, k — Gaussian gravitation constant, c — the speed of light, $3/c^2 = 1.00069809 \times 10^{-4}$, R — the planetary disturbing function, a — the semi-major axis of the orbit, $1/a = 2/r - (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/k^2$.

The solar term in Eq. (1) is modyfied by including the relativistic effects (Sitarski 1983).

By integrating numerically the equations of motion we can obtain values of the position $\mathbf{r}$ and of the velocity $\dot{\mathbf{r}}$ of the comet, and hence we can compute the values of orbital elements for an arbitrary moment, *e.g.* for a known perihelion time. A value of the mean anomaly M can be computed for the observed perihelion time and the following observational equation can be written for such an observation moment,

$$\sum_{i=1}^{6} \frac{\partial M}{\partial E_i} \Delta E_i + \sum_{j=1}^{n} \frac{\partial M}{\partial a_j} \Delta a_j = \Delta M \tag{2}$$

where ΔE_i are the corrections to the six orbital elements E_1, and Δa_j are the corrections to the constant nongravitational parameters a_j. Since for the perihelion time we have $M = 0$, so for the observed perihelion time T the deviation $\Delta M = -M_T$ if by M_T we denote the mean anomaly computed for the moment T.

To improve the initial parameters of motion we should know values of differential coefficients in observational equations, in this purpose we have to integrate the differential equation for the deviation $\Delta\mathbf{r}$ from the true position of the comet, being a consequence of the inaccuracy of initial data:

$$\Delta\ddot{\mathbf{r}} + k^2\frac{\Delta\mathbf{r}}{r^3} - 3k^2\frac{\mathbf{r}}{r^5}(\mathbf{r}\cdot\Delta\mathbf{r}) = \frac{\partial}{\partial\mathbf{r}}\left(\frac{\partial R}{\partial\mathbf{r}}\cdot\Delta\mathbf{r}\right) + \frac{\dot{a}(t)}{2a}\Delta\dot{\mathbf{r}} + \sum_{j=1}^{n}\frac{\partial\dot{a}}{\partial a_j}\frac{\dot{\mathbf{r}}}{2a}\Delta a_j. \tag{3}$$

Substituting in Eq. (3)

$$\Delta\mathbf{r} = \sum_{i=1}^{6}\mathbf{G}_i\Delta E_i + \sum_{j=1}^{n}\mathbf{G}_{6+j}\Delta a_j, \tag{4}$$

we obtain a set of differential equations for $\mathbf{G}_i$ ($i = 1, \ldots, 6$) and for $\mathbf{G}_{6+j}$ ($j = 1, \ldots, n$) a numerical integration of which allows to calculate the values of coefficients for connection of $\Delta\mathbf{r}$ with the corrections to the initial parameters of motion (Sitarski 1971, 1981).

We can express $\Delta\mathbf{r}$ by residuals $\Delta\alpha$ and $\Delta\delta$, corresponding to the observed position of the comet on the sky – right ascension α and declination δ (Brouwer and Clemence 1961, p. 234; Sitarski 1971). To use perihelion times to the orbit improvement in practical computations, let us write Eq. (2) in the equivalent form

$$\frac{\partial M}{\partial\mathbf{r}}\cdot\Delta\mathbf{r} + \frac{\partial M}{\partial\dot{\mathbf{r}}}\cdot\Delta\dot{\mathbf{r}} = \Delta M, \tag{5}$$

making use of (4) and taking into account that

$$\Delta\dot{\mathbf{r}} = \sum_{i=1}^{6}\dot{\mathbf{G}}_i\Delta E_i + \sum_{j=1}^{n}\dot{\mathbf{G}}_{6+j}\Delta a_j. \tag{6}$$

Formulae for $\partial M/\partial\mathbf{r}$ and $\partial M/\partial\dot{\mathbf{r}}$ were derived by the present author (Sitarski 1987) basing on the known relations for the eccentricity e and the eccentric anomaly E (Brouwer and Clemence 1961, p. 48):

$$\begin{aligned} e\sin E &= \frac{\sqrt{a^{-1}}}{k}(\mathbf{r}\cdot\dot{\mathbf{r}}) = F(\mathbf{r}, \dot{\mathbf{r}}), \\ e\cos E &= 1 - a^{-1}r = G(\mathbf{r}, \dot{\mathbf{r}}), \\ a^{-1} &= \frac{2}{r} - \frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})}{k^2}. \end{aligned} \tag{7}$$

Using the relations (7) and Kepler's equation we find:

$$
\begin{aligned}
&e\frac{\partial E}{\partial \mathbf{r}} = \frac{\partial F}{\partial \mathbf{r}}\cos E - \frac{\partial G}{\partial \mathbf{r}}\sin E,\\
&e\frac{\partial E}{\partial \dot{\mathbf{r}}} = \frac{\partial F}{\partial \dot{\mathbf{r}}}\cos E - \frac{\partial G}{\partial \dot{\mathbf{r}}}\sin E,\\
&\frac{\partial F}{\partial \mathbf{r}} = \frac{1}{k\sqrt{a^{-1}}}\left[a^{-1}\dot{\mathbf{r}} - \frac{(\mathbf{r}\cdot\dot{\mathbf{r}})}{r^3}\mathbf{r}\right],\\
&\frac{\partial G}{\partial \mathbf{r}} = \frac{2-a^{-1}r}{r^2}\mathbf{r},\\
&\frac{\partial F}{\partial \dot{\mathbf{r}}} = \frac{1}{k\sqrt{a^{-1}}}\left[a^{-1}\mathbf{r} - \frac{(\mathbf{r}\cdot\dot{\mathbf{r}})}{k^2}\dot{\mathbf{r}}\right],\\
&\frac{\partial G}{\partial \dot{\mathbf{r}}} = \frac{2r}{k^2}\dot{\mathbf{r}},\\
&\frac{\partial M}{\partial \mathbf{r}} = (1-e\cos E)\frac{\partial E}{\partial \mathbf{r}},\\
&\frac{\partial M}{\partial \dot{\mathbf{r}}} = (1-e\cos E)\frac{\partial E}{\partial \dot{\mathbf{r}}},
\end{aligned}
\qquad (8)
$$

what allows to get numerical values of $\partial M/\partial E_i$ and $\partial M/\partial a_j$ in Eq. (2).

We can use the positional observations of the comet from some apparitions, and the perihelion times from some one else, to solve a joint set of observational equations by the least squares method. It is easily possible since the right ascension α and declination δ as well as the mean anomaly M are the angular quantities.

3. Numerical Test of the Method

The presented method has been successfully applied in case of Comet P/Encke (Sitarski 1987). It also was used for Comet Halley in an attempt to explain a source of the discordance of results obtained by Yeomans and Kiang (1981) and by Landgraf (1984); then we included the nongravitational effects in the comet's motion using the Marsden's parameters (Sitarski and Ziołkowski 1986, 1987). Here we linked 27 apparitions of Comet Halley including the nongravitational effects in the form of $\dot{a}$, assuming that $\dot{a} = \text{const}$ over two millennia of the considered comet's motion.

To improve the orbit we selected the 250 best positional observations made during 1835 Aug. 21 – 1986 March 13 and used 24 observed perihelion times from the interval 1835 AD – 12 BC. We solved 524 observational equations by the least squares method to correct the seven parameters: six orbital elements and $\dot{a}$. The equations of motion and a set of differential equations for $\mathbf{G}_i$ ($i = 1, \ldots, 7$) were integrated by the recurrent power series method (Sitarski 1979, 1981).

The process of orbit improvement required many iterations. Finally we obtained the following result:

$$\text{Epoch of osculation: 1835 Nov. 18.0 ET}$$

$$\left.\begin{array}{lll} T = 1835\,\text{Nov. } 16.43808\ \text{ET} & \omega = 110^\circ\!.68775 \\ q = 0.58655146\ \text{a. u.} & \Omega = 56.80626 \\ e = 0.96739880 & i = 162.26134 \end{array}\right\} 1950.0 \qquad (9)$$

$$\dot{a} = (+6.3154 \pm 0.0012) \times 10^{-8}\ \text{a. u./day}$$

Starting from the above elements, the equations of the comet's motion were integrated backwards till 1500 BC. We obtained a very good agreement with the results by Yeomans and Kiang (v. Tab. 5 and 6 in Section 6) who also assumed that the Marsden's nongravitational parameters, used in their computations, were constant with time. This accordance fully confirms the numerical correctness of both results as obtained by the different authors applying the entirely different methods of computations. Therefore, a conclusion is justified that both methods, with $\dot{a}$ and with Marsden's parameters A_1, A_2 as well, are equivalent for modelling the nongravitational effects in the long-term comet's motion.

4. Analysis of Nongravitational Effects

4.1. Computation of discrete values of $\dot{a}$

Using the perihelion times alone for the orbit improvement, we are not able to correct all the orbital elements, but the two ones which can influence significantly the values of predicted perihelion times. We chose the initial value of the true anomaly v_0 and of the perihelion distance q_0 since we had ready formulae allowing to start numerical integration of differential equations for $\mathbf{G}_v$ and $\mathbf{G}_q$ (Sitarski 1971, 1987). Thus to find a value of $\dot{a}$ from the observational equations, we determined three parameters: Δv_0, Δq_0, $\dot{a}$ (or $\Delta\dot{a}$) while the elements e, ω, Ω, i were not corrected.

As observational data we took the instants of old perihelion passages of Comet Halley as deduced by Kiang and published in the paper by Yeomans and Kiang (1981). We also used the orbital elements of the comet given by those authors for epoch near to the perihelion times. Linking consecutively four or five perihelion times from the interval 1986 AD − 87 BC, we obtained values of $\dot{a}$ for 24 moments in the considered interval of two millennia. The moment for an individual value of $\dot{a}$ was calculated as a mean value of the used perihelion times. The values of $\dot{a}$ were obtained during two iterations at least (sometimes three iterations were necessary). The results are given in Table 1.

After graphical presentation of the $\dot{a}$ values from Tab. 1 *versus* the time (Fig. 1), it was evident that they cannot be represented by any mean constant value

Table 1

Values of the nongravitational parameter $\dot{a}$ as obtained by the linkages of four or five consecutive perihelion times of Comet Halley; $\dot{a}(t)$ was approximated by a parabolic function. The first value of $\dot{a}$ (for 1910.76) was computed when linking the last three apparitions of the comet using the positional observations.

No	Observation interval	Middle moment	$10^8\dot{a}$ determ.	Mean error	Weight	$10^8\dot{a}$ adjusted
1	1835 - 1986	1910.765	+6.3822	0.0043	21.03	+6.2293
2	1759 - 1986	1872.872	+6.4864	0.0778	14.68	+6.2869
3	1682 - 1910	1797.023	+6.5035	0.0727	15.70	+6.3891
4	1607 - 1835	1721.405	+6.3586	0.0103	18.93	+6.4733
5	1531 - 1759	1645.357	+5.7047	0.1591	7.18	+6.5403
6	1378 - 1682	1531.519	+5.7343	0.1670	6.84	+6.6074
7	1301 - 1607	1455.346	+6.3448	0.3905	2.92	+6.6301
8	1222 - 1531	1378.337	+7.3128	0.4144	2.76	+6.6350
9	1145 - 1456	1301.068	+6.8704	0.5164	2.21	+6.6216
10	1066 - 1378	1223.025	+6.1383	0.3667	3.11	+6.5894
11	989 - 1301	1145.192	+6.8052	0.4492	2.54	+6.5387
12	837 - 1222	1052.261	+7.1915	0.2305	4.95	+6.4538
13	760 - 1145	959.789	+6.8506	0.1104	10.34	+6.3430
14	684 - 1066	867.677	+5.9845	2.5845	0.44	+6.2066
15	607 - 989	775.081	+3.7464	2.8481	0.40	+6.0432
16	530 - 837	684.081	+4.3012	1.8481	0.62	+5.8570
17	451 - 760	606.944	+2.8825	3.1172	0.37	+5.6792
18	374 - 684	529.693	+3.8696	2.8103	0.41	+5.4829
19	295 - 607	451.804	+5.6872	2.4451	0.47	+5.2664
20	218 - 530	374.040	+5.9184	3.0008	0.38	+5.0317
21	141 - 451	296.137	+4.6268	1.3000	0.88	+4.7779
22	66 - 374	219.056	+6.5921	1.8775	0.61	+4.5085
23	-11 - 295	142.182	+3.4647	1.3639	0.84	+4.2217
24	-86 - 218	66.039	+2.9879	1.1373	1.00	+3.9197

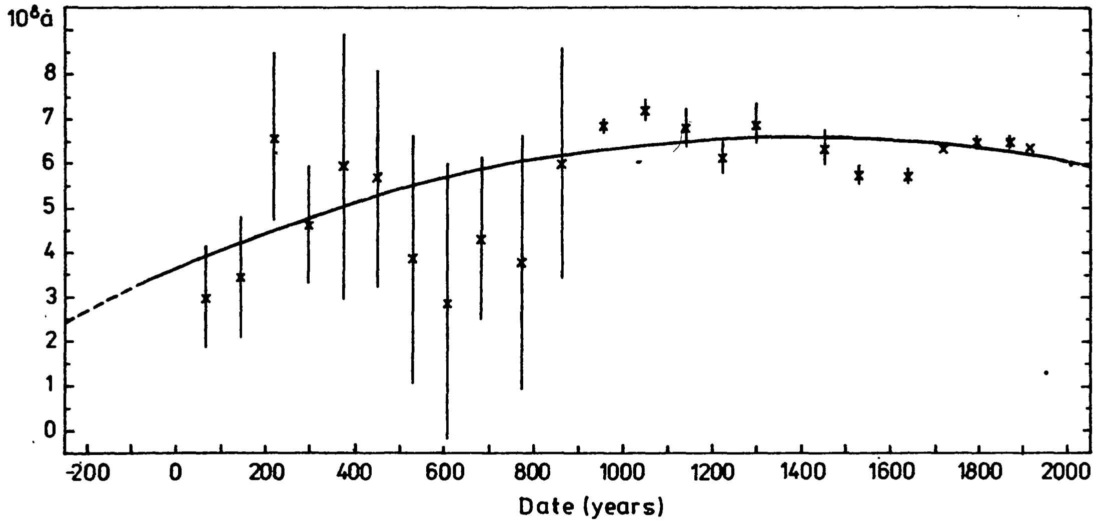

Fig. 1. Nongravitational parameter $\dot{a}$ versus the time for Comet Halley. The crosses denote values of $\dot{a}$ obtained by the linkages of four or five consecutive perihelion times of the comet; the limits of $\dot{a}$ values, determined by their mean errors, are marked. The curve presents the parabolic approximation of $\dot{a}(t)$.

of $\dot{a}$, and that the simplest reasonable regular approximation for $\dot{a}(t)$ would be a parabolic one:

$$\dot{a}(t) = a_0(1+a_1 t+a_2 t^2). \tag{10}$$

Numerical values of parameters of the parabola (10) were computed by the least squares method using the determined values of $\dot{a}$ from Tab. 1 weighted according to their mean errors. The following result was obtained:

$$\begin{aligned} a_0 &= (+6.1470 \pm 0.0376) \times 10^{-8} \text{ a.u./day} \\ a_1 &= (+7.7125 \pm 0.3766) \times 10^{-7} \\ a_2 &= (-1.8714 \pm 0.1001) \times 10^{-12} \end{aligned} \tag{11}$$

In expression (10) t is in days, and $t = 0$ for the epoch: 832 Feb. 25.0 ET = JD 2025000.5.

4.2. *Determination of a_0, a_1, a_2 from observational equations*

The empirical values (11) of a_0, a_1, a_2 may serve as initial data to improve them along with orbital elements on a basis of observational equations. However, the problem is not quite simple. To compose observational equations we are using a great number of very exact positional observations as well as scarcely 24 uncertain perihelion times. Thus the 500 equations based on positional observations strongly overweight the 24 equations from perihelion times. It is clear that the least squares method will tend to adjust the corrected parameters in favour of positional observations, on the other hand, the accurate positional observations should be the main base for orbit correction.

To examine the problem we made the following attempts to improve the orbit and nongravitational parameters (11):

(*a*) We took 250 positional observations from 1835 Aug. 21 – 1986 March 13 and 24 perihelion times from 1835 AD – 12 BC. We accepted the parabolic form (10) for $\dot{a}(t)$, but corrected only the parameter a_0 along with the six orbital elements; we obtained a new a_0 value very close to its initial value given in (11).

(*b*) We used 295 positional observations from 1835 Aug. 21 – 1987 April 20 and 24 perihelion times as in case (*a*), but now improved the nine quantities: six orbital elements and three nongravitational parameters a_0, a_1, a_2.

(*c*) We took only 57 positional observations from the same interval as in case (*b*) and 24 perihelion times, improving again the nine quantities.

(*d*) We used 27 perihelion times alone from 1986 AD – 87 BC to improve four quantities: one orbital element and a_0, a_1, a_2. We accepted orbital elements of the comet for 1986, and during the iterative process of orbit correction we kept the five elements constant except for the perihelion distance q which was corrected together with the nongravitational parameters (11).

Thus we obtained four different parabolic approximations for $\dot{a}(t)$ in respect of the different observational material used for correcting the parameters of motion. These four solutions are given in Table 2 and graphically presented on Fig. 2.

We see that there is a problem how to find the best values of a_0, a_1, a_2 from the observational equations. In the four considered cases we used the observational material from the same two-millennia interval, but it appeared

Table 2

Four parabolic approximations of $\dot{a}(t)$ for Comet Halley obtained in respect of use of different observational material.

Case	(a)	(b)	(c)	(d)
Number of positional obs. used	250	295	57	0
$10^8 a_0$	+6.03 ± 0.00	+5.73 ± 0.07	+5.66 ± 0.10	+5.79 ± 0.25
$10^7 a_1$	+7.71 } const.	+6.99 ± 0.92	+9.19 ± 1.56	+8.99 ± 1.79
$10^{12} a_2$	−1.87 } const.	−1.09 ± 0.16	−1.60 ± 0.29	−2.48 ± 0.77
$10^8 \dot{a}_{max}$	+6.51	+6.38	+6.41	+6.26
Date of $\dot{a}_{max}$	1396	1713	1617	1328

that the result of determination of the shape of parabola $\dot{a}(t)$ depended on the number of the positional observations used for the orbit improvement. An explanation may be that the accurate positional observations as well as the uncertain perihelion times have been treated equivalently in the process of

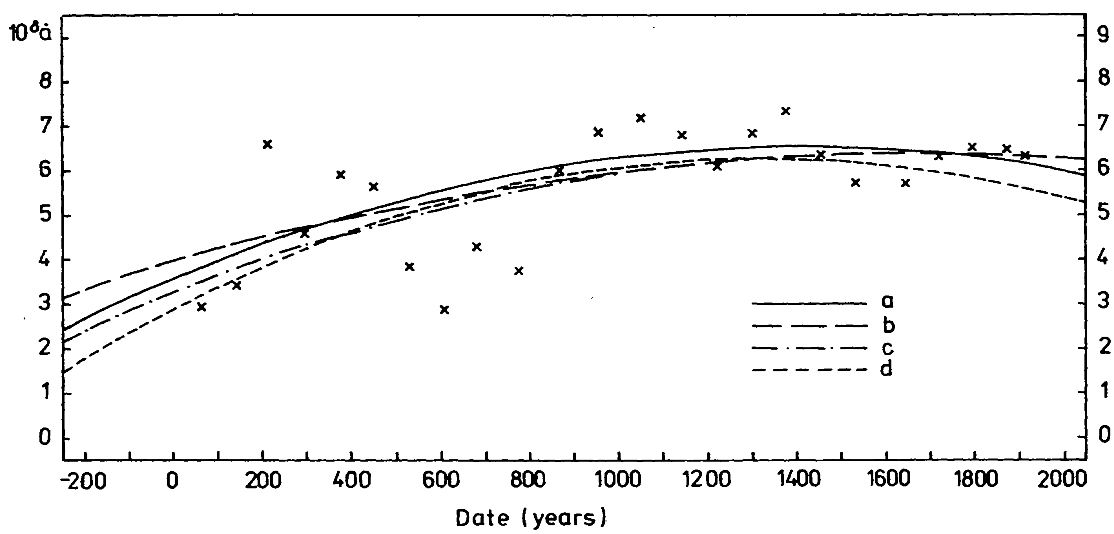

Fig. 2. Four approximations of $\dot{a}(t)$ obtained for Comet Halley: (*a*) the parabola adjusted to the empirical values of $\dot{a}$ marked by crosses; (*b*) the parabola determined using 295 positional observations and 24 perihelion times; (*c*) the parabola determined from 57 positional observations and 24 perihelion times; (*d*) the parabola determined from 27 perihelion times alone.

correcting the dynamical parameters of motion. However, an accuracy of the perihelion times deduced for the ancient returns of the comet is different if compare *e.g.* the first and the second millennium of the comet's motion. It well is visible on Fig. 1 where the exactness of observational material used for determination of the discrete values of $\dot{a}$ is revealed in the marked values of the mean errors of $\dot{a}$. Therefore, the problem of an appropriate weighting of the full observational material of Comet Halley, being not yet solved, requires a careful consideration in the future.

5. Improvement of the Orbit

To improve the orbit of Comet Halley using the full observational material, we accepted a parabolic approximation for the nongravitational effects. We decided to use the parabola $\dot{a}(t)$ as adjusted to the 24 empirical values of $\dot{a}$ and to keep its shape unchanged during the process of orbit improvement, but correcting the parameter a_0 along with orbital elements. An indication for such a decision may be that the values of parameters a_0, a_1, a_2 given by (11) were determined from the *weighted* discrete values of $\dot{a}$, hence in this case the inequivalence of the used observational material in some extent has been taken into account when calculating the values of nongravitational parameters for $\dot{a}(t)$.

We selected the 300 best positional observations of Comet Halley made during 1835 Aug. 21 – 1987 May 1 and used 25 moments of perihelion passages of the comet observed during 1835 AD – 87 BC. We accepted the parabolic form (10) for $\dot{a}(t)$ and took the initial values of a_0, a_1, a_2 given by (11). We corrected the six orbital elements and the parameter a_0 by the least squares method in the iterative process. We obtained the following results:

Epoch of osculation: 1835 Nov. 18.0 ET

$$\begin{array}{ll} T = 1835\text{ Nov. } 16.43614\text{ ET} & \left.\begin{array}{l}\omega = 110^\circ.70267 \\ \Omega = 56.82437 \\ i = 162.27767\end{array}\right\} 1950.0 \\ q = 0.58654655\text{ a.u.} & \\ e = 0.96739929 & \end{array} \tag{12}$$

$$a_0 = (+6.03001 \pm 0.00013) \times 10^{-8}\text{ a.u./day}$$

$$\dot{a}(t) = a_0(1 + a_1 t + a_2 t^2), \quad a_1 = +7.71255 \times 10^{-7}, \quad a_2 = -1.87143 \times 10^{-12},$$

t is counted in days from the epoch: 832 Feb. 25.0 ET = 2025000.5 JD.

The solution (12) was taken as input data for integrating the equations of motion of the comet by the recurrent power series method till 1500 BC. To avoid a dangerous extrapolation of $\dot{a}(t)$ far from the observation interval, the constant value $\dot{a} = +2.48424 \times 10^{-8}$ a.u./day was kept for the integration before 240 BC. The results of integration are given in Table 3.

Table 3

Orbital evolution for Comet Halley with parabolic $\dot{a}(t)$; equinox 1950.0.

No		T (ET)	q (AU)	e	Peri.	Node	Incl.	P (YR)	Epoch	$10^8 \dot{a}$
1		1835 Nov. 16.4361	0.5865465	0.9673993	110°.7027	56°.8244	162°.2777	76.32	1835 Nov. 18	6.22
2		1759 Mar. 12.2056	0.5843956	0.9676945	110.7071	56.5517	162.3911	76.94	1759 Mar. 21	6.31
3		1682 Sep. 14.4790	0.5824815	0.9679420	109.2158	54.8690	162.2834	77.45	1682 Aug. 31	6.39
4		1607 Oct. 24.9954	0.5834349	0.9675151	107.5357	53.0639	162.9221	76.11	1607 Nov. 13	6.44
5		1531 Aug. 23.6782	0.5809488	0.9677753	106.9600	52.3510	162.9339	76.55	1531 Aug. 14	6.48
6		1456 June 8.1029	0.5794097	0.9680218	105.8198	51.1621	162.9066	77.13	1456 June 28	6.50
7		1378 Nov. 9.6458	0.5759048	0.9684015	105.2868	50.3250	163.1265	77.81	1378 Nov. 5	6.51
8		1301 Oct. 25.2171	0.5723528	0.9689600	104.4914	49.4572	163.0895	79.18	1301 Nov. 9	6.50
9		1222 Sep. 29.6804	0.5738187	0.9688740	103.8387	48.6096	163.2053	79.15	1222 Oct. 15	6.46
10		1145 Apr. 20.6005	0.5743573	0.9688207	103.7007	48.3692	163.2365	79.06	1145 Apr. 2	6.41
11		1066 Mar. 22.6833	0.5740165	0.9689032	102.4723	46.9426	163.1254	79.31	1066 Mar. 8	6.35
12		989 Sep. 7.6975	0.5813913	0.9679232	101.4842	45.8826	163.4109	77.16	989 Sep. 28	6.26
13		912 July 19.2823	0.5795828	0.9681151	100.7862	44.9782	163.3227	77.50	912 July 14	6.16
14		837 Feb. 28.3124	0.5817032	0.9678440	100.1111	44.2624	163.4593	76.94	837 Mar. 10	6.04
15		760 May 20.5271	0.5811660	0.9679021	100.0071	44.0203	163.4553	77.04	760 June 2	5.90
16		684 Oct. 2.4741	0.5788646	0.9682027	99.1626	43.1385	163.4302	77.67	684 Sep. 29	5.75
17		607 Mar. 15.0359	0.5800798	0.9680937	98.8111	42.5993	163.4886	77.52	607 Mar. 18	5.57
18		530 Sep. 27.3074	0.5747866	0.9687663	97.5908	41.3113	163.4067	78.95	530 Oct. 8	5.38
19		451 June 27.9566	0.5729007	0.9689719	97.0371	40.5494	163.4915	79.34	451 June 25	5.16
20		374 Feb. 15.3526	0.5763389	0.9686415	96.5191	39.9177	163.5546	78.79	374 Mar. 1	4.94
21		295 Apr. 20.0218	0.5750045	0.9688145	95.2456	38.4471	163.3796	79.17	295 Apr. 25	4.68
22		218 May 17.7621	0.5805269	0.9680371	94.1459	37.2403	163.5865	77.40	218 Apr. 29	4.42
23		141 Mar. 22.5348	0.5821923	0.9679093	93.6960	36.5562	163.4500	77.27	141 Mar. 24	4.14
24		66 Jan. 25.5679	0.5841206	0.9676122	92.6512	35.4638	163.5893	76.59	66 Feb. 6	3.85
25	BC 12	-11 Oct. 8.9174	0.5861971	0.9674366	92.5563	35.2378	163.6016	76.38	-11 Oct. 8	3.53
26	87	-86 Aug. 3.4181	0.5845615	0.9677443	90.7852	33.3633	163.3543	77.15	-86 July 14	3.20
27	164	-163 Oct. 23.1333	0.5834313	0.9677513	89.1315	31.4215	163.7154	76.95	-163 Nov. 15	2.85
28	240	-239 Mar. 22.5472	0.5836792	0.9677973	87.8851	29.9260	163.4266	77.17	-239 Mar. 19	2.48
29	315	-314 Feb. 13.3095	0.5861218	0.9674395	86.5836	28.5637	163.6003	76.37	-314 Feb. 1	2.48
30	392	-391 Dec. 15.2225	0.5868705	0.9673932	86.5000	28.3362	163.5952	76.36	-391 Dec. 22	2.48
31	467	-466 Dec. 2.0055	0.5857330	0.9676782	85.1296	26.9115	163.3809	77.14	-466 Dec. 16	2.48
32	543	-542 Apr. 13.9426	0.5869496	0.9675232	84.0512	25.5855	163.6046	76.83	-542 Apr. 19	2.48
33	620	-619 Oct. 16.1398	0.5883052	0.9674303	82.8292	24.0930	163.2888	76.77	-619 Sep. 30	2.48
34	695	-694 Dec. 4.1096	0.5899868	0.9671957	81.6540	22.8497	163.4142	76.27	-694 Dec. 13	2.48
35	771	-770 Oct. 10.5896	0.5891595	0.9672982	81.4576	22.5065	163.4022	76.47	-770 Sep. 23	2.48
36	846	-845 May 26.6869	0.5874067	0.9676780	80.3074	21.2939	163.1182	77.47	-845 May 20	2.48
37	923	-922 Aug. 6.0734	0.5876585	0.9675636	79.0805	19.8252	163.3060	77.11	-922 Aug. 12	2.48
38	1000	-999 Oct. 13.0393	0.5887733	0.9676367	77.7555	18.2412	162.8303	77.60	-999 Sep. 25	2.48
39	1075	-1074 Jan. 22.1063	0.5925150	0.9671424	75.6830	16.0350	162.9723	76.58	-1074 Jan. 22	2.48
40	1152	-1151 July 9.2522	0.5910164	0.9673359	75.5414	15.7407	162.9157	76.97	-1151 July 5	2.48
41	1227	-1226 Apr. 11.8141	0.5917578	0.9673468	74.6633	14.8243	162.8325	77.15	-1226 Apr. 10	2.48
42	1304	-1303 May 22.6951	0.5929804	0.9672527	74.5469	14.5525	162.8459	77.05	-1303 May 24	2.48
43	1380	-1379 May 1.9432	0.5905408	0.9677534	73.7614	13.7001	162.5878	78.37	-1379 Apr. 13	2.48
44	1458	-1457 June 1.4811	0.5932673	0.9674701	72.8451	12.5741	162.6916	77.88	-1457 June 2	2.48

6. Discussion and Conclusions

We improved the orbit of Comet Halley including the nongravitational effects in the comet's motion as approximated by the parabolic function for $\dot{a}(t)$. We obtained a new solution for the long-term nongravitational motion of Comet Halley; it was possible due to the new approach to the procedure of orbit improvement based on the observational material collected during 30 returns of the comet. We got one system of dynamical parameters of motion well representing all the observed perihelion times of the comet.

The linkage of all the observed apparitions of Comet Halley was not a simple process from the numerical point of view. It should be emphasized that the numerical integration of the variation equation (3) with the substitution (4) is the only way to obtain good values of differential coefficients in observational equations. The old method of varying the appropriate parameters and integrating several times the equations of motion certainly would

Table 4

Values of partial derivatives for the perihelion times of Comet Halley (an exponent form is used, *e.g.* $-1.2\,E-2$ means -1.2×10^{-2}).

Date	$\partial M/\partial x$	$\partial M/\partial y$	$\partial M/\partial z$	$\partial M/\partial \dot{x}$	$\partial M/\partial \dot{y}$	$\partial M/\partial \dot{z}$	$\partial M/\partial a_0$	$\partial M/\partial a_1$	$\partial M/\partial a_2$
1835	-1.2E-2	-6.3E-3	-4.8E-3	+1.7E-1	-2.1E-1	-5.8E-3	+2.7E-5	+5.1E-7	+1.9E-1
1759	+4.8E+2	-8.6E+2	-7.7E+1	-3.0E+4	-1.7E+4	-1.2E+4	-8.1E+3	-1.5E+2	-5.6E+7
1682	+4.7E+2	-8.4E+2	-7.5E+1	-2.9E+4	-1.7E+4	-1.2E+4	-2.4E+4	-4.3E+2	-1.5E+8
1607	+2.0E+3	-3.5E+3	-3.2E+2	-1.2E+5	-7.0E+4	-5.0E+4	-1.3E+5	-2.3E+3	-8.0E+8
1531	+2.2E+3	-3.9E+3	-3.5E+2	-1.3E+5	-7.7E+4	-5.5E+4	-1.7E+5	-2.9E+3	-9.8E+8
1456	+1.5E+3	-2.8E+3	-2.5E+2	-9.5E+4	-5.5E+4	-3.9E+4	-1.6E+5	-2.6E+3	-8.5E+8
1378	+9.5E+2	-1.7E+3	-1.5E+2	-5.9E+4	-3.4E+4	-2.4E+4	-1.7E+5	-2.6E+3	-7.8E+8
1301	+2.9E+2	-5.3E+2	-4.9E+1	-1.8E+4	-1.0E+4	-7.4E+3	-1.8E+5	-2.5E+3	-6.9E+8
1222	-7.9E+2	+1.4E+3	+1.3E+2	+4.9E+4	+2.8E+4	+2.0E+4	+4.2E+4	+9.1E+2	+3.4E+8
1145	-1.6E+3	+2.9E+3	+2.6E+2	+9.9E+4	+5.7E+4	+4.1E+4	+2.3E+5	+3.8E+3	+1.2E+9
1066	-1.4E+3	+2.6E+3	+2.3E+2	+8.8E+4	+5.1E+4	+3.6E+4	+2.6E+5	+4.3E+3	+1.3E+9
989	-8.7E+2	+1.6E+3	+1.4E+2	+5.4E+4	+3.1E+4	+2.2E+4	+1.9E+5	+3.3E+3	+1.0E+9
837	+1.4E+3	-2.6E+3	-2.3E+2	-8.9E+4	-5.1E+4	-3.7E+4	-2.7E+5	-3.9E+3	-1.2E+9
760	-1.4E+4	+2.5E+4	+2.3E+3	+8.8E+5	+5.0E+5	+3.6E+5	+2.7E+6	+3.8E+4	+1.2E+10
684	-4.3E+4	+7.7E+4	+6.9E+3	+2.7E+6	+1.5E+6	+1.1E+6	+8.0E+6	+1.2E+5	+3.6E+10
607	-2.3E+4	+4.2E+4	+3.7E+3	+1.4E+6	+8.2E+5	+5.9E+5	+4.6E+6	+6.6E+4	+2.0E+10
530	+9.7E+3	-1.8E+4	-1.6E+3	-6.0E+5	-3.5E+5	-2.5E+5	-2.8E+6	-3.9E+4	-1.2E+10
451	+3.6E+4	-6.4E+4	-5.7E+3	-2.2E+6	-1.3E+6	-9.0E+5	-8.1E+6	-1.2E+5	-3.6E+10
374	+4.0E+4	-7.3E+4	-6.5E+3	-2.5E+6	-1.4E+6	-1.0E+6	-8.7E+6	-1.2E+5	-3.9E+10
295	-5.6E+4	+1.0E+5	+9.0E+3	+3.5E+6	+2.0E+6	+1.4E+6	+1.1E+7	+1.5E+5	+4.7E+10
218	-2.3E+5	+4.1E+5	+3.7E+4	+1.4E+7	+8.1E+6	+5.8E+6	+4.4E+7	+6.3E+5	+2.0E+11
141	-3.8E+5	+6.9E+5	+6.2E+4	+2.4E+7	+1.4E+7	+9.7E+6	+7.3E+7	+1.0E+6	+3.2E+11
66	-5.0E+5	+9.0E+5	+8.0E+4	+3.1E+7	+1.8E+7	+1.3E+7	+8.7E+7	+1.3E+6	+3.9E+11
-11	-2.1E+5	+3.8E+5	+3.4E+4	+1.3E+7	+7.5E+6	+5.3E+6	+3.7E+7	+5.3E+5	+1.6E+11

not suffice and the iterative process of orbit improvement would become divergent. In Tab. 4 are presented the numerical values of partial derivatives for the instants of perihelion passages of the comet to show their enormous values reached for the moments far from the initial epoch of integration. However, the iterative process of orbit improvement appeared to be convergent although it required many iterations.

Tables 5 and 6 contain the comparison of our results with those by other authors. The data in Table 6 show that our solution is much closer to the Landgraf's results than to the Yeomans' and Kiang's ones; it also is shown graphically on Fig. 3. Our earlier conclusion is confirmed that the secular change of the nongravitational effects, included in the process of orbit improvement, may considerably influence the results of prediction of the past perihelion times of the comet (Sitarski and Ziołkowski 1986). It is visible in Table 6 if we compare our two solutions, with constant $\dot{a}$ and with parabolic $\dot{a}(t)$; the differences between both solutions are shown on Fig. 3 where they are marked by crosses since our results with constant $\dot{a}$ in fact coincide with those by Yeomans and Kiang.

It is rather unreal to find a secular change in nongravitational parameters (and then to extrapolate it over two millennia!) only in a basis of the positional

Table 5

Representation of 30 observed perihelion times of Comet Halley obtained by different authors. The data given in brackets were not used in the process of orbit improvement.

No	$T_{observed}$	Yeomans and Kiang	Landgraf	Sitarski with $\dot{a}$ constant	Sitarski with $\dot{a}(t)$ parabolic
		$T_{obs} - T_{cal}$	$T_{obs} - T_{cal}$	$T_{obs} - T_{cal}$	$T_{obs} - T_{cal}$
	d	d	d	d	d
1	1986 Feb. 9.46	–	- 0.05	0.00	0.00
2	1910 Apr. 20.18	0.00	0.00	0.00	0.00
3	1835 Nov. 16.44	0.00	0.00	0.00	0.00
4	1759 Mar. 13.05	- 0.01	- 0.01	+ 0.23	+ 0.84
5	1682 Sep. 15.27	- 0.01	- 0.01	+ 0.25	+ 0.79
6	1607 Oct. 27.56	+ 0.02	+ 0.04	+ 1.10	+ 2.56
7	1531 Aug. 25.8	- 0.44	- 0.46	+ 0.73	+ 2.12
8	1456 June 9.1	- 0.53	- 0.40	+ 0.31	+ 1.00
9	1378 Nov. 9.02	- 1.67	- 1.60	- 1.39	- 0.63
10	1301 Oct. 24.53	- 1.05	- 0.66	- 1.23	- 0.69
11	1222 Sep. 30.8	+ 1.98	+ 2.25	+ 1.41	+ 1.12
12	1145 Apr. 21.25	+ 2.69	+ 3.13	+ 1.82	+ 0.65
13	1066 Mar. 23.5	+ 2.57	+ 3.43	+ 2.24	+ 0.82
14	989 Sep. 9.0	+ 3.31	+ 4.91	+ 3.43	+ 1.30
15	912 July 9.5	- 9.17	- 7.50	(- 8.59)	(- 9.78)
16	837 Feb. 28.27	–	- 0.15	- 0.14	- 0.04
17	760 May 22.5	+ 1.83	+ 1.88	+ 1.56	+ 1.97
18	684 Sep. 28.5	- 4.27	- 2.93	- 5.29	- 3.97
19	607 Mar. 13.0	- 2.48	- 0.57	- 3.41	- 2.04
20	530 Sep. 26.7	- 0.43	+ 1.08	- 1.54	- 0.61
21	451 June 24.5	- 3.75	- 2.73	- 4.26	- 3.46
22	374 Feb. 17.4	+ 1.06	+ 2.11	+ 1.19	+ 2.05
23	295 Apr. 20.5	+ 0.10	- 0.13	+ 0.69	+ 0.49
24	218 May 17.5	- 0.22	- 0.21	+ 0.78	- 0.26
25	141 Mar. 22.35	- 0.08	+ 1.27	+ 0.45	- 0.18
26	66 Jan. 26.5	+ 0.54	+ 4.60	+ 0.36	+ 0.93
27	-11 Oct. 5.5	- 5.35	- 0.50	- 6.37	- 3.42
28	-86 Aug. 2.5	- 3.96	- 1.04	(- 5.25)	- 0.92
29	-163 Nov. 17(?)	+ 4.43	+17.89	(+ 0.05)	(+24.87)
30	-239 Mar. 30.5	-55.62	-17.02	(-56.64)	(+ 7.95)

observations made during the few last apparitions of the comet. Although Landgraf (1984) found a linear term in A_2 by some numerical speculations, however, his next paper (Landgraf 1986) revised the earlier results and presented the new solution which as a matter of fact confirmed the results by Yeomans and Kiang (1981) as got on the assumption that A_1 and A_2 were constant with time. The latter assumption was justified if intend to find a preliminary orbit well representing all the observed apparitions. However, it would be highly improbable that the activity of the comet could be stable during its 30 revolutions around the Sun, and it was very risky to make conclusions on the assumption that the nongravitational forces remained constant over two millennia (*e.g.* Yeomans 1984, 1986).

Taking into consideration our recent results, there is rather no doubt that

Table 6

Prediction of 20 perihelion times of Comet Halley for the years BC according to the different authors.

No	Yeomans and Kiang	Landgraf	Sitarski with $\dot{a}$ constant	Sitarski with $\dot{a}(t)$ parabolic
1	12.78	12.77	12.78	12.77
2	87.60	87.59	87.60	87.59
3	164.87	164.83	164.87	164.81
4	240.40	240.29	240.40	240.22
5	315.69	315.37	315.68	315.12
6	391.70	391.32	391.69	392.95
7	466.55	466.28	466.54	467.92
8	540.36	542.96	540.31	543.28
9	616.57	618.72	616.52	620.79
10	690.06	692.02	690.03	695.92
11	763.59	769.09	763.60	771.77
12	836.35	846.38	836.41	846.40
13	911.38	924.14	911.46	923.59
14	986.92	1002.78	986.98	1000.78
15	1059.92	1082.97	1059.93	1075.06
16	1129.25	1159.45	1129.23	1152.52
17	1198.36	1237.27	1198.32	1227.28
18	1266.68	1316.32	1266.64	1304.39
19	1334.65	1394.78	1334.62	1380.33
20	1404.79	1473.18	1404.79	1458.41

the nongravitational parameters of motion of Comet Halley are not constant with time and that we are able to detect the variations of nongravitational effects by investigation of the orbital motion of the comet. It must be said that we did *not* assume any variability of the nongravitational parameter $\dot{a}$: the changes of $\dot{a}$ with time were found in applying our new method to study the nongravitational motion of the comet. We admited the parabolic form for $\dot{a}(t)$, but it was some kind of averaging the detected irregular variations of $\dot{a}$ over two millennia by means of the simplest continuous function of time. Then the

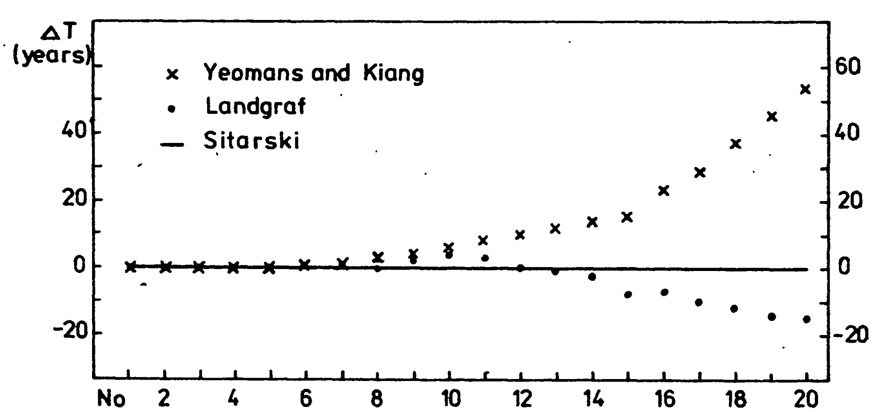

Fig. 3. Graphical presentation of differences ΔT, referred to the Sitarski's solution, for the 20 predicted perihelion times T of Comet Halley for the years BC obtained by different authors.

three constant parameters of the parabola $\dot{a}(t)$ have been determined by the leat-squares method basing on the 24 empirical values of $\dot{a}$.

Weissman (1987) writes with reference to the Yeomans' and Kiang's work: "The derived value for the transverse component of the nongravitational force on the Halley orbit has been remarkably constant over the past 30 apparitions since 240 BC; the radial component of the nongravitational force has varied somewhat, but is not well determined. This suggests a constant rotation rate and rotation pole orientation, and highly repeatable outgassing. This is in sharp contrast to the complex rotational models that have now been suggested for Halley ... It appeares that this paradox will not be resolved very soon". However, the present work shows that investigations of the orbital motion of Comet Halley allow to find variations of nongravitational effects. Analyzing the data presented on Fig. 1 we may conclude that the fluctuations of $\dot{a}$ found for the last millennium of the comet's motion presumably represent some real variations, however, it is too early to speculate for interpretation of these variations.

Investigations of the nongravitational motion of Comet Halley were just started and some problems should be resolved before the work on the comet's motion would be continued. The positional observations from 1759 and 1682 must be used, but a number of normal places should be created to use reasonably thousands of positional observations made during the last five apparitions of the comet. A reexamination of the old perihelion times would be desirable since there is a controversy between Kiang (1972) and Hasegawa (1979) in interpretation of some ancient records concerning the old observations of Comet Halley. It is necessary to elaborate a mathematical method for weighting the full observational material of the comet to use it in a homogeneous way for the orbit improvement. Then we shall be ready to undertake a new attempt to link all the apparitions of Comet Halley by one system of orbital elements.

The paper by Yeomans and Kiang (1981) is commonly known as presenting a successful result of modelling the long-term motion of Comet Halley. The predicted perihelion times of the comet before 240 BC sometimes are quoted after those authors (Freitag 1984, Weissman 1986). Of course, Yeomans and Kiang could not suppose that small secular changes in nongravitational forces would influence the predicted perihelion times so considerably and that their model of Comet Halley's motion for the long past should be revised. Nevertheless, the work by Yeomans and Kiang has a historical importance: indeed it was not a final word concerning the studies of motion of Comet Halley, but it was the first remarkable step starting the investigations of the nongravitational long-term motion of the famous comet.

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