First Order Planetary Perturbations with Elliptic Functions

CAROL A. WILLIAMS[1]
Department of Mathematics, University of South Florida, Tampa, FL 33620
THOMAS VAN FLANDERN[2]
United States Naval Observatory, Washington, D.C. 20390
EDWARD A. WRIGHT[3]
Department of Physics, University of South Florida, Tampa, FL 33620

(Received May 21, 1987; accepted May 21, 1987)

Abstract. The differential equations of planetary theory are solved analytically to first order for the two-dimensional case, using only Jacobian elliptic functions and the elliptic integrals of the first and second kind. This choice of functions leads to several new features potentially of importance for planetary theory. The first of these is that the solutions do not require the expansion of the reciprocal of the distance between two planets, even for those variables which depend on two angular arguments. A second result is that the solution is free from small divisors with the exception of two special resonances. In fact, not only are the solutions for resonant orbits free from small divisors, the perturbations for all variables are expressible in closed form. A subset of the resonant orbits maintains this form and in addition has the remarkable feature that the first order perturbations are purely periodic; they contain no secular terms. A solution for the 1:3 resonance case is given as an example.

1. Introduction.

The motivation for this study came from problems encountered in developing an analytic solution for the orbit of Pluto. In addition to the small divisor, $3n - 2n'$, expected because of the resonance in mean longitudes between Neptune and Pluto [5], there is another serious difficulty with the expansion of $1/\Delta$, where Δ is the distance between these two planets. Several studies have been made of Pluto's orbit, yet it has eluded successful analytic representation. Petrovskaia [10] gives an expansion of $1/\Delta$ which is valid in spite of the intersection of the orbits when they are projected into two dimensions. Seeking to avoid the cumbersome nature of this expansion, Chapront [4] describes a special technique of fitting a series to DE200, the numerical integration of Standish and Williams [12]. Besides these difficulties, there are other problems with the analytic representation of Pluto's orbit. There is a second resonance found by Williams and Benson [13] in the argument of the perihelion, and confirmed by

[1]also, Department of Astronomy, University of Florida, Gainesville, FL 32611

[2]present address: 6327 Western Avenue N.W., Washington, D.C. 20015

[3]present address: Simmonds Precision Products, Inc., Instrument Systems Division, Panton Road, Vergennes VT 05491

Celestial Mechanics **40** (1987), 367–391.

Kinoshita and Nakai [8]. Nacozy and Diehl [9] made some progress in understanding the motion by fitting numerical results to periodic solutions of the third kind in the restricted three body problem, and successfully representing the motion of its perihelion by adding perturbations of other planets besides Neptune. The last problem is that the numerical values of the eccentricity and inclination are larger than those of other planets. This was a more serious difficulty when computers were not available to do algebraic computation and is no longer a major factor. Pluto is more conspicuous by its absence from analytic developments than by its presence, for example in research at the Bureau des Longitudes, Paris (Bretagnon [1]) and at the U.S.Nautical Almanac Office, United States Naval Observatory.

The work presented in this paper was designed to avoid expansion of $1/\Delta$. It will be seen that in so doing we have removed all small divisors from the solution except $n - n'$ and $2n - n'$. Although an analytic representation of the motion of Pluto was the motivating factor for developing this theory, no specific representation of its orbit is given here. It was felt that it should not be handled here, considering the additional details to which one should pay attention. This solution, except for Pluto, is completely general. It is expressed entirely in terms of Jacobian elliptic functions and Legendre elliptic integrals of the first and second kind. Furthermore, in many places it is in closed form. No use is made of the classical d'Alembert series; in that respect it would be interesting to compare our solution with classical developments.

At the time this research was carried out, Richardson also obtained a solution for planetary perturbations using elliptic functions and integrals and avoiding expansion of $1/\Delta$. His results were published in 1982 [11]. His solutions, however, are limited to variables whose derivatives depend on the synodic angle only. We show in this paper that the procedures can be extended to variables which depend on two angular frequencies.

2. The Equations of Motion

The analysis starts with the differential equations for the perturbation of one planet by another, given by

$$\ddot{\mathbf{r}} + \frac{G(M+m)}{r^3}\mathbf{r} = \nabla \mathcal{R} \tag{1}$$

where

$$\mathcal{R} = Gm'\left(\frac{1}{\Delta} - \frac{\mathbf{r}\cdot\mathbf{r}'}{r'^3}\right).$$

In these equations, G is the gravitational constant, M the mass of the sun, m and $\mathbf{r}$, the mass and position vector of the disturbed planet. Primed quantities refer to the disturbing planet. The distance between the planets is found from the relation,

$$\Delta = \sqrt{\|\mathbf{r}\|^2 + \|\mathbf{r}'\|^2 - 2\mathbf{r}\cdot\mathbf{r}'}. \tag{2}$$

We shall use the method of variation of elements to develop the perturbations. In the application of this method to planetary theory, the appropriate differential equations are developed from Eq.(1) in many standard texts. The elements which we choose as variables are

$$\begin{array}{lcl} a, & & \text{the semi-major axis,} \\ \epsilon, & & \text{the constant of mean longitude,} \\ h & = & e \sin \varpi, \\ k & = & e \cos \varpi, \end{array}$$

e being the eccentricity and ϖ the longitude of perihelion. Here, ϵ is associated with the perturbed mean longitude,

$$\lambda = \int dt \int \dot{n} dt + \epsilon, \tag{3}$$

introduced to avoid mixed secular terms in the perturbations. (See Brouwer and Clemence [2, p.285] for a discussion of this variable.) A convenient form of the equations adapted from Danby [6, pp.250ff] is

$$\begin{aligned} \dot{a} &= \frac{2}{n\eta}\left(eR\sin v + \frac{p}{r}T\right), \\ \dot{\epsilon} &= \frac{\eta}{na}\left[-R\left(2\eta\frac{r}{p} + \frac{e}{1+\eta}\cos v\right) + eT\left(1+\frac{r}{p}\right)\sin v\right], \\ \dot{h} &= \frac{\eta}{na}\left\{-R\cos(v+\varpi) + T\left[\left(1+\frac{r}{p}\right)\sin(v+\varpi) + h\frac{r}{p}\right]\right\}, \\ \dot{k} &= \frac{\eta}{na}\left\{+R\sin(v+\varpi) + T\left[\left(1+\frac{r}{p}\right)\cos(v+\varpi) + k\frac{r}{p}\right]\right\}, \end{aligned} \tag{4}$$

where v is the true anomaly, p the semi-latus rectum and $\eta = \sqrt{1-e^2}$. R and T are the radial and transversal components of the acceleration; with S as the synodic angle,

$$\begin{aligned} R &= -Gm'\left[\frac{r}{\Delta^3} - (\frac{1}{\Delta^3} - \frac{1}{r'^3})r'\cos S\right], \\ T &= -Gm'\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right)r'\sin S. \end{aligned}$$

The integration of Eqs.(4) is done by Picard iteration, where we take the reference solution for the first order to be coplanar, circular orbits for both m and m'. With this approximation Eq.(2) becomes

$$\Delta = \sqrt{a^2 + a'^2 - 2aa'\cos S}. \tag{5}$$

As we proceed, it is necessary to keep in mind that eventually we wish to have Eqs.(4) in a form which is readily adapted to integration with elliptic functions. In this spirit, define two quantities

$$\kappa^2 = \frac{4\alpha}{(1+\alpha)^2} \qquad \text{and} \qquad 2\phi = \pi - S; \tag{6}$$

then rewrite Eq.(5) as

$$\Delta = \tilde{a}(1+\alpha)\sqrt{(1-\kappa^2\sin^2\phi)} \tag{7}$$

with the auxiliary quantities

$$\tilde{a} = \max(a, a') \qquad \text{and} \qquad \alpha = \min(a/a', a'/a).$$

Since the square root function in Eq.(7) appears frequently in the later developments, we introduce

$$\delta = \sqrt{(1-\kappa^2\sin^2\phi)}. \tag{8}$$

When evaluated along the reference orbits, all of the coefficients of R and T in Eqs.(4) will be constants. In fact in R and T themselves, the only terms which remain as functions of the time are those which depend on ϕ and δ. To show this explicitly, we first define

$$\begin{aligned} R_1 &= -\frac{Gm'}{\tilde{a}^3(1+\alpha)^3}\,a, \\ R_2 &= -\frac{Gm'}{\tilde{a}^3(1+\alpha)^3}\,a', \\ R_3 &= +\frac{Gm'}{a'^2}. \end{aligned} \tag{9}$$

Then the components of the acceleration become

$$\begin{aligned} R &= \frac{R_1}{\delta^3} + \left(\frac{R_2}{\delta^3} + R_3\right)\cos 2\phi, \\ T &= \left(\frac{R_2}{\delta^3} + R_3\right)\sin 2\phi. \end{aligned} \tag{10}$$

Substituting Eqs.(10) into Eqs.(4) and letting $e = e' = 0$ everywhere yields the equations in their final form

$$
\begin{aligned}
\dot{a} &= -\frac{2a}{3n}\dot{n} = \frac{2}{n}\left(\frac{R_2}{\delta^3} + R_3\right)\sin 2\phi, \\
\dot{\epsilon} &= -\frac{2}{na}\left[\frac{R_1}{\delta^3} + \left(\frac{R_2}{\delta^3} + R_3\right)\cos 2\phi\right], \\
\dot{h} &= -\frac{1}{na}\left[\frac{R_1}{\delta^3}\cos\lambda - \frac{1}{2}\left(\frac{R_2}{\delta^3} + R_3\right)\left[\cos(\lambda + 2\phi) - 3\cos(\lambda - 2\phi)\right]\right], \\
\dot{k} &= +\frac{1}{na}\left[\frac{R_1}{\delta^3}\sin\lambda + \frac{1}{2}\left(\frac{R_2}{\delta^3} + R_3\right)\left[3\sin(\lambda + 2\phi) - \sin(\lambda - 2\phi)\right]\right].
\end{aligned} \tag{11}
$$

3. Quadratures Involving a Single Frequency

By inspection and Eq.(8) it is seen that the first two of Eqs.(11) depend on the angular variable ϕ only, while $\dot{h}$ and $\dot{k}$ depend on both ϕ and λ. A more traditional method of integrating these latter equations would proceed with an expansion of $1/\delta$ in powers of κ. This can be avoided by introducing a new variable, u, defined by

$$u = \int_0^\phi \frac{dx}{\sqrt{(1 - \kappa^2 \sin^2 x)}}. \tag{12}$$

We then have

$$
\begin{aligned}
\phi &= am(u, \kappa), \\
\sin\phi &= \mathrm{sn}\, u, \\
\cos\phi &= \mathrm{cn}\, u, \\
\delta &= \mathrm{dn}\, u.
\end{aligned} \tag{13}
$$

The independent variable in Eqs.(11) is, of course, t. We wish to transform the independent variable to either ϕ or u to perform the integrations. From the second of Eqs.(6) we see that $2\dot{\phi} = -\dot{S}$. When evaluated along the reference orbits, we obtain

$$2\dot{\phi} = -(\dot{\lambda} - \dot{\lambda}') = n' - n, \tag{14}$$

where we have made use of the fact that to first order $\dot{\varpi} = \dot{\varpi}' = 0$. From Eq.(14) and Eq.(12) we derive

$$(n' - n)dt = 2d\phi = 2\mathrm{dn}u du. \tag{15}$$

Eq.(15) implies that the introduction of ϕ or u as an independent variable replacing t will lead to the divisor $n - n'$ in the perturbations. Both of these choices for the transformation of the independent variable eliminate the possibility of applying this planetary theory to orbits with a 1 : 1 resonance, but one should add that a rotation of the coordinate system normally would suffice to handle this case. Thus there is no need for applying this theory to the 1 : 1 resonance.

The first two of Eqs.(11) can be integrated with the aid of Table I which contains quadratures from Byrd and Friedman [3] taken from their formulas [# 315.02, #320.02, #360.11] as well as some obvious quadratures. We can already observe that, since the expressions in Table I are in closed form, the first order perturbations for a and ϵ will also be in closed form, but we shall examine those more closely in Section 6.

TABLE I

$$\int \frac{1}{\delta^3}\, d\phi = \frac{1}{\kappa'^2}\left[E(u) - \kappa^2 \frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}\right]$$

$$\int \frac{\cos 2\phi}{\delta^3}\, d\phi = \frac{1}{\kappa^2}\left[2u - \left(1 + \frac{1}{\kappa'^2}\right)\left(E(u) - \kappa^2 \frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}\right)\right]$$

$$\int \frac{\sin 2\phi}{\delta^3}\, d\phi = \frac{2}{\kappa^2\,\mathrm{dn}u}$$

$$\int \cos 2\phi\, d\phi = \mathrm{sn}u\,\mathrm{cn}u$$

$$\int \sin 2\phi\, d\phi = \frac{1}{2}\left(\mathrm{sn}^2 u - \mathrm{cn}^2 u\right)$$

Two quantities in Table I which may need definition are $E(u)$, the Legendre elliptic integral of the second kind and κ', the complementary modulus.

4. Quadratures Involving Two Frequencies

The integration of Eqs.(11) containing the two angular variables ϕ and λ is considered in this section. We first write all linear combinations of λ and ϕ as multiples of ϕ. This is possible since to first order ϕ and λ are both linear functions of t. Define three numbers, ω_j, and a phase angle, ϕ_0 according to

$$\begin{aligned} \lambda &= \omega_1\phi + \phi_0, \\ \lambda - 2\phi &= \omega_2\phi + \phi_0, \\ \lambda + 2\phi &= \omega_3\phi + \phi_0, \end{aligned} \qquad (16)$$

where

$$\omega_1 = \frac{2n}{n' - n}, \qquad \omega_2 = \omega_1 - 2, \qquad \omega_3 = \omega_1 + 2 \tag{17}$$

and

$$\phi_0 = \epsilon - \frac{\omega_1}{2}(\pi + \epsilon' - \epsilon).$$

From Eqs.(11) and (16), the right hand sides of the equations for $\dot{h}$ and $\dot{k}$, in exponential notation, will contain terms of the form

$$\exp i(\omega_j \phi + \phi_0) \qquad \text{and} \qquad \exp i(\omega_j \phi + \phi_0)/\delta^3$$

which reduce to the forms

$$\exp(i\phi_0), \qquad \exp(i\phi_0)/\delta^3, \qquad \exp(i\omega_j\phi), \qquad \exp(i\omega_j\phi)/\delta^3.$$

Their integration after the transformation from t to ϕ by Eq.(15) requires the following quadratures:

$$\int d\phi, \qquad \int \frac{1}{\delta^3} d\phi, \qquad \int \exp(i\omega_j\phi) d\phi \tag{18}$$

and

$$\int \frac{\exp(i\omega_j\phi)}{\delta^3} d\phi \tag{19}$$

The quadratures in Eq.(18) are immediately obtained either by inspection or from Table I. The following discussion concerns the solution of the quadrature in Eq.(19).

In order to avoid expansion of $1/\delta^3$ in powers of κ, we examine the possibility of integrating with elliptic functions of ϕ or u, as was done in Section 3. This will require expansion of the numerator of Eq.(19) in powers of $\cos\phi$. In order to do this, we start with the Fourier expansion of $\cos\omega_j\phi$ and $\sin\omega_j\phi$. Begin by dropping the subscript of ω_j. It will be recovered later. Write

$$\cos\omega\phi = \sigma \sum_{j=0}^{\infty} a_j \cos j\phi, \qquad \sin\omega\phi = \sigma \sum_{j=1}^{\infty} b_j \sin j\phi. \tag{20}$$

The coefficients of the Fourier series of Eqs.(20) are found from the definite integrals

$$\sigma a_j = \frac{1}{2\pi}\int_0^{2\pi} \cos\omega\phi \cos j\phi d\phi, \qquad \sigma b_j = \frac{1}{2\pi}\int_0^{2\pi} \sin\omega\phi \sin j\phi d\phi$$

from which we obtain

$$
\begin{aligned}
\sigma &= \frac{\sin 2\omega\pi}{2\omega\pi}, \\
a_j &= \frac{\omega^2}{\omega^2 - j^2}, \qquad j \geq 0, \\
b_j &= \frac{j\omega}{\omega^2 - j^2}, \qquad j \geq 0.
\end{aligned} \tag{21}
$$

Powers of $\cos\phi$ are introduced into Eqs.(20) with the Chebyshev polynomials $(n \geq 0)$

$$
\begin{aligned}
T_n(\cos\phi) &= \cos n\phi = \frac{n}{2}\sum_{m=0}^{\lfloor n/2 \rfloor}(-1)^m (n-m)\binom{n-m}{m}(2\cos\phi)^{n-2m}, \\
U_n(\cos\phi) &= \frac{\sin(n+1)\phi}{\sin\phi} = \sum_{m=0}^{\lfloor n/2 \rfloor}(-1)^m \binom{n-m}{m}(2\cos\phi)^{n-2m}.
\end{aligned} \tag{22}
$$

The desired series will have the forms

$$
\cos\omega\phi = \sum_{n=0}^{\infty} A_n \cos^n\phi \qquad \text{and} \qquad \frac{\sin\omega\phi}{\sin\phi} = \sum_{n=1}^{\infty} B_n \cos^{n-1}\phi.
$$

The coefficients $A_n \equiv A_n(\omega)$ and $B_n \equiv B_n(\omega)$ are determined by combining the series of Eqs.(20) and Eqs.(22) to give

$$
\begin{aligned}
A_0 &= \sigma, \\
A_n/\omega &= B_n/n \\
&= \sigma\omega\frac{2^{n-1}}{n!}\sum_{j=n}^{\infty}(-1)^{j-n}\frac{(2j-n)}{\omega^2-(2j-n)^2}\frac{(j-1)!}{(j-n)!}.
\end{aligned} \tag{23}
$$

The next step in evaluating the quadrature of Eq.(19) is to transform the independent variable from ϕ to u using Eqs.(13), and use the series in Eq.(23). Considering the even functions first, write

$$
\int \frac{\cos\omega\phi}{\delta^3}\,d\phi = \int \sum_{n=0}^{\infty} A_n \frac{\cos^n\phi}{\delta^3}\,d\phi = \sum_{n=0}^{\infty} A_n \int \frac{\mathrm{cn}^n u}{\mathrm{dn}^2 u}\,du. \tag{24}
$$

From Byrd and Friedman [3, #355.01], we obtain the quadrature necessary for complete evaluation of Eq.(24), with the recurrence relation

$$\int \frac{\mathrm{cn}^n u}{\mathrm{dn}^2 u}\, du = \frac{1}{\kappa^2} \int \mathrm{cn}^{n-2} u\, du - \gamma \int \frac{\mathrm{cn}^{n-2} u}{\mathrm{dn}^2 u}\, du, \qquad n \geq 2 \tag{25}$$

where

$$\kappa'^2 = 1 - \kappa^2 \qquad \text{and} \qquad \gamma = (\kappa'/\kappa)^2.$$

That this is indeed a recurrence relation we can see by putting

$$I_n = \int \frac{\mathrm{cn}^n u}{\mathrm{dn}^2 u}\, du \qquad \text{and} \qquad C_n = \int \mathrm{cn}^n u\, du, \tag{26}$$

so that we can write Eq.(25) as

$$I_n = \frac{1}{\kappa^2} C_{n-2} - \gamma I_{n-2}, \qquad n \geq 2. \tag{27}$$

To complete the recurrence relation we borrow from Byrd and Friedman [3, # 312.05 and # 312.06] and obtain the following formula, for $n \geq 4$,

$$\begin{aligned} C_n &= \frac{1}{n-1} \left[(n-3)\gamma C_{n-4} - (n-2)(\gamma - 1) C_{n-2} \right. \\ &\qquad \left. + \mathrm{cn}^{n-3} u \,\mathrm{sn} u\, \mathrm{dn} u/\kappa^2 \right]. \end{aligned} \tag{28}$$

The recurrence will start with

$$\begin{aligned}
C_0 &= u, && \#312.00 \\
C_1 &= (1/\kappa) \sin^{-1}(\kappa\, \mathrm{sn} u), && \#312.01 \\
C_2 &= (1/\kappa^2)[E(u) - \kappa'^2 u], && \#312.02 \\
C_3 &= (1/2\kappa^3)[(\kappa^2 - \kappa'^2) \sin^{-1}(\kappa\, \mathrm{sn} u) + \kappa \mathrm{sn} u \mathrm{dn} u], && \#312.03 \\
I_0 &= (1/\kappa'^2) E(u) - (\kappa/\kappa')^2 (\mathrm{sn} u\, \mathrm{cn} u/\mathrm{dn} u), && \#315.02 \\
I_1 &= \mathrm{sn} u/\mathrm{dn} u. && \#354.01
\end{aligned} \tag{29}$$

The second function in the quadrature appearing in Eq.(19) is developed in a similar manner. We first write from Eqs.(13) and (23)

$$\begin{aligned} \int \frac{\sin \omega\phi}{\delta^3}\, d\phi &= \int \sum_{n=1}^{\infty} B_n \frac{\sin\phi \cos^{n-1}\phi}{\delta^3}\, d\phi \\ &= \sum_{n=1}^{\infty} B_n \int \frac{\mathrm{sn} u\, \mathrm{cn}^{n-1} u}{\mathrm{dn}^2 u}\, du. \end{aligned} \tag{30}$$

From Byrd and Friedman [3, # 355.01], we obtain

$$\int \frac{\mathrm{sn}\,u\,\mathrm{cn}^n u}{\mathrm{dn}^2 u}\,du = \frac{1}{\kappa^2}\int \mathrm{sn}\,u\,\mathrm{cn}^{n-2}u\,du - \gamma\int \frac{\mathrm{sn}\,u\,\mathrm{cn}^{n-2}u}{\mathrm{dn}^2 u}\,du. \tag{31}$$

If we define

$$J_n = \int \frac{\mathrm{sn}\,u\,\mathrm{cn}^n u}{\mathrm{dn}^2 u}\,du \qquad \text{and} \qquad S_n = \int \mathrm{sn}\,u\,\mathrm{cn}^n u\,du, \tag{32}$$

we can rewrite Eq.(31) in a form similar to Eq.(27), namely,

$$J_n = \frac{1}{\kappa^2}S_{n-2} - \gamma J_{n-2} \qquad n \geq 2. \tag{33}$$

The integration of S_n, since it was not found in Byrd and Friedman, is developed here. By integration by parts, considering that

$$\mathrm{sn}\,u\,\mathrm{cn}\,u\,du = -(1/\kappa^2)d(\mathrm{dn}\,u),$$

we obtain

$$\int \mathrm{sn}\,u\,\mathrm{cn}^n u\,du = -\frac{1}{\kappa^2}\mathrm{cn}^{n-1}u\,\mathrm{dn}\,u - \frac{n-1}{\kappa^2}\int \mathrm{sn}\,u\,\mathrm{cn}^{n-2}u\,\mathrm{dn}^2 u\,du.$$

With the identity

$$\mathrm{cn}^2 u = \gamma + (1/\kappa^2)\mathrm{dn}^2 u,$$

this becomes

$$S_n = -\frac{1}{n\kappa^2}\mathrm{cn}^{n-1}u\,\mathrm{dn}\,u - \frac{n-1}{n}\gamma S_{n-2}, \qquad n \geq 2. \tag{34}$$

The starting values for the recurrence relations Eqs.(33) and (34), taken from Byrd and Friedman [3], are

$$\begin{aligned} S_0 &= (1/\kappa)\ln(\mathrm{dn}\,u - \kappa\,\mathrm{cn}\,u), && \#310.01 \\ S_1 &= -(1/\kappa^2)\mathrm{dn}\,u, && \#360.03 \\ J_0 &= -(1/\kappa'^2)(\mathrm{cn}\,u/\mathrm{dn}\,u), && \#353.01 \\ J_1 &= 1/(\kappa^2\mathrm{dn}\,u). && \#360.11 \end{aligned} \tag{35}$$

TABLE II

Planets		α	κ^2	κ'^2	$(\kappa/\kappa')^2$	$(\kappa'/\kappa)^2$
Mercury-	Venus	0.535	0.908	0.092	9.908	0.101
	Earth	0.387	0.805	0.195	4.122	0.243
	Mars	0.254	0.646	0.354	1.826	0.548
	Jupiter	0.074	0.258	0.742	0.347	2.879
	Saturn	0.040	0.149	0.851	0.175	5.702
	Uranus	0.020	0.078	0.922	0.084	11.899
	Neptune	0.013	0.050	0.950	0.053	18.917
	Pluto	0.010	0.038	0.962	0.040	25.031
Venus-	Earth	0.723	0.974	0.026	37.789	0.026
	Mars	0.475	0.873	0.127	6.881	0.145
	Jupiter	0.139	0.429	0.571	0.750	1.333
	Saturn	0.075	0.261	0.739	0.353	2.833
	Uranus	0.038	0.140	0.860	0.163	6.143
	Neptune	0.024	0.092	0.908	0.101	9.896
	Pluto	0.018	0.071	0.929	0.076	13.167
Earth-	Mars	0.656	0.957	0.043	22.223	0.045
	Jupiter	0.192	0.541	0.459	1.178	0.849
	Saturn	0.104	0.342	0.658	0.520	1.923
	Uranus	0.052	0.188	0.812	0.232	4.311
	Neptune	0.033	0.125	0.875	0.142	7.024
	Pluto	0.025	0.096	0.904	0.107	9.389
Mars-	Jupiter	0.293	0.701	0.299	2.343	0.427
	Saturn	0.159	0.473	0.527	0.899	1.113
	Uranus	0.079	0.273	0.727	0.375	2.669
	Neptune	0.051	0.184	0.816	0.225	4.445
	Pluto	0.039	0.143	0.857	0.167	5.995
Jupiter-	Saturn	0.543	0.912	0.088	10.376	0.096
	Uranus	0.271	0.671	0.329	2.041	0.490
	Neptune	0.173	0.503	0.497	1.012	0.988
	Pluto	0.132	0.411	0.589	0.698	1.432
Saturn-	Uranus	0.500	0.889	0.111	7.981	0.125
	Neptune	0.319	0.733	0.267	2.751	0.364
	Pluto	0.243	0.628	0.372	1.691	0.591
Uranus-	Neptune	0.638	0.951	0.049	19.530	0.051
	Pluto	0.485	0.880	0.120	7.336	0.136
Neptune-	Pluto	0.760	0.981	0.019	53.023	0.019

Although the solution for the perturbations of h and k are obtained without expanding $1/\delta$ (traditionally done in powers of α), nevertheless the expansion of $\cos\omega_j\phi$ introduces a power series not in α but in $\gamma = (\kappa'/\kappa)^2$. This is due to the fact that the relations given by Eqs.(27), (28), (33) and (34) contain the factor γ. In this regard, one should note that γ is not necessarily a small quantity, as one can see from Table II where the values of γ are given for all pairs of planets. For Neptune-Pluto, it has a value less than 0.02, but for some pairs it is ≥ 1. In order to compare all of these parameters, Table II lists the values of $\alpha, \kappa^2, \kappa'^2, (\kappa/\kappa')^2$, and $(\kappa'/\kappa)^2$. In the following discussion let us restrict our attention to values of α which may occur in planetary theories, namely $0 \leq \alpha \leq 1$. The value of α associated with $\gamma = 1$ is $\alpha_1 = 3 - \sqrt{8} \approx 0.17$. For values of $\alpha > \alpha_1$, $\gamma < 1$ and as $\alpha \rightarrow 1$, $\gamma \rightarrow 0$. For Neptune-Pluto, κ'^2 and $(\kappa'/\kappa)^2$ are the smallest parameters available for an expansion. Thus it appears that expansion of the numerator of Eq.(19), rather than its denominator, is preferable for the Neptune-Pluto problem. At the other end of the spectrum, where α is small, $(\kappa'/\kappa)^2 > 1$. In that region, $(\kappa/\kappa')^2$ would be a more suitable expansion parameter if one wants to use elliptic functions to avoid expanding $1/\delta$. For this type of development, the integrals of Eqs.(24) and (30) should be developed in powers of sn$u$ rather than cn$u$. However, it must be pointed out that $(\kappa/\kappa')^2$ is $> \alpha$ throughout the range of the table.

Consider the value $\alpha_2 = 1/3$, where $(\kappa'/\kappa)^2 < \alpha$ whenever $\alpha > \alpha_2$. The value $\alpha = \alpha_1$ represents a demarcation between expansion parameters if elliptic functions are to be used. The value $\alpha = \alpha_2$ is more important if numerical convergence is the criterion for the developments. Whenever $\alpha < \alpha_2$ the classical expansions of $1/\delta$ with the Laplacian coefficients expressed as hypergeometric series in powers of α are preferred. When $\alpha > \alpha_2$, we contend that the expansions in powers of γ as outlined in this paper are more suitable. We shall say more about this in the next section.

5. First Order Perturbations of the Elements

The data needed to construct the solution of Eqs.(11) are contained in either Table I or Section 4. Consider first the variables a and ϵ. From the differential equations we obtain

$$\int \dot{a}\, dt = \frac{2}{n}\left(R_2 \int \frac{\sin 2\phi}{\delta^3}\, dt + R_3 \int \sin 2\phi\, dt\right).$$

Substituting for dt from Eq.(15) and using Table I leads to the expression

$$a(t) - a(0) = \frac{1}{n(n'-n)}\left[\frac{4R_2}{\kappa^2 \mathrm{dn} u} + R_3(\mathrm{sn}^2 u - \mathrm{cn}^2 u)\right]_{u(0)}^{u},$$

where the R_i are defined by Eqs.(9).

If we choose $\phi = 0$ at $t = 0$, then $u(0) = 0$. The corresponding synodic angle is $\lambda(0) - \lambda'(0) = \pi$. The osculating value of $a(t)$ at $t = 0$ is $a(0)$. Since sn$u$, cn$u$, and dn$u$ are periodic functions of u, $a(t)$ is a combination of a constant mean value and a periodic perturbation. There are no secular terms in the semi-major axis, as expected. In the following expression, ϕ is introduced from Eqs.(13), giving

$$a(t) = a(0) + a_0 \left[a_1(1 - 1/\delta) + a_2(1 - \cos 2\phi)\right], \tag{36}$$

where

$$a_0 = \frac{1}{n(n-n')}, \quad a_1 = \frac{4R_2}{\kappa^2}, \quad \text{and} \quad a_2 = -R_3.$$

The a_i are to be evaluated from the parameters defining the reference orbits.

A similar solution is obtained for the constant of mean longitude, namely

$$\epsilon(t) = \epsilon(0) + \epsilon_0 \left[\epsilon_1 u + \epsilon_2 \left(E(u) - \kappa^2 \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}\right) + \epsilon_3 \mathrm{sn}\,u\,\mathrm{cn}\,u\right] \tag{37}$$

where

$$\begin{aligned} \epsilon_0 &= \frac{4}{an(n-n')}, \qquad \epsilon_1 = \frac{2R_2}{\kappa^2}, \\ \epsilon_2 &= \frac{R_1}{\kappa'^2} - \frac{R_2}{\kappa^2}\left(1 + \frac{1}{\kappa'^2}\right), \quad \text{and} \quad \epsilon_3 = R_3. \end{aligned}$$

The coefficients of the secular terms in $\epsilon(t)$ may be incorporated into the first order approximation of the averaged mean motion; this is done in Section 6.

It would be somewhat involved to compare the results in Eqs.(36) and (37) with the results of Richardson [11]. The methods used in the two papers are very different. Richardson works in a Hamiltonian context. Besides, he takes the more general problem of many planets. Further, he gives the generator of the averaging function, but does not give the specific transformation for the variables. Our results appear to be equivalent to his.

The integration of $\dot{h}$ and $\dot{k}$ makes use of the developments in Section 5. Considering $h(t)$ first, we write from Eq.(11)

$$\begin{aligned} h(t) - h(0) = &-\frac{1}{na}\left[R_1 \int_0^t \frac{\cos\lambda}{\delta^3}\,d\tau - \frac{R_2}{2}\int_0^t \frac{\cos(\lambda+2\phi) - 3\cos(\lambda-2\phi)}{\delta^3}\,d\tau \right. \\ &\left. -\frac{R_3}{2}\int_0^t \left[\cos(\lambda+2\phi) - 3\cos(\lambda-2\phi)\right] d\tau\right]. \end{aligned}$$

First, let us change the independent variable to ϕ, adopt Eqs.(16) and set $\nu = 2/[an(n-n')]$ We may then write

$$h(t) = h(0) + \nu(H_1 + H_2)|_0^\phi \tag{38}$$

and define H_1 and H_2 with the equations:

$$\begin{aligned} H_1(\phi) - H_1(0) &= R_1 \int_0^\phi \frac{\cos(\omega_1 x + \phi_0)}{\delta^3}\,dx \\ &+ \frac{R_2}{2}\left[3\int_0^\phi \frac{\cos(\omega_2 x + \phi_0)}{\delta^3}\,dx - \int_0^\phi \frac{\cos(\omega_3 x + \phi_0)}{\delta^3}\,dx\right], \quad (39) \\ H_2(\phi) - H_2(0) &= \frac{R_3}{2}\int_0^\phi [3\cos(\omega_2 x + \phi_0) - \cos(\omega_3 x + \phi_0)]\,dx. \end{aligned}$$

Let us consider the development of H_1 first. As an example, starting with the first quadrature in H_1 and, for the moment, ignoring the constant factor R_1

$$\int_0^\phi \frac{\cos(\omega_1 x + \phi_0)}{\delta^3}\,dx = \cos\phi_0 \int_0^\phi \frac{\cos\omega_1 x}{\delta^3}\,dx - \sin\phi_0 \int_0^\phi \frac{\sin\omega_1 x}{\delta^3}\,dx.$$

Adopting the form of the quadratures given by Eqs.(24) and (30) and using the notation introduced in Eqs.(26) and (32), the first term in H_1 is written,

$$\begin{aligned} R_1 \int_0^\phi \frac{\cos(\omega_1 x + \phi_0)}{\delta^3}\,dx &= R_1 \cos\phi_0 \sum_{n=0}^{\infty} A_n(\omega_1)[I_n(u) - I_n(0)] \\ &- R_1 \sin\phi_0 \sum_{n=1}^{\infty} B_n(\omega_1)[J_{n-1}(u) - J_{n-1}(0)] \end{aligned}$$

where I_n and J_n are given by the recurrence relations of Section 4. Inspection of the starting values in Eqs.(29) and (35) indicates that $I_n(0) = 0$ while $J_n(0) \neq 0$, for all n. Incorporating all of the terms in Eq.(39) gives

$$H_1(u) = \sum_{n=0}^{\infty} [Q_{11}(n) I_n(u) + Q_{12}(n) J_{n-1}(u)] \tag{40}$$

where we define

$$\begin{aligned} Q_{11}(n) &= +[R_1 A_n(\omega_1) + (R_2/2)(3A_n(\omega_2) - A_n(\omega_3))]\cos\phi_0, \\ Q_{12}(n) &= -[R_1 B_n(\omega_1) + (R_2/2)(3B_n(\omega_2) - B_n(\omega_3))]\sin\phi_0. \end{aligned}$$

It is necessary to define $B_0(\omega_j) = 0$ in order to extend the summation to include $n = 0$ as it is done in Eq.(40).

The quantity H_2 in Eq.(39) is easily integrated to give

$$H_2(\phi) = \frac{R_3}{2}\left(\frac{3}{\omega_2}\sin(\omega_2\phi + \phi_0) - \frac{1}{\omega_3}\sin(\omega_3\phi + \phi_0)\right). \tag{41}$$

The ω_2 and ω_3 appear as divisors in this expression and it is important to see whether they can take on the value 0. From their definition in Eq.(17) we have

$$\begin{aligned}\omega_2 &= \omega_1 - 2 = 0 \quad \Leftrightarrow \quad n/n' = 1/2,\\ \omega_3 &= \omega_1 + 2 = 0 \quad \Leftrightarrow \quad n' = 0.\end{aligned}$$

The latter of these two results is meaningless to our discussion and therefore the only resonance which could produce a small divisor or a secular term in the perturbations of the eccentric variables is $n/n' = 1/2$. Including the additional constraint imposed by the divisor $n - n'$, we discover that these two are the only small divisors appearing in the solutions.

The perturbations in $k(t)$ lead to the same conclusions. Following the developments for $h(t)$ we eventually obtain

$$\begin{aligned}k(t) &= k(0) - \nu(K_1 + K_2)|_0^\phi,\\ K_1(u) &= \sum_{n=0}^{\infty}[Q_{21}(n)I_n(u) + Q_{22}J_{n-1}(u)],\\ K_2(\phi) &= \frac{R_3}{2}\left(\frac{1}{\omega_2}\cos(\omega_2\phi + \phi_0) - \frac{3}{\omega_3}\cos(\omega_3\phi + \phi_0)\right)\end{aligned} \tag{42}$$

together with

$$\begin{aligned}Q_{21}(n) &= [R_1 A_n(\omega_1) - (R_2/2)(A_n(\omega_2) - 3A_n(\omega_3))]\sin\phi_0,\\ Q_{22}(n) &= [R_1 B_n(\omega_1) - (R_2/2)(B_n(\omega_2) - 3B_n(\omega_3))]\cos\phi_0.\end{aligned}$$

A clearer picture of the perturbations appears when we study the functions $I_n(u)$ and $J_n(u)$. Table III gives these functions for $2 \leq n \leq 9$, expressed in terms of the starting values of Eqs.(29) and (35) and of the Jacobian elliptic functions.

From this table, one may observe that the coefficients of I_0, I_1, J_0 and J_1, are the first four terms of $-\gamma/(1+\gamma)$ expanded in powers of γ. One can also expect as the table suggests, that the other terms of the J_n's, decrease somewhat rapidly in powers of γ. (Recall that $\gamma = (\kappa'/\kappa)^2$.) The same is not true for the remaining terms in the first half of the table —a point which suggests that a large number of terms may be necessary in applications.

From the starting values, we observe that J_0, S_0, J_1, and S_1 are purely periodic. So also are I_1, C_1 and C_3. Secular terms come from I_0, C_0 and C_2 which depend on the elliptic integrals u and $E(u)$. If one recalls Eq.(24), it is easily seen that these secular terms arise from the even powers of $\cos\phi$. Had we used an averaging method instead of Picard iteration, we might have cured the problem.

TABLE III

	I_2	I_4	I_6	I_8
I_0	$-\gamma$	γ^2	$-\gamma^3$	γ^4
C_0/κ^2	1	$-\gamma$	$\gamma(1+3\gamma)/3$	$\gamma(4-9\gamma-15\gamma^2)/15$
C_2/κ^2		1	$(2-5\gamma)/3$	$(8-17\gamma+33\gamma^2)/15$
$snu\,cnu\,dnu/\kappa^4$			$1/3$	$(4-9\gamma)/15$
$snu\,cn^3u\,dnu/\kappa^4$				$1/5$
	I_3	I_5	I_7	I_9
I_1	$-\gamma$	γ^2	$-\gamma^3$	γ^4
C_1/κ^2	1	$-\gamma$	$\gamma(1+2\gamma)/2$	$\gamma(5-11\gamma-12\gamma^2)/12$
C_3/κ^2		1	$(3-7\gamma)/4$	$(15-32\gamma+57\gamma^2)/24$
$snu\,cn^2u\,dnu/\kappa^4$			$1/4$	$-(5-11\gamma)/24$
$snu\,cn^4u\,dnu/\kappa^4$				$1/6$
	J_2	J_4	J_6	J_8
J_0	$-\gamma$	γ^2	$-\gamma^3$	γ^4
S_0/κ^2	1	$-3\gamma/2$	$15\gamma^2/8$	$-35\gamma^3/16$
$cnu\,dnu/\kappa^4$		$-1/2$	$7\gamma/8$	$-19\gamma^2/16$
$cn^3u\,dnu/\kappa^4$			$-1/4$	$11\gamma/24$
$cn^5u\,dnu/\kappa^4$				$-1/6$
	J_3	J_5	J_7	J_9
J_1	$-\gamma$	γ^2	$-\gamma^3$	γ^4
S_1/κ^2	1	$-5\gamma/3$	$11\gamma^2/5$	$-93\gamma^3/35$
$cn^2u\,dnu/\kappa^4$		$-1/3$	$3\gamma/5$	$-29\gamma^2/35$
$cn^4u\,dnu/\kappa^4$			$-1/5$	$13\gamma/35$
$cn^6u\,dnu/\kappa^4$				$-1/7$

Table III also contains the information necessary to separate the various terms appearing in $h(t)$ and $k(t)$ according to the categories of terms: constant, secular and periodic. Consider Eqs.(38), (40) and (41) to obtain

$$\begin{aligned} h(t) &= h(0) + \nu(h_0 + h_s + h_p), \\ h_0 &= -(H_1(0) + H_2(0)), \\ h_s &= \left(\sum_{n=0}^{\infty} Q_{11}(2n) I_{2n}(u) \right)_{sec}, \\ h_p &= H_1(u) + H_2(\phi) - h_0 - h_s. \end{aligned} \tag{43}$$

Following the same procedure, we also obtain

$$\begin{aligned} k(t) &= k(0) - \nu(k_0 + k_s + k_p), \\ k_0 &= -(K_1(0) + K_2(0)), \\ k_s &= \left(\sum_{n=0}^{\infty} Q_{21}(2n) I_{2n}(u) \right)_{sec}, \\ k_p &= K_1(u) + K_2(\phi) - k_0 - k_s. \end{aligned} \tag{44}$$

Because of the choice of reference orbits we should substitute the value $e = 0$, into the right hand sides of Eqs.(43) and (44). This condition is equivalent to $h(0) = 0$ and $k(0) = 0$, but the first order perturbations induce a non-zero value for the eccentricity of $\mathcal{O}(m')$, as expected.

6. First Order Perturbations in Mean Longitude.

The perturbed mean longitude is developed from its definition in Eq.(3). The value of $\dot{n}$ is derived from $\dot{a}$ using Eqs.(11) and then integrated according to Eq.(36). This procedure yields

$$\int_0^t \dot{n}(\tau)\, d\tau = n(t) - n(0) = n_0 \left[n_1(1 - 1/\delta) + n_2(1 - \cos 2\phi) \right], \tag{45}$$

where

$$n_i = -\frac{3n}{2a} a_i.$$

The integration of Eq.(45) leads to the quadrature

$$\int_0^t n(\tau)\, d\tau = [n(0) + n_0(n_1 + n_2)]t + 2n_0(n_1\, u + n_2 \,\mathrm{sn}\, u \,\mathrm{cn}\, u)/(n - n'). \tag{46}$$

An expression for λ is obtained by adding Eq.(37) and Eq.(46), yielding

$$\begin{aligned}\lambda &= [n(0) + n_0(n_1 + n_2)]t + \epsilon(0) + \left(\epsilon_0\epsilon_1 + \frac{2n_0n_1}{n - n'}\right) u \\ &+ \epsilon_0\epsilon_2 \left(E(u) - \kappa^2 \frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}\right) + \left(\epsilon_0\epsilon_3 + \frac{2n_0n_2}{n - n'}\right) \mathrm{sn}u\,\mathrm{cn}u.\end{aligned} \tag{47}$$

The elliptic integrals in Eq.(47) contain both secular and periodic parts. The relationships between t, u and ϕ are taken from Eqs.(13)and(15). We may use this information to separate the secular and periodic perturbations in the mean longitude. Note that $\lambda(0) = \epsilon(0)$. From the theory of elliptic integrals, we know that

$$E(u) = (E/K)\,u + Z(u),$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, and $Z(u)$, the Jacobi *zeta* function, is purely periodic in u. Further,

$$\phi = \mathrm{am}u = (\pi/2K)\,u + \mathrm{am}^*u,$$

where am^*u is the purely periodic part of the amplitude function. Therefore both u and $E(u)$ can be expressed as sums of terms linear in ϕ plus periodic terms. Writing the elliptic integrals this way and setting $2\phi = (n' - n)t$, we obtain the final form

$$\lambda = \lambda_0 + \lambda_1 t + \lambda_2 Z(u) + \lambda_3 \mathrm{sn}u\,\mathrm{cn}u\,\mathrm{nd}u + \lambda_4 \mathrm{sn}u\,\mathrm{cn}u + \lambda_5 \mathrm{am}^*u. \tag{48}$$

The λ_i are given by the relations

$$\begin{aligned}\lambda_0 &= \epsilon(0), \\ \lambda_1 &= n(0) + n_0(n_1 + n_2) + \epsilon_0(n' - n)(\epsilon_1 K + \epsilon_2 E)/\pi - 2Kn_0n_1/\pi, \\ \lambda_2 &= \epsilon_0\epsilon_2, \\ \lambda_3 &= -\kappa^2\epsilon_0\epsilon_2, \\ \lambda_4 &= \epsilon_0\epsilon_3 + 2n_0n_2/(n - n'), \\ \lambda_5 &= -2\epsilon_0(\epsilon_1 K + \epsilon_2 E)/\pi - 4n_0n_1K/[\pi(n - n')].\end{aligned}$$

Since the mean longitude is often a critical variable in computations, the closed form expression given in Eq.(48) may prove useful in many applications. Of particular interest also is the closed form expression for the averaged mean motion, λ_1.

7. The Resonance Case.

From the values of the coefficients of the Fourier series given by Eqs.(21), we see that $\cos\omega\phi$ and $\sin\omega\phi$ contain the divisor $\omega^2 - j^2$. If ω has a value near an integer this would be a small divisor. From Eqs.(17) we can determine which frequencies of the mean longitudes would lead to integer values of the ω_i's. There are only two classes of solutions, one given by

$$\begin{aligned} n/n' &= p/(p+1), \\ \omega_1 = 2p, \quad \omega_2 &= 2(p-1) \quad \text{and} \quad \omega_3 = 2(p+1), \end{aligned} \tag{49}$$

and the other by

$$\begin{aligned} n/n' &= (2p-1)/(2p+1), \\ \omega_1 = 2p-1, \quad \omega_2 &= 2p-3 \quad \text{and} \quad \omega_3 = 2p+1; \end{aligned} \tag{50}$$

in both cases, p is a positive integer. These solutions still hold if n/n' is replaced by n'/n.

If the ω_j's satisfy either Eq.(49) or (50) then $\cos\omega_j\phi$ and $\sin\omega_j\phi/\sin\phi$ can be expressed as power series in $\cos\phi$ directly from the Chebyshev polynomials in Eqs.(22) without the use of the Fourier series of Eq.(20). This avoids the use of the Fourier coefficients altogether; thus no small divisors will appear in the developments.

If Eq.(49) or Eq.(50) is satisfied exactly, we can write $\omega = N$, where N is a positive integer, and express the real part of the quadrature of Eq.(19) by

$$\int \frac{\cos\omega\phi}{\delta^3}\,d\phi = \int \frac{\cos N\phi}{\delta^3}\,d\phi = I_N^*. \tag{51}$$

Eq.(51) can be considered a definition of I_N^*. Similarly, J_N^* can be defined by

$$\int \frac{\sin\omega\phi}{\delta^3}\,d\phi = \int \frac{\sin N\phi}{\delta^3}\,d\phi = J_N^*. \tag{52}$$

Using the Chebyshev polynomials and the definitions of I_n and J_n in Eqs.(26) and (32), we write for $N \geq 1$

$$\begin{aligned} I_N^* &= \frac{N}{2}\sum_{m=0}^{\lfloor N/2\rfloor}(-1)^m\frac{1}{N-m}\binom{N-m}{m}2^{N-2m}I_{N-2m}, \\ J_N^* &= \sum_{m=0}^{\lfloor (N-1)/2\rfloor}(-1)^m\binom{N-m-1}{m}2^{N-2m-1}J_{N-2m-1}. \end{aligned} \tag{53}$$

If $N = 0$, $\cos N\phi = 1$ and $\sin N\phi = 0$. Thus from Eqs.(51) and (52) we have

$$I_0^* = \int (1/\delta^3)\, d\phi \quad \text{and} \quad J_0^* = \text{constant}.$$

In most problems, we do not have exact resonance but may be merely close to resonance, with small divisors presenting a problem of convergence. The transition from exact resonance to its neighborhood is studied in classical developments by many methods including expanding trigonometric terms into power series in the time. This transition may be studied here without the introduction of power series. We begin the investigation by defining a quantity β according to

$$\omega = N + \beta \quad \text{and} \quad -0.5 \leq \beta \leq 0.5. \tag{54}$$

We shall need the Fourier series of Eqs.(20) with ω replaced by β:

$$\begin{aligned} \cos \beta\phi &= \sigma_\beta \sum_{j=0}^{\infty} a_j(\beta) \cos j\phi, \\ \sin \beta\phi &= \sigma_\beta \sum_{j=1}^{\infty} b_j(\beta) \sin j\phi, \end{aligned}$$

Further we define $\sigma_\beta = \sin 2\beta\pi / 2\beta\pi$. After some algebra we have

$$\begin{aligned} \cos \omega\phi = \sigma_\beta \cos N\phi + \frac{\sigma_\beta}{2} \sum_{1 \leq j \leq \infty} &\{[a_j(\beta) + b_j(\beta)] \cos(N + j)\phi \\ &+ [a_j(\beta) - b_j(\beta)] \cos(N - j)\phi\}. \end{aligned}$$

With this form of $\cos \omega\phi$ substituted into the real part of the quadrature in Eq.(19) , we use Eqs.(51) and (52) to obtain the desired result

$$\int \frac{\cos(N+\beta)\phi}{\delta^3}\, d\phi = \sigma_\beta \left[I_N^* + \beta \sum_{1 \leq j \leq \infty} \left(\frac{I_{N+j}^*}{\beta - j} + \frac{I_{N-j}^*}{\beta + j} \right) \right]. \tag{55}$$

After a similar set of operations is applied to the integration of the imaginary part of Eq.(19), we obtain

$$\int \frac{\sin(N+\beta)\phi}{\delta^3}\, d\phi = \sigma_\beta \left[J_N^* + \beta \sum_{1 \leq j \leq \infty} \left(\frac{J_{N+j}^*}{\beta - j} + \frac{J_{N-j}^*}{\beta + j} \right) \right]. \tag{56}$$

Inspection of Eqs.(53) for I_N^* and J_N^* reveals that they contain no small divisors and that they do not depend on β. Therefore, as $\beta \to 0$, the quadratures in Eqs.(55) and (56) approach the values given in Eqs.(51) and (52), respectively. Thus the solution is well behaved as resonance is approached.

An important resonance in the solar system is that between Jupiter and Saturn where

$$n/n' = 2/5,$$

in which case

$$\omega_1 = 4/3, \quad \omega_2 = -2/3 \quad \text{and} \quad \omega_3 = 10/3.$$

Since the ω_j's are not integers, the developments of Section 4 can be applied and will not produce small divisors at first order. However, elliptical orbits are normally adopted as reference orbits when one is deriving the first order perturbations of the Jupiter-Saturn problem. We adopt circular reference orbits in this paper and it is not clear that the absense of small divisors would persist if we were to adopt elliptical orbits instead.

Although it is interesting that no small divisors appear in our developments, unfortunately there are, instead, secular terms. However, if Eq.(49) or (50) holds, the perturbations will be in closed form because of Eqs.(51) and (52). Further, in the case of Eq.(50), the solution will contain no secular terms. The next section discusses this feature in detail.

8. An Example of a Resonance Solution.

The variables $a(t)$, $\epsilon(t)$, and $\lambda(t)$ are given in closed form for any orbit and the solutions given by Eqs. (36), (37), and (48) are applicable whether we have resonance or not. Therefore, we shall concentrate on the solution for the variables $h(t)$ and $k(t)$. Let us choose, as an example, the resonance $n/n' = 1/3$. In this case, we have $\omega_1 = 1$, $\omega_2 = -1$ and $\omega_3 = 3$. We begin with Eq.(39) appearing in the development of the perturbations of $h(t)$ and write

$$\begin{aligned} H_1(u) - H_1(0) \quad = \quad & [R_1 I^*_{\omega_1}(u) + (R_2/2)(3I^*_{\omega_2}(u) - I^*_{\omega_3}(u))]\cos\phi_0 \\ & - [R_1 J^*_{\omega_1}(u) + (R_2/2)(3J^*_{\omega_2}(u) - J^*_{\omega_3}(u))]\sin\phi_0 \\ & + [R_1 J^*_{\omega_1}(0) + (R_2/2)(3J^*_{\omega_2}(0) - J^*_{\omega_3}(0))]\sin\phi_0. \end{aligned}$$

From the definition of I^*_N and J^*_N in Eq.(53) we see that $I^*_N = I^*_{-N}$ and $J^*_N = -J^*_{-N}$. Thus

$$H_1(u) = \cos\phi_0[(R_1 + 3R_2/2)I^*_1 - R_2 I^*_3/2] - \sin\phi_0[(R_1 - 3R_2/2)J^*_1 - R_2 J^*_3/2].$$

Again, from Eq.(53), we can express I^*_N and J^*_N in terms of I_n and J_n. Then using Table III together with the starting values for I_1, C_1, J_0 and S_0 in Eqs.(29)

and (35), we obtain

$$
\begin{aligned}
I_1^* &= \frac{\mathrm{sn}u}{\mathrm{dn}u}, \\
I_3^* &= -(4\gamma+3)\frac{\mathrm{sn}u}{\mathrm{dn}u} + \frac{4}{\kappa^3}\sin^{-1}(\kappa\mathrm{sn}u), \\
J_1^* &= -\frac{\mathrm{cn}u}{\kappa'^2\mathrm{dn}u}, \\
J_3^* &= \frac{1+4\gamma}{\kappa'^2}\frac{\mathrm{cn}u}{\mathrm{dn}u} + \frac{4}{\kappa^3}\ln(\mathrm{dn}u - \kappa\mathrm{cn}u).
\end{aligned} \tag{57}
$$

Combining the results of Eq.(57) with Eqs.(38), (39), and (41) and writing the results in terms of ϕ from Eq.(13), we obtain

$$
\begin{aligned}
h(t) = h(0) + \nu\{ &- H_1(0) - H_2(0) \\
&+ [R_1 + (3+2\gamma)R_2]\cos\phi_0 \sin\phi/\delta \\
&+ [R_1 - (1-2\gamma)R_2]\sin\phi_0\cos\phi/(\kappa'^2\delta) \\
&- (2R_2/\kappa^3)\cos\phi_0\sin^{-1}(\kappa\sin\phi) \\
&+ (2R_2/\kappa^3)\sin\phi_0\ln(\delta - \kappa\cos\phi) \\
&+ (R_3/6)[9\sin(\phi-\phi_0) - \sin(3\phi+\phi_0)]\}.
\end{aligned} \tag{58}
$$

In a similar manner we derive from Eqs.(42),

$$
\begin{aligned}
k(t) = k(0) - \nu\{ &- K_1(0) - K_2(0) \\
&+ [R_1 - (5+6\gamma)R_2]\sin\phi_0\sin\phi/\delta \\
&- [R_1 - (1+6\gamma)R_2]\cos\phi_0\cos\phi/(\kappa'^2\delta) \\
&+ (6R_2/\kappa^3)\sin\phi_0\sin^{-1}(\kappa\sin\phi) \\
&+ (6R_2/\kappa^3)\cos\phi_0\ln(\delta - \kappa\cos\phi) \\
&- (R_3/2)[\cos(\phi-\phi_0) + \cos(3\phi+\phi_0)]\}.
\end{aligned} \tag{59}
$$

Note that the expressions for the first order perturbations of the eccentric variables $h(t)$ and $k(t)$ are in closed form, are purely periodic and do not contain any small divisors. This will be the case for any resonance which leads to the ω_j's being odd integers. From Eq.(50) we find that this corresponds exclusively to all resonances of the form $n/n' = (2p-1)/(2p+1)$.

9. Conclusions.

The planetary theory presented here has at least four interesting features not found together in other theories. They are:

1. avoidance of expansion of $1/\Delta$ in powers of α,
2. absense of small divisors with the exception of $n - n'$ and $2n - n'$,
3. purely periodic solutions in closed form for orbits satisfying the resonance condition $n/n' = (2p-1)/(2p+1)$,
4. compactness.

The closed form expressions for the perturbations in a, ϵ and consequently λ have been known from the work of previous authors. It is interesting that by continuing to force the solution for the eccentric variables to be expressed in Jacobian elliptic functions and Legendre elliptic integrals, we obtain solutions at exact resonance which are in the form of finite truncations of the infinite series used in the non-resonance cases.

One of the difficulties with this solution is the presence of secular terms introduced into the perturbations of the eccentric variables. Since the source of the secular terms is known, there is some hope that this secular feature can be controlled in some way. An averaging method may prove useful in this regard.

Numerical tests of the theory have not been performed. Since our reference orbits are circular and small divisors are not present, numerically generated periodic orbits of the first kind in the general three body problem might provide good comparisons. Such orbits have been generated by Hadjidemetriou [7].

Because they are analytic and in closed form, the periodic solutions for resonant orbits may be useful as reference orbits for minor planet studies or for linear stability analyses.

In order to be of practical value, the planetary perturbations presented in this paper should be extended to three dimensions. Another necessary extension of this theory is the inclusion of perturbations proportional to the eccentricity. Such extensions would require expansion of $1/\Delta$, not in powers of α, but in powers of e. Although these classical types of expansions are not expected to introduce any difficulties in the use of elliptic functions, nevertheless they would produce enormous complications best handled with computer automated algebra.

A complete planetary theory in terms of elliptic functions has not appeared up to now. It would seem that the avoidance of these functions may have been due to the difficulty of computing with them. Numerically, there are no difficulties with their evaluation. Algebraically, however, they are expected to generate difficulties at higher orders. The lack of availability of computer software to perform algebraic and calculus operations on elliptic functions and integrals has led to a postponement of a second order solution. Hopefully, this is only a temporary situation. The second order solution should shed considerable light on how well this theory can be expected to perform.

Perhaps the most interesting and significant feature of this perturbation method is that it nicely avoids most of the cumbersome developments that have plagued planetary theory for a long time.

Acknowledgments.

The research was initiated at the United States Naval Observatory where the first author spent the Spring term of 1981 as a contractor to the Nautical Almanac Office. It was continued at the University of South Florida with partial support from its Division of Sponsored Research. The final draft of the manuscript was completed while the first author was a summer guest at the National Bureau of Standards as a Mathematician in the Center for Applied Mathematics. The first and third authors wish to thank the Laboratory for Computer Science at M.I.T. for computer time on the MACSYMA consortium computer. We also gratefully acknowledge the assistance of Ellen Golden and Jeff Golden at M.I.T..

C. A. Williams and T. Van Flandern acknowledge extensive discussions with André Deprit who also provided generous help with the preparation of the manuscript. Dr. Bruce Miller is to be thanked for having produced the data for Table III on a Lisp workstation at the National Bureau of Standards. Additional discussions with Peter Kammeyer, David Richardson, Ken Seidelmann, and Victor Slabinski proved very helpful.

References

[1] Bretagnon, P.: 1982, *Astron. Astrophys.* **84**, 329.

[2] Brouwer, D. and Clemence, G. M.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York.

[3] Byrd, P. and Friedman, M.: 1971, *Handbook of Elliptic Integrals for Engineers and Scientists*, Second edition, Springer-Verlag, New York.

[4] Chapront, J.: 1984, *Celest. Mech.* **34**, 165.

[5] Cohen, C. J. and Hubbard, E. C.: 1965, *Astron. J.* **70**, 10.

[6] Danby, J. M. A.: 1962, *Fundamentals of Celestial Mechanics*, Macmillan, New York.

[7] Hadjidemetriou, J. D.: 1976, *Astrophys. Space Science* **40**, 201.

[8] Kinoshita, H. and Nakai, H.: 1984, *Celest. Mech.* **34**, 203.

[9] Nacozy, P. and Diehl, R. E.: 1974, *Celest. Mech.* **8**, 445.

[10] Petrovskaia, M. S.: 1970, *Celest. Mech.* **3**, 121.

[11] Richardson, D. L.: 1982, *Celest. Mech.* **26**, 187.

[12] Standish, E. M. and Williams, J. G.: 1982, *Development Ephemeris DE200-LE200*, Jet Propulsion Laboratory.

[13] Williams, J. G. and Benson, G. S.: 1971, *Astron. J.* **76**, 167.