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On spinor field equations of BUCHDAHL and WÜNSCH

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The field equations of BUCHDAHL and WÜNSCH for the description of fields of particles with nonzero rest mass and arbitrary spin in curved space-times are considered. The Lagrangian, the field equations and the energy-momentum tensor for fields of integer spin are formulated by means of real tensor fields.

Es werden die Feldgleichungen von BUCHDAHL und WÜNSCH für die Beschreibung von Partikeln mit Ruhemassen ungleich Null und beliebigem Spin in gekrümmter Raum-Zeit untersucht. Die Lagrange-, die Feldgleichungen und der Energie-Impuls-Tensor für Felder mit ganzzahligem Spin werden mittels eines realen Tensorfeldes formuliert.

Key words: Relativity — spinor field equations

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1. Introduction

We begin our considerations with the following spinor field equations in Minkowski space-time:

$$\begin{aligned} &\partial^{D}_{\dot{X}_0}\varphi_{DA_1\ldots A_n\dot{X}_1\ldots\dot{X}_k} + \mu\chi_{A_1\ldots A_n\dot{X}_0\dot{X}_1\ldots\dot{X}_k} = 0\,, \\ &\partial^{\dot{Z}}_{A_0}\chi_{A_1\ldots A_n\dot{Z}\dot{X}_1\ldots\dot{X}_k} - \mu\varphi_{A_0A_1\ldots A_n\dot{X}_1\ldots\dot{X}_k} = 0\,. \end{aligned} \tag{1}$$

φ and χ are the field spinors which have to be symmetric in their dotted and undotted indices. The spin s and the rest mass m are defined by

$$s = \frac{k+n+1}{2} \qquad (k, n = 0, 1, 2, \ldots)\,, \qquad m = -i\mu\sqrt{2} \qquad (m \neq 0)\,, \tag{2}$$

respectively. If we note these definitions then the system (1) describes fields of particles with spin $s \geqq 1/2$ and nonvanishing rest mass m in Minkowski space-time, i.e. if we have not interactions with other fields (DIRAC 1936).

The situation becomes more difficult if there are interactions. If we have additionally an electromagnetic field, then we must replace the derivatives ∂_l in (1) by $\partial_l - ieA_l$, where A and e are the electromagnetic potential and the charge, respectively. But this replacement leads to inconsistent equations if the spin is greater than one (FIERZ and PAULI 1939). If we consider the system (1) in a curved space-time and replace the partial by the covariant derivatives (minimal gravitational coupling) then we get in general inconsistent equations for $s > 1$, too (BUCHDAHL 1958, 1962; WÜNSCH 1978).

From these results the question arises, if it is possible to modify the system (1) in the case of interactions in order to obtain consistent equations for arbitrary spin. Because we are mostly interested in gravitational interactions in the following we consider the spinor field equations in a Riemannian space-time (M, g).

One possible answer to our question was given by BUCHDAHL (1982). He suggested the following first-order system

$$\begin{aligned} &\nabla^{D}_{\dot{X}}\varphi_{DA_1\ldots A_n} + \mu\chi_{A_1\ldots A_n\dot{X}} = 0\,, \\ &\nabla^{\dot{Z}}_{A}\chi_{A_1\ldots A_n\dot{Z}} - \mu\varphi_{AA_1\ldots A_n} = \frac{n(n-1)}{\mu(n+1)}\,\varepsilon_{A(A_1}\psi^{EFG}_{\ldots A_2}\varphi_{|EFG|A_3\ldots A_n)}\,, \end{aligned} \tag{3}$$

where ε and ψ denote the Levi-Civita and the Weyl spinor, respectively. BUCHDAHL showed that the system (3) is consistent in arbitrarily curved space-times. Furthermore, it reduces to (1) for $k = 0$ if $n = 0$ or $n = 1$ (i.e. in the case of Dirac's or Proca's equation) or if the underlying space-time is conformally flat. But the gravitational field seems to be nonminimally coupled in the first-order system (3) since this contains the Weyl spinor explicitly.

WÜNSCH (1985) presented the system

$$\begin{aligned} &\nabla^{D}_{\dot{X}}\varphi_{DA_1\ldots A_n} + \mu\chi_{A_1\ldots A_n\dot{X}} = 0\,, \\ &\nabla^{\dot{Z}}_{(A}\chi_{A_1\ldots A_n)\dot{Z}} - \mu\varphi_{AA_1\ldots A_n} = 0 \end{aligned} \tag{4}$$

and proved, that this is equivalent to the system of BUCHDAHL (3). Because (4) does not contain any curvature spinor explicitly it is minimally coupled to gravitation. Furthermore, the existence and uniqueness of the solution of Cauchy's problem are proved in WÜNSCH (1985). This paper also contains the remark that the form (4) of the field equations allows the coupling to an electromagnetic field by $\nabla_l \to \nabla_l - ieA_l$ and that the arising system is consistent, too (see also ILLGE 1988a).

These results show that (4) is a likely system to describe fields of particles with nonzero rest mass and nonzero spin in an arbitrarily curved space-time.

This paper especially deals with the system (4) for bosonic fields. In this case the field equations are the Euler-Lagrange equations of an action problem (ILLGE 1986). Because of the integer spin the Lagrangian and the equations (4) can be formulated by means of tensor fields. The tensor fields belonging to the spinor fields φ and χ are complex fields (ILLGE 1987), but an exact analysis shows that it is possible to restrict oneself to their real parts. In this way we obtain the Lagrangian, the field equations and the energy-momentum tensor for fields of integer spin, formulated by means of real tensor fields.

2. Definitions and former results

2.1. Definitions and notations

Let (M, g) be a four-dimensional Riemannian space-time of class C^∞ whose metric g has the signature $(+---)$. All investigations are of local nature; therefore we restrict M to a suitable coordinate neighbourhood. The signs of the curvature tensor and of the Ricci tensor are determined by the Ricci identity $\nabla_{[i}\nabla_{j]}T_k = -\frac{1}{2} R_{ijkl}T_l$ and $R_{il} := g^{jk}R_{ijkl}$, respectively. In the following we use the two-component spinor calculus (see e.g. PENROSE and RINDLER 1984). For any point $x \in M$ let $S(x)$ denote the spinor space at x. We choose a fix coordinate system $\{\eta^A | A = 1, 2\}$ in $S(x)$ and denote the coordinates of a spinor φ of type (n, k) by

$$\varphi_{A_1\ldots A_n\dot{X}_1\ldots\dot{X}_k}, \qquad A_1, \ldots, A_n \in \{1, 2\}, \qquad \dot{X}_1, \ldots, \dot{X}_k \in \{\dot{1}, \dot{2}\}.$$

Further let $S_{n,k}$ denote the set of all spinor fields of type (n, k) and of class C^∞ on M and $\mathscr{S}_{n,k} \subseteqq S_{n,k}$ the set of the symmetric spinor fields of type (n, k).

There is a 1:1-correspondence between (complex) tensors of rank n and spinors of type (n, n) arranged by the connecting quantities $\sigma^i{}_{A\dot{X}}$. We denote this correspondence symbolically by "$\leftrightarrow$". If $X \leftrightarrow \chi$ then χ is called the spinor equivalent of X. Moreover, we use the notations of (WÜNSCH 1978, 1985; PENROSE and RINDLER 1984).

2.2. Lagrangian and energy-momentum tensor for fields of integer spin

In the case of integer spin the field equations (4) are derivable by means of an action principle. Let $\Omega \subseteqq M$ be a domain with a sufficiently smooth boundary and the action I defined by

$$I = \int_\Omega L \,\mathrm{d}V,$$

where the Lagrangian density L is the real part of

$$\overset{c}{L} = a\chi^{A_2\ldots A_{2s}\dot{X}}\,\nabla^{A_1}_{\dot{X}}\,\varphi_{A_1\ldots A_{2s}} + b\varphi^{A_1\ldots A_{2s}}\,\nabla^{\dot{X}}_{A_1}\chi_{A_2\ldots A_{2s}\dot{X}} + \frac{1}{2}\,\mu(a+b)\,(\chi^{A_2\ldots A_{2s}\dot{X}}\chi_{A_2\ldots A_{2s}\dot{X}} - \varphi^{A_1\ldots A_{2s}}\varphi_{A_1\ldots A_{2s}}) \tag{5}$$

with $a = \text{const.}$, $b = \text{const.}$, $a + b \neq 0$, $s = 1, 2, \ldots$, $(n = 2s - 1)$. Then (4) are the Euler-Lagrange equations of the variational problems $\delta I/\delta\chi = 0$ and $\delta I/\delta\varphi = 0$ (ILLGE 1986).

The knowledge of the Lagrangian to the system (4) makes it possible to compute the energy-momentum tensor by means of the well-known formula (see e.g. SCHMUTZER 1968)

$$T_{kl} = \frac{2}{\sqrt{|g|}}\frac{\delta(L\sqrt{|g|})}{\delta g^{kl}}.$$

The energy-momentum tensor T_{kl} for bosonic fields described by the Lagrangian density (5) reads as the real part of the tensor equivalent of the spinor

$$\begin{aligned}\overset{c}{T}_{K\dot{K}L\dot{L}} = (a+b)\Big\{ & s\varphi_{KLA_3\ldots A_{2s}}\,\nabla_{A_2(\dot{K}}\chi^{A_2\ldots A_{2s}}{}_{\dot{L})} + (s-1)\,\chi^{A_2\ldots A_{2s}}{}_{(\dot{K}}\nabla_{\dot{L})A_2}\varphi_{KLA_3\ldots A_{2s}} - \\ & - \frac{s-1}{2}\,(\varphi_{KA_2\ldots A_{2s}}\,\nabla^{A_2}{}_{\dot{L}}\chi_L{}^{A_3\ldots A_{2s}}{}_{\dot{K}} + \varphi_{LA_2\ldots A_{2s}}\,\nabla^{A_2}{}_{\dot{K}}\chi_K{}^{A_3\ldots A_{2s}}{}_{\dot{L}}) + \\ & + s\mu\left(\chi_{A_3\ldots A_{2s}K\dot{K}}\chi^{A_3\ldots A_{2s}}{}_{L\dot{L}} - \frac{1}{2}\,\varepsilon_{KL}\varepsilon_{\dot{K}\dot{L}}\chi_{A_2\ldots A_{2s}\dot{X}}\chi^{A_2\ldots A_{2s}\dot{X}}\right)\Big\}\end{aligned} \tag{6}$$

(ILLGE 1987).

3. Description of bosonic fields by means of real tensor fields

3.1. The special case $s = 1$ (Proca's equation)

From (4) one obtains Proca's equation by setting $n = 1$:

$$\nabla^{D}_{\dot{X}}\varphi_{DA} + \mu\chi_{A\dot{X}} = 0\,, \qquad \nabla^{\dot{Z}}_{(A}\chi_{B)\dot{Z}} - \mu\varphi_{AB} = 0\,, \tag{7}$$

where the spinor field χ has to be hermitean (WÜNSCH 1978). The symmetrization in the second equation of (7) can be omitted because the first equation implies the vanishing of the divergence of χ.

We define tensor fields Φ and A by

$$\Phi_{kl} \leftrightarrow \varphi_{AB}\varepsilon_{\dot{X}\dot{Y}} + \bar{\varphi}_{\dot{X}\dot{Y}}\varepsilon_{AB}\,, \qquad A_k \leftrightarrow \chi_{A\dot{X}}\,.$$

Φ is a real, antisymmetric tensor of second rank (a bivector), whilst A is a real vector because of the hermitecity of χ. Elementary calculations show that the system (7) is equivalent to

$$\frac{1}{2}\nabla^i\Phi^*_{ik} + \frac{m}{\sqrt{2}}A_k = 0\,, \qquad -\nabla_{[k}A_{l]} + \frac{m}{2\sqrt{2}}\Phi^*_{kl} = 0\,, \tag{8}$$

where Φ^* denotes the dual bivector of Φ (see SCHMUTZER 1968) and m is the rest mass (cf. (2)). By defining $F_{kl} := \frac{m}{\sqrt{2}}\Phi^*_{kl}$ we obtain Proca's equation in the usual form

$$F_{kl} = 2\nabla_{[k}A_{l]}\,, \qquad \nabla^i F_{ik} + m^2 A_k = 0\,. \tag{9}$$

The Lagrangian density (5) reads in tensor form

$$L = -\tilde{a}A^j\,\nabla^i F_{ij} + \tilde{b}F^{ij}\,\nabla_i A_j - \frac{1}{2}(\tilde{a} + \tilde{b})\left(\frac{1}{2}F^{ij}F_{ij} + m^2 A^i A_i\right); \tag{10}$$

the real constants $\tilde{a}$ and $\tilde{b}$ have to satisfy the condition $\tilde{a} + \tilde{b} \neq 0$. If one takes the field equations (9) into account then the Lagrangian density (10) reads

$$L = \frac{\tilde{b} - \tilde{a}}{2}\left(\frac{1}{2}F^{ij}F_{ij} - m^2 A^i A_i\right). \tag{10a}$$

Now one can determine the energy-momentum tensor in terms of the fields F_{ij} and A_i from (10) by variation of the metric or from (6) by computation of the tensor equivalents. The result is

$$T_{kl} = (\tilde{a} + \tilde{b})\left\{F_k{}^i F_{il} + \frac{1}{4}g_{kl}F^{ij}F_{ij} - m^2\left[A_k A_l - \frac{1}{2}g_{kl}A^i A_i\right]\right\}. \tag{11}$$

Most of these formulas are already known from the literature (CHRISTENSEN 1978).

3.2. Definition of suitable tensor fields for $s = 2, 3, \ldots$

The tensorial description of the fields for $s = 2, 3, \ldots$ requires some preliminary definitions and lemmata.

Definition 1: Let $\varphi \in \mathscr{S}_{2s,0}$ be any symmetric spinor. The tensor Φ defined by

$$\Phi_{k_1\ldots k_{2s}} \leftrightarrow \varphi_{A_1\ldots A_{2s}}\varepsilon_{\dot{X}_1\dot{X}_2}\cdots\varepsilon_{\dot{X}_{2s-1}\dot{X}_{2s}} + \bar{\varphi}_{\dot{X}_1\ldots\dot{X}_{2s}}\varepsilon_{A_1A_2}\cdots\varepsilon_{A_{2s-1}A_{2s}}$$

is called a bivector of rank s. Further let χ be any member of the set $\mathscr{S}_{2s-1,1}$. The tensor X defined by

$$\chi_{lk_3\ldots k_{2s}} \leftrightarrow \chi_{A_3\ldots A_{2s}B\dot{Y}}\varepsilon_{\dot{X}_3\dot{X}_4}\cdots\varepsilon_{\dot{X}_{2s-1}\dot{X}_{2s}} + \bar{\chi}_{\dot{X}_3\ldots\dot{X}_{2s}\dot{Y}B}\varepsilon_{A_3A_4}\cdots\varepsilon_{A_{2s-1}A_{2s}}$$

is called a vector-bivector of rank s.

In the paper (ILLGE 1988b) it was shown:

Lemma 1: A real tensor Φ of rank $2s$ is a bivector of rank s if and only if it has the following symmetries:

i) $\Phi_{k_1\ldots k_{2\nu-2}(k_{2\nu-1}k_{2\nu})k_{2\nu+1}\ldots k_{2s}} = 0\,,$

ii) $\Phi_{k_1\ldots k_{2\nu-2}k_{2\varrho-1}k_{2\varrho}k_{2\nu+1}\ldots k_{2\varrho-2}k_{2\nu-1}k_{2\nu}k_{2\varrho+1}\ldots k_{2s}} = \Phi_{k_1\ldots k_{2s}}$

for all $\nu, \varrho \in \{1, \ldots, s\}$,

iii) ${}^*\Phi^*_{k_1\ldots k_{2s}} \equiv \frac{1}{4}e_{k_1k_2}{}^{l_1l_2}e_{k_3k_4}{}^{l_3l_4}\Phi_{l_1l_2l_3l_4k_5\ldots k_{2s}} = -\Phi_{k_1\ldots k_{2s}}\,,$

iv) $g^{k_1k_3}g^{k_2k_4}\Phi_{k_1\ldots k_{2s}} = g^{k_1k_3}g^{k_2k_4}\Phi^*_{k_1\ldots k_{2s}} = 0\,.$

A real tensor X of rank $2s - 1$ is a vector-bivector of rank s if and only if it has the following symmetries:

v) The properties i) ... iv) with respect to the last $2s - 2$ indices,

vi) $g^{lk_3} X_{lk_3 \ldots k_{2s}} = g^{lk_3} X^*_{lk_3 \ldots k_{2s}} = 0$.

Definition 2: Let Φ be any bivector field of rank s and X any vector-bivector field of rank s. Then we define the derivatives $\delta\Phi$ and $\mathrm{d}X$ respectively by

$$(\delta\Phi)_{lk_3 \ldots k_{2s}} := \nabla^i \Phi_{ilk_3 \ldots k_{2s}},$$

$$(\mathrm{d}X)_{k_1 \ldots k_{2s}} := -\frac{1}{s} \sum_{v=1}^{s} (\nabla_{[k_{2v-1}} X_{k_{2v}]k_1 \ldots \hat{k}_{2v-1} \hat{k}_{2v} \ldots k_{2s}} - \nabla_{[k_{2v-1}} {}^* X^*_{k_{2v}]k_1 \ldots \hat{k}_{2v-1} \hat{k}_{2v} \ldots k_{2s}}).$$

The symbols $\hat{k}_{2v-1} \hat{k}_{2v}$ denote that these indices are to be omitted.

It is easy to verify that the derivative $\delta\Phi$ of a bivector of rank s is a vector-bivector of rank s. Vice versa the derivative $\mathrm{d}X$ of a vector-bivector of rank s is a bivector of rank s. Moreover, the operator $-\frac{1}{2} d$ is the (formal) adjoint of the operator δ (ILLGE 1988b). In the same paper we have shown:

Lemma 2: Let Φ be some bivector field of rank s and X some vector-bivector field of rank s. Further let φ and χ denote the spinor fields belonging to Φ and X according to definition 1. Then we have

$$(\delta\Phi)_{lk_3 \ldots k_{2s}} \leftrightarrow \nabla^D_{\dot{Y}} \varphi_{DBA_3 \ldots A_{2s}} \varepsilon_{\dot{X}_3 \dot{X}_4} \ldots \varepsilon_{\dot{X}_{2s-1} \dot{X}_{2s}} + \nabla^{\dot{Z}}_B \bar{\varphi}_{\dot{Z}\dot{Y}\dot{X}_3 \ldots \dot{X}_{2s}} \varepsilon_{A_3 A_4} \ldots \varepsilon_{A_{2s-1} A_{2s}},$$

$$(\mathrm{d}X)_{k_1 \ldots k_{2s}} \leftrightarrow \nabla^{\dot{Z}}_{(A_1} \chi_{A_2 \ldots A_{2s})\dot{Z}} \varepsilon_{\dot{X}_1 \dot{X}_2} \ldots \varepsilon_{\dot{X}_{2s-1} \dot{X}_{2s}} + \nabla^D_{(\dot{X}_1} \bar{\chi}_{\dot{X}_2 \ldots \dot{X}_{2s})D} \varepsilon_{A_1 A_2} \ldots \varepsilon_{A_{2s-1} A_{2s}}.$$

3.3. Tensorial description of the field for $s = 2, 3, \ldots$

Let us return to the field equations (4). Let Φ and X denote the bivector of rank s and vector-bivector of rank s belonging to the spinor fields φ and χ according to definition 1. Using lemma 2 it is easy to show that the system (4) for $n = 2s - 1$ ($s = 2, 3, \ldots$) is equivalent to the system of tensor equations

$$\begin{aligned} &(\delta\Phi)_{lk_3 \ldots k_{2s}} - \frac{m}{\sqrt{2}} X^*_{lk_3 \ldots k_{2s}} = 0, \\ &(\mathrm{d}X)_{k_1 \ldots k_{2s}} + \frac{m}{\sqrt{2}} \Phi^*_{k_1 \ldots k_{2s}} = 0. \end{aligned} \tag{12}$$

Using the formulas of section 3.2. one obtains the Lagrangian density immediately from (5):

$$L = \tilde{a}(X, \delta\Phi) + \frac{\tilde{b}}{2} (\Phi, \mathrm{d}X) + \frac{m}{2\sqrt{2}} (\tilde{a} + \tilde{b}) \left(\frac{1}{2} (\Phi, \Phi^*) - (X, X^*)\right), \tag{13}$$

where the real constants $\tilde{a}$ and $\tilde{b}$ have to satisfy $\tilde{a} + \tilde{b} \neq 0$ (note $(\Phi, \mathrm{d}X) = -2(\Phi, \nabla X)$). In (13) we have used the usual notation for the scalar product of tensors: Let A and B be two tensors of rank r then (A, B) is defined by

$$(A, B) = A_{i_1 \ldots i_r} B^{i_1 \ldots i_r}.$$

The Lagrangians (10) and (13) and the field equations (9) and (12) have an entirely equal form. But the energy-momentum tensor for fields of integer spin $s \geqq 2$ will become more difficult than (11) if we compute it by variation of the metric. For example the term $F^{ij} \nabla_i A_j$ in (10) does not contain Christoffel symbols because of the antisymmetry of the field F. Contrary to this the corresponding term in (13) contains derivatives of the metric.

Because the energy-momentum tensor is already known in its spinor form (6) it is sufficient to convert it into tensor form by means of definition 1. The result is:

$s = 2$:

$$\begin{aligned} T_{kl} = 2(\tilde{a} + \tilde{b}) \Big\{ &\Phi_{(k|}{}^{i_2 i_3 i_4} \nabla_{i_3} [X_{i_2|l) i_4} + X_{i_4|l) i_2}] - 2\Phi_{(k}{}^{i_2}{}_{l)}{}^{i_4} \nabla^{i_3} X_{i_2 i_3 i_4} - X^{i_2 i_3 i_4} \nabla_{i_3} \Phi_{(k|i_2|l) i_4} + \\ &+ \frac{m}{\sqrt{2}} \left[g_{kl} \left(\frac{1}{2} (X, X^*) - \frac{1}{8} (\Phi, \Phi^*) \right) - X_{(k}{}^{i_3 i_4} X^*_{l) i_3 i_4} \right] \Big\} \end{aligned} \tag{14}$$

$s \geqq 3$:

$$\begin{aligned} T_{kl} = 2(\tilde{a} + \tilde{b}) \Big\{ &(s-1)\, \Phi_{(k|}{}^{i_2 \ldots i_{2s}} \nabla_{i_3} X_{i_2|l) i_4 \ldots i_{2s}} - s\Phi_{(k}{}^{i_2}{}_{l)}{}^{i_4 \ldots i_{2s}} \nabla^{i_3} X_{i_2 i_3 \ldots i_{2s}} - \\ &-(s-1)\, X^{i_2 i_3 \ldots i_{2s}} \nabla_{i_3} \Phi_{(k|i_2|l) i_4 \ldots i_{2s}} + \frac{m}{\sqrt{2}} \left[\frac{s}{4} g_{kl} (X, X^*) - \frac{s}{2} X_{(k}{}^{i_3 \ldots i_{2s}} X^*_{l) i_3 \ldots i_{2s}} \right] \Big\} \end{aligned} \tag{15}$$

Comparing the formulas of section 3.3. with those of section 2.2. we observe that the Lagrangian, the field equations and the energy-momentum tensor formulated by means of spinor or real tensor fields have essentially the same form. However, the spinor calculus seems to be more suitable for calculations because of the complicated symmetries of the field tensors (see lemma 1).

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