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GEOMETRIC, DYNAMIC, ORBITAL AND PHOTOMETRIC DATA ON METEOROIDS FROM PHOTOGRAPHIC FIREBALL NETWORKS

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ГЕОМЕТРИЧЕСКИЕ, ДИНАМИЧЕСКИЕ, ОРБИТАЛЬНЫЕ И ФОТОМЕТРИЧЕСКИЕ ДАННЫЕ ПО МЕТЕОРИОДАМ, СФОТОГРАФИРОВАННЫМ В БОЛИДНОЙ СЕТИ

В работе приведены методы и математические выражения для обработки фотографий болидов полученных из многопостовой программы. Из них можно получить геометрические, динамические, орбитальные и фотометрические данные. Этими методами обрабатываются снимки болидов полученные по программе Европейской сети.

Methods and procedures of computing geometric, dynamic, orbital and photometric data from multistation photographic records of fireballs are presented and corresponding mathematical formulae are put forth. These methods and procedures are currently used in evaluating photographic records from the European Fireball Network.

Key words: meteoroids: photographic fireballs — reduction of photographs: geometric, dynamic, orbital and photometric data.

1. Introduction

Since 1947 photographic observations of meteors were carried out at the Ondřejov Observatory (Ceplecha, 1977). They became a systematic double-station program using cameras with rotating shutter since 1951. The success of photographing a meteorite fall (Příbram meteorites) in 1959 initiated a multistation fireball network program now covering a large part of Central Europe territory, each station equipped with an all-sky mirror camera or with a fish-eye camera. During this period, we developed a number of reduction methods and procedures, which we used to compute the geometric, dynamic, orbital and photometric data from our photographic records.

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We published only a small part of these methods and some of them were already changed with the use of big computers. During the last decade, we enclosed all of the actually used methods into one computational program, FIRBAL, ($\approx$4000 Fortran statements). This paper contains equations applied in the majority of our present-day computational methods for reducing the data from fireball photographs, with some preference of methods suitable for cameras with fish-eye objectives. Because the methods for fish-eye objectives located at many stations are the most general case, the other simpler possibilities can be easily derived from them.

We publish preliminary data on photographic fireballs, computed by methods of this paper, in SEAN Bulletin. Due to possibilities of the computer program FIRBAL, the preliminary data are mostly very close to the final revised values from all stations published later in the Bull. Astron. Inst. Czechosl.

2. The Formulation of the Problem

The European Network for observation of fireballs yields photographic records of the entire sky hemisphere taken with fish-eye objectives (Zeiss Distagon $f/3{\cdot}5, f = 30$ mm) with a field of view of 180° at several (at least two) stations. The diameter of the 180° image is about 80 mm. The cameras are fixed and pointed vertically (zenith is approximatelly equal to the center of projection). Each record we are interested in contains also the image of the fireball as seen from different locations at time T. The fireball image is occulted (broken) by a rotating shutter close to the focal plane at regular time intervals (in our case: 12·5 times per second with half-time exposure and half-time occultation of the image). For each of these time marks, we want to compute the height above sea-level, the distance from the station and the relative distance (length) along the fireball trajectory. We want to determine the most probable average trajectory of the fireball. Direction of the trajectory against the fireball flight is traditionally called its radiant. Using the time marks, we want to compute the average distance along the trajectory, the height, the velocity and the deceleration, all these values represented as function of time. We want to determine the brightness of the fireball at individual trajectory points. From the velocity as function of time, we want to determine the initial velocity (no-atmosphere velocity). The observed direction of the fireball trajectory and the absolute value of the initial velocity define the velocity vector. We correct this vector for the Earth's rotation and gravity (zenith attraction) and proceed to the geocentric value of the velocity vector. Subtraction of the Earth's velocity vector from it yields the heliocentric velocity vector of the meteoroid at the point of its orbit, where the Earth was located at time T. These data are enough for computing the meteoroid orbit.

The first step to achieve all this is the computation of unknown directions (right ascension and declination or azimuth and zenith distance) to the individual points of the fireball trail on each photograph. We use stars as fiducial points in this respect and we also use star images for the photometric calibrations.

3. The Astrometric Positional Determinations from Fish-Eye Photographs

We measure the rectangular coordinates x, y of each point on the photographic image; in our case we use a Zeiss Ascorecord device. About 15 to 20 stars (their apparent positions) are used for the definition of the conversion of x, y into a, z, the azimuth and zenith distance. This means 30 to 40 independent equations to be fitted to by the least-squares method. Our definition of coordinate orientation is given by the x-axis pointing to the north, y-axis pointing to the west and the origin of x, y coordinates in approximate zenith (which is identical with the approximate center of projection). The conversion formulae are then

$$(1) \qquad \tan(a - a_0) = (y - y_0)/(x - x_0)$$

and

$$(2) \qquad z = U + Vr + S\exp(Dr)\,,$$

where

$$(3) \qquad r^2 = (x - x_0)^2 + (y - y_0)^2$$

is the distance, r, from the center of projection to the point (x, y). These equations contain 7 unknowns to be determined, a_0, x_0, y_0, U, V, S, D.

The fixed cameras give us stars as star trails. The time of the beginning (of the end) of the exposure defines the azimuths and zenith distances of the beginning (of the end) of the star trails, a_i(cat), z_i(cat), $i = 1, 2, \ldots, n$, where n is the number of positional stars used and "cat" stands for "catalogue" denoting the catalogue α_i, δ_i of stars used for the computation of a_i, z_i.

We can solve equations (1), (2), (3) by linearizing them into the gradients of the unknowns, Δa_0, Δx_0, Δy_0, ΔU, ΔV, ΔS, ΔD. Starting with $a_0 = 0$, $x_0 = 0$. $y_0 = 0$, $U = 0$, $V = 0{\cdot}0323$ arc mm^{-1}, $S = 0{\cdot}00327$ arc, $D = 0{\cdot}113$ as the first approximation (V is the scale of the image in zenith and S, D define the "shortening" of the scale with r; the given numbers hold for Zeiss Distagon $f/3{\cdot}5$, $f = 30$ mm). We can proceed to n equations from differentiating equation (1) and using it to n stars. This first approximation corresponds to the orientation and position of the image in the Zeiss-Ascorecord (x, y)-coordinate system mentioned above. The resulting equations (4) contain a multiplying factor, $\sin z_i$(cat), which converts differences in azimuths to the great-circle angular distances comparable to differences in z_i:

$$(4) \qquad (\sin z_i(\mathrm{cat}))\,\Delta a_0 + \\ + ((y_i - y_0)\sin z_i(\mathrm{cat})/r_i^2)\,\Delta x_0 +$$

$$+ ((x_i - x_0) \sin z_i(\text{cat})/r_i^2)\, \Delta y_0 =$$
$$= (a_i(\text{cat}) - a_i(\text{com})) \sin z_i(\text{cat})\,,$$

where $i = 1, 2, \ldots, n$ and where $a_1(\text{com})$ is the azimuth computed from (1) with the approximate values of a_0, x_0, y_0 (all equal to zero at the first step, but Δa_0, Δx_0, Δy_0 at the second step and so on). We have also another n equations from differentiating equation (2) with definition (3):

$$(5) \quad ((V + DS \exp(Dr_i))(x_0 - x_i)/r_i)\, \Delta x_0 + + ((V + DS \exp(Dr_i))(y_0 - y_i)/r_i)\, \Delta y_0 + + \Delta U + r_i\, \Delta V + (\exp(Dr_i))\, \Delta S + + (r_i S \exp(Dr_i))\, \Delta D = z_i(\text{cat}) - z_i(\text{com})\,,$$

where $z_{i(\text{com})}$ is the zenith distance computed from (2) and (3) with the approximate values of x_0, y_0, U, V, S, D. Altogether we have $2n$ linear equations for 7 unknowns Δa_0, Δx_0, Δy_0, ΔU, ΔV, ΔS, ΔD and we can proceed to the least-squares equivalent of the 7 normal equations for these 7 unknowns. If we denote the starting values of a_0, x_0, y_0, U, V, S, D as "old", we can define the next approximation, the "new" values:

$$(6) \quad \begin{aligned} a_0(\text{new}) &= a_0(\text{old}) + \Delta a_0 \\ x_0(\text{new}) &= x_0(\text{old}) + \Delta x_0 \\ y_0(\text{new}) &= y_0(\text{old}) + \Delta y_0 \\ U(\text{new}) &= U(\text{old}) + \Delta U \\ V(\text{new}) &= V(\text{old}) + \Delta V \\ S(\text{new}) &= S(\text{old}) + \Delta S \\ D(\text{new}) &= D(\text{old}) + \Delta D \end{aligned}$$

The next approximation starts again with equations (4) and (5) with the new values of a_0, x_0, y_0, U, V, S, D. The resulting Δa_0, Δx_0, Δy_0, ΔU, ΔV, ΔS, ΔD define the next new values of all 7 unknowns by (6). This procedure is repeated until the values of Δa_0, Δx_0, Δy_0, ΔU, ΔV, ΔS, ΔD are sufficiently close to zero (less than a prescribed small value, ε). The resulting a_0, x_0, y_0, U, V, S, D and equations (1), (2) and (3) then define any conversion of measured x, y into a, z (or vice versa). Our practical experience with solutions for thousands of good images taken with the above mentioned fish-eye objective speaks for quick convergence in almost all cases and the precision (standard deviation of one measured position) is mostly close to 1 minute of arc, even in cases of almost 90° difference in z_i. The positional stars are usually chosen (if possible) close to the fireball image covering thus more than the whole interval of a and z of the fireball.

The above procedure is convergent in cases of enough positional stars. If the night is less transparent, we sometimes can use only few of the brightest stars. The tactics we then use within the computer program FIRBAL is the following: we start with the full number of 7 unknowns (4 stars and more) and if the procedure cannot find the solution for them, we drop one of the unknowns (the first to be dropped is D) and compute with 6 unknowns and again, if it is not successful, we use 5 unknowns and so on. For the dropped unknowns, we use the standard values. They are close to the values given by the above equation (4), but we determine them for each camera separately, using several good images, because each objective differs from the others in a recognizable way. Another possibility in our computational program is the solution separately for azimuths (3 unknowns of equation (1)) and for zenith distances (4 unknowns of equation (2) and (3) with x_0, y_0 taken from the solution of (1)). The sequence of automatic choice of "lower" procedures if the "higher" procedure does not succeed in finding the solution, is the following: all 7 unknowns; 3 unknowns of equation (1) and 4 unknowns of equation (2) + (3) separately; 3 unknowns of equation (1) and 3 unknowns of equation (2) + (3) (D is chosen as standard value); 2 unknowns a_0, x_0, ($y_0 = 0$) of equation (1) and 2 unknowns U, V, (D = standard value) of equations (2) + (3); 1 unknown a_0 of equation (1) and 1 unknown V of equations (2) + (3); the last regime corresponds to "no stars available" and only the standard values of V, S, D are taken in this case with the standard values of $a_0 = x_0 = y_0 = U = 0$.

We have also incorporated another special regime of work of the positional reduction procedure in the program FIRBAL, if there are only the declination trails of stars available (bad weather at the beginning and end of an exposure, making any time data spurious). We then use the star trails in the meridian section for defining the zenith distances and for determining all 4 unknowns of equation (2) + (3) with parameters of equation (1) all zeroes: $a_0 = x_0 = y_0 = 0$. This regime can also work with a smaller number of stars (star trails) available, dropping gradually and automatically D, S, U, V and using their standard values.

4. Conversion of Measured Coordinates x, y into α, δ and Definition of Geocentric Rectangular System

Now any measured point, and specifically any measured point of the fireball trail with coordinates x, y (Ascorecord system) can be converted into a, z

by means of equations (1), (2) and (3) using the resulting values a_0, x_0, y_0, U, V, S, D. The standard deviations of these 7 unknowns can either be used to determine the standard deviations of any value computed from double-station fireball record or they can be used for computation of statistical weights if records from more than two stations are combined. Azimuth and zenith distance, a, z, the local sidereal time at the station, ϑ_S, of the fireball instant, and the geographical coordinates of the station, φ_S, λ_S, then define the right ascension and declination, α, δ of any measured point and the geocentric position of the station.

All the computations are performed in geocentric coordinates. The conversion of geographic latitude, φ, into geocentric latitude, φ', and the value of the geocentric radius vector at the zero height level, R, we perform by using the following formulae:

$$\begin{aligned}(7)\qquad \varphi' = \varphi &- 0{\cdot}1924240867° \sin 2\varphi + \\ &+ 0{\cdot}000323122° \sin 4\varphi - \\ &- 0{\cdot}0000007235° \sin 6\varphi\end{aligned}$$

$$R = \left(40680669{\cdot}86 \frac{1 - 0{\cdot}0133439554 \sin^2 \varphi}{1 - 0{\cdot}006694385096 \sin^2 \varphi}\right)^{1/2} \text{ in km.}$$

The rectangular geocentric system of coordinates is then given by the following definition:

$$(8)\qquad \begin{aligned} X &= (R + h) \cos \varphi' \cos \vartheta \\ Y &= (R + h) \cos \varphi' \sin \vartheta \\ Z &= (R + h) \sin \varphi' . \end{aligned}$$

Any unit vector in direction of α, δ can be written in the same system of coordinates

$$(9)\qquad \begin{aligned} \xi &= \cos \delta \cos \alpha \\ \eta &= \cos \delta \sin \alpha \\ \zeta &= \sin \delta . \end{aligned}$$

5. The Fireball Trajectory

We strictly separate our measurements of time marks (breaks of the fireball image) from the measurements of the apparent fireball trail (great circle). The great circle of the fireball trail is thus determined independently of measuring the time marks. The wire setting is usually pointed to approximate centers of 10 or more suitable dashes regularly spaced along the whole fireball trail. Overexposed parts are omitted from these measurements. Thus the plane containing the station and the fireball trajectory is defined independently of the time marks and, moreover, independently of those parts of the trail, where the quality of the image is poor. The geometric precision of the resulting trajectory is significantly increased this way.

Each point measured for the determination of the fireball trail can be represented from (9) as ξ_i, η_i, ζ_i, $i = 1, 2, \ldots, k$, where k is the number of all measured points on the fireball trail. If a, b, c is a unit vector perpendicular to the average plane containing the average fireball trajectory, then

$$(10)\qquad a\xi_i + b\eta_i + c\zeta_i = \Delta_i ,$$

where $\Delta_i = 0$ in the ideal case of all measured points being exactly on the same great circle (each (ξ_i, η_i, ζ_i) being perpendicular to (a, b, c)). But Δ_i are small values to be minimized by the choice of unknown vector a, b, c. From the condition

$$\sum_{i=1}^{k} \Delta_i^2 = \text{minimum} ,$$

we can derive the solution of the unknown vector (a, b, c) as (symbol [ ] stands for $\sum_{i=1}^{k}$):

$$(11)\qquad \begin{aligned} a' &= [\xi_i\eta_i] [\eta_i\zeta_i] - [\eta_i^2] [\xi_i\zeta_i] \\ b' &= [\xi_i\eta_i] [\xi_i\zeta_i] - [\xi_i^2] [\eta_i\zeta_i] \\ c' &= [\xi_i^2] [\eta_i^2] - [\xi_i\eta_i]^2 \\ d' &= (a'^2 + b'^2 + c'^2)^{1/2} \\ a &= a'/d' \\ b &= b'/d' \\ c &= c'/d' . \end{aligned}$$

Substituting (a, b, c) into (8) written for station A, where the fireball trajectory was photographed, yields the geocentric position of the plane containing the station A and the trajectory:

$$(12)\qquad a_A\xi + b_A\eta + c_A\zeta + d_A = 0 ,$$

where

$$(13)\qquad - d_A = a_AX_A + b_AY_A + c_AZ_A$$

is the distance of the plane from the Earth's center.

The fireball is photographed from $N \geqq 2$ stations. Any pair of them now define two planes, each plane containing the fireball trajectory. The intersection of these two planes is exactly the fireball trajectory as defined by the fireball photographs taken from these two stations. If one of these stations is A and the other B, we can derive from (12) the intersection of these two planes:

$$\xi_R = (b_A c_B - b_B c_A)/d$$
$$\eta_R = (a_B c_A - a_A c_B)/d \quad (14)$$
$$\zeta_R = (a_A b_B - a_B b_A)/d \, ,$$

where

$$d = ((b_A c_B - b_B c_A)^2 + (a_B c_A - a_A c_B)^2 + (a_A b_B - a_B b_A)^2)^{1/2} \quad (15)$$

It is an easy task to convert ξ_R, η_R, ζ_R into α_R, δ_R the right ascension and declination of the fireball radiant (the direction against the flight of the fireball), if we solve equation (9). If this computed point, α_R, δ_R, is below the horizon ($z_R > 90°$), then it is the "anti-radiant" and the computation proceeds just by changing the sign of the vector ξ_R, η_R, ζ_R. We need also (for computation of statistical weight of the intersection) the value of the angle Q of the two planes given by their normal vectors (a_A, b_A, c_A), (a_B, b_B, c_B).

$$\cos Q_{AB} = |a_A a_B + b_A b_B + c_A c_B| : ((a_A^2 + b_A^2 + c_A^2)(a_B^2 + b_B^2 + c_B^2))^{1/2} \, . \quad (16)$$

If Q is very small, such intersection of planes looses its statistical significance. The statistical weight of the intersection of two planes A, B is proportional to $\sin^2 Q_{AB}$.

6. Projection of any Measured Point of the Fireball Trajectory onto the Average Fireball Trajectory as Defined from 2 Stations

Even the points, which define the fireball trajectory from station A do not lie exactly in the plane containing station A and the fireball trajectory: they differ from it by an angle ψ_i, which can be computed for each point from (10): $\sin \psi_i = \Delta_i$. Also all the other measured points on the fireball record from station A do not fit exactly this plane. Such points are the beginning and the end of the trail, and mostly the time marks. The next task is to find the best (perpendicular) projection of such points measured on the station-A record onto the average fireball trajectory defined by ξ_R, η_R, ζ_R computed from (14) (from station A and B records).

If n is the suffix of any measured point, we can compute ξ_n, η_n, ζ_n of this point from measured x_n, y_n (Ascorecord system of coordinates) using (1), (2), (3), conversion of a_n, z_n into α_n, δ_n, and (9). The position of station A from (8), (X_A, Y_A, Z_A) with (ξ_n, η_n, ζ_n) defines a straight line deviating somewhat from the (station A, fireball trajectory) plane. We define the plane perpendicular to the (station A, fireball trajectory) plane and containing the straight line (X_A, Y_A, Z_A), (ξ_n, η_n, ζ_n). The intersection of this plane with the fireball trajectory is the point we are searching for (the closest point to the measured point, which lies on the average fireball trajectory defined from 2 stations). This plane perpendicular to the (station A, fireball trajectory) plane can be written as

$$a_n \xi + b_n \eta + c_n \zeta + d_n = 0 \, . \quad (17)$$

The vector (a_n, b_n, c_n) and d_n can be computed from

$$a_n = \eta_n c_A - \zeta_n b_A$$
$$b_n = \zeta_n a_A - \xi_n c_A \quad (18)$$
$$c_n = \xi_n b_A - \eta_n a_A$$
$$d_n = -a_n X_A - b_n Y_A - c_n Z_A \, .$$

The intersection X_n, Y_n, Z_n is then given by the three planes that should contain this point:

$$a_A \xi + b_A \eta + c_A \zeta + d_A = 0$$
$$a_B \xi + b_B \eta + c_B \zeta + d_B = 0 \quad (19)$$
$$a_n \xi + b_n \eta + c_n \zeta + d_n = 0$$

and the distance of this point from station A is given

$$r_n = ((X_n - X_A)^2 + (Y_n - X_A)^2 + (Z_n - Z_A)^2)^{1/2} \, . \quad (20)$$

Good checks of numerical computations are the conditions

$$X_n = r_n \xi_n \, , \quad Y_n = r_n \eta_n \, , \quad Z_n = r_n \zeta_n \, . \quad (21)$$

Now the corrected vector $(\xi'_n, \eta'_n, \zeta'_n)$ is given as the intersection of two planes (a_A, b_A, c_A) and (a_n, b_n, c_n):

$$\xi_{An} = b_n c_A - c_n b_A$$
$$\eta_{An} = c_n a_A - a_n c_A$$
$$\zeta_{An} = a_n b_A - b_n a_A$$
$$\xi'_n = \xi_{An}/l_{An} \quad (22)$$
$$\eta'_n = \eta_{An}/l_{An}$$
$$\zeta'_n = \zeta_{An}/l_{An} \, , \quad \text{where}$$
$$l_{An} = (\xi_{An}^2 + \eta_{An}^2 + \zeta_{An}^2)^{1/2}$$

is the length of the vector product, just to keep the vectors to be of unit length. The sign of the vector ξ'_n, η'_n, ζ'_n is defined by the condition that α'_n computed from (9) differs only by a small value from α_n. If this difference is close to 180°, then we only change the sign of the resulting ξ'_n, η'_n. ζ'_n.

The projection of this point (X_n, Y_n, Z_n) (corresponding to the direction $(\xi'_n, \eta'_n, \zeta'_n)$ from station A)

onto the Earth's surface can be computed by solving equation (8), which yields φ'_n, ϑ_n, $R + h_n$. By means of (7) the geocentric latitude, φ'_n can be converted to geographic latitude, φ_{nc} and also R can be computed. The height, h_n, of the point above zero level results by subtracting R. The local sidereal time, ϑ_n, defines the longitude of the point, λ_n (sidereal time intervals are equal to longitude intervals). The only correction, which remains to be done, is the correction of φ_{nc} for the *vertical* projection (and not in the direction of radius vector) of the point (X_n, Y_n, Z_n) to the zero level height to obtain φ_n of this vertical projection

(23) $$\varphi_n = \varphi_{nc} + h_n(\varphi'_n - \varphi_{nc})/(R + h_n)\,.$$

The same procedure (17) to (23) holds also for station B. If the fireball is photographed from two stations only, we arrived at the end of our computations of the fireball trajectory. We use equations (17) to (23) for each of the measurable time marks from station A. We have h_n, r_n, φ_n, λ_n. We can compute distances, l_n, along the fireball trajectory, say, from the first time mark X_1, Y_1, Z_1:

(24) $$l_n = [(X_n - X_1)^2 + (Y_n - Y_1)^2 + (Z_n - Z_1)^2]^{1/2}$$

and analyze these distances as function of time (Pecina and Ceplecha, 1983, 1984) and determine velocities and decelerations at any point of the fireball trajectory. We can do the same for the time marks of station B. We can also compute h, r, φ, λ, l of the beginning and the terminal points of the luminous trajectory of the fireball from station A and from station B.

7. The Average Fireball Trajectory in Case of Photographs Taken from More than 2 Different Stations

If the number of different stations with photographic record of the fireball trajectory is greater than $2 (N > 2)$, then we have N planes, each containing the fireball trajectory and the corresponding station. Because these planes differ slightly from the exact positions due to errors in measurements, we have $\binom{N}{2}$ intersections of these planes. The weighted average intersection of all these planes is the average fireball trajectory we are searching for.

We already computed (a_A, b_A, c_A) and d_A (equations (12) and (13)) defining the plane of the fireball trajectory from station A: this is an "average" trajectory in the sense of using all points measured on the fireball trails from station A and B. The station-B plane of fireball trajectory from the same equations is given by (a_B, b_B, c_B) and d_B. We will compute the direction of the intersection of these two planes, (ξ_R, η_R, ζ_R), as a preliminary direction of the average trajectory from all the stations (it differs very little from the final direction of the trajectory, so that no iterative procedure is necessary). A plane perpendicular to (ξ_R, η_R, ζ_R) is:

(25) $$\xi_R\xi + \eta_R\eta + \zeta_R\zeta + d_R = 0$$

and intersects the two planes from any two stations S and L at a point, which lies on the fireball trajectory defined by these two planes

(26) $$a_S\xi + b_S\eta + c_S\zeta + d_S = 0$$
$$a_L\xi + b_L\eta + c_L\zeta + d_L = 0\,.$$

We now define two different d_R of equation (25) by means of two different points, plane (25) should go through: the beginning point of the luminous fireball trajectory from station A: $(X_{BEG}, Y_{BEG}, Z_{BEG})$, and the terminal point of the luminous fireball trajectory from station A: $(X_{TER}, Y_{TER}, Z_{TER})$.

(27) $$d_R(\mathrm{BEG}) = -\xi_R X_{BEG} - \eta_R Y_{BEG} - \zeta_R Z_{BEG}$$

(28) $$d_R(\mathrm{TER}) = -\xi_R X_{TER} - \eta_R Y_{TER} - \zeta_R Z_{TER}$$

The solution of (25) and (26) with $d_R = d_R(\mathrm{BEG})$ yields a point $(X_{LS}(\mathrm{BEG}), Y_{LS}(\mathrm{BEG}), Z_{LS}(\mathrm{BEG}))$, which corresponds to the intersection of the fireball trajectory as defined from stations L and S with the standard plane (25) (perpendicular to the direction of the approximate fireball trajectory as defined from stations A and B). This point has the statistical weight $G_{LS} = g_L g_S \sin^2 Q_{LS}$, where Q_{LS}, the angle of the two planes, can be computed from (16) and g_L and g_S are the statistical weights of the fireball records from station L and S, respectively: in case of equal positional accuracy, g_L is proportional to the apparent recorded total length of the fireball trail as photographed from station L, and g_S is proportional to the same value from station S. (g_L and g_S can be taken directly as lengths of the fireball trail in millimeters of the record).

We now have $\binom{N}{2}$ different points of all combinations L, S with the statistical weight G_{LS}. The weighted average of these points is the first point on the average trajectory of the fireball $(X(\mathrm{BEG}), Y(\mathrm{BEG}), Z(\mathrm{BEG}))$ somewhere close to the beginning of the trajectory. The second point on the average trajectory, $(X(\mathrm{TER}), Y(\mathrm{TER}), Z(\mathrm{TER}))$, which we need for the definition of the average-trajectory position and direction, can be computed from $(X_{LS}(\mathrm{TER}), Y_{LS}(\mathrm{TER}), Z_{LS}(\mathrm{TER}))$, if (25) and (26) are solved with $d_R(\mathrm{TER})$. This point lies somewhere close to the termination of the fireball luminous trajectory. The difference of coordinates of these two points yields the direction of the *average trajectory* (the average radiant) as it corresponds to the fireball records from all N stations.

$$\begin{aligned}\bar{\xi}_R &= X(\mathrm{BEG}) - X(\mathrm{TER})\\ \bar{\eta}_R &= Y(\mathrm{BEG}) - Y(\mathrm{TER}) \qquad (29)\\ \bar{\zeta}_R &= Z(\mathrm{BEG}) - Z(\mathrm{TER})\,.\end{aligned}$$

Using (9), we can solve also for $\bar{\alpha}_R$, $\bar{\delta}_R$ of this average radiant.

8. Projection of any Measured Point of the Fireball Trajectory at any station *S* onto the Average Fireball Trajectory Defined from All *N* Stations

In section 6 we proceeded to equations (19), which can be used to compute the geocentric rectangular coordinates X_n, Y_n, Z_n of any point on the fireball trajectory. The measured direction, which generally does not point exactly to the fireball trajectory, was perpendicularly projected onto the trajectory. In section 6 the fireball trajectory was defined from two stations. The same procedure as in Section 6 can be used, if the trajectory is defined as an average trajectory from all N stations by $(X(\mathrm{BEG}), Y(\mathrm{BEG}), Z(\mathrm{BEG}))$ and $(\bar{\xi}_R, \bar{\eta}_R, \bar{\zeta}_R)$, from equation (29). If we have any point on the fireball trajectory measured from any station S (ξ_n, η_n, ζ_n), then the direction to it deviates, generally, from the direction to the average trajectory by angle ψ_n: $\sin\psi_n = \Delta_n$ defined by equation (10) with unknown a_S, b_S, c_S, which would correspond to the plane containing station S and the average fireball trajectory $(X(\mathrm{BEG}), Y(\mathrm{BEG}), Z(\mathrm{BEG}))$, $(\bar{\xi}_R, \bar{\eta}_R, \bar{\zeta}_R)$. The vector (a_S, b_S, c_S) can be computed from:

$$\begin{aligned}i &= \bar{\eta}_R(Z_S - Z(\mathrm{BEG})) - \bar{\zeta}_R(Y_S - Y(\mathrm{BEG}))\\ j &= \bar{\zeta}_R(X_S - X(\mathrm{BEG})) - \bar{\xi}_R(Z_S - Z(\mathrm{BEG}))\\ k &= \bar{\xi}_R(Y_S - Y(\mathrm{BEG})) - \bar{\eta}_R(X_S - X(\mathrm{BEG}))\\ m &= (i^2 + j^2 + k^2)^{1/2}\\ a_S &= i/m \qquad (30)\\ b_S &= j/m\\ c_S &= k/m\\ d_S &= -a_S X_S - b_S Y_S - c_S Z_S\,.\end{aligned}$$

The plane containing the station S and the average trajectory is then

$$a_S\xi + b_S\eta + c_S\zeta + d_S = 0\,. \qquad (31)$$

The plane, which is perpendicular to plane (31) and contains the measured point (ξ_n, η_n, ζ_n) (the point, which lies close to the average fireball trajectory, deviating by ψ_n), can be written as equation (17), where a_n, b_n, c_n, d_n are given by (18), but written with (a_S, b_S, c_S), (X_S, Y_S, Z_S) instead of (a_A, b_A, c_A), (X_A, Y_A, Z_A).

The plane perpendicular to (31) and containing the average meteor trajectory is given by

$$a_R\xi + b_R\eta + c_R\zeta + d_R = 0\,, \qquad (32)$$

where

$$\begin{aligned}i_R &= \bar{\eta}_R c_S - \bar{\zeta}_R b_S\\ j_R &= \bar{\zeta}_R a_S - \bar{\xi}_R c_S\\ k_R &= \bar{\xi}_R b_S - \bar{\eta}_R a_S \qquad (33)\\ m_R &= (i_R^2 + j_R^2 + k_R^2)^{1/2}\\ a_R &= i_R/m_R\\ b_R &= j_R/m_R\\ c_R &= k_R/m_R\\ d_R &= -a_R X(\mathrm{BEG}) - b_R Y(\mathrm{BEG}) - c_R Z(\mathrm{BEG})\,.\end{aligned}$$

The intersection of the three planes, (31), (17) and (32), gives the solution $\xi = \bar{X}_n$, $\eta = \bar{Y}_n$, $\zeta = \bar{Z}_n$, which is the point $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$ of the perpendicular projection of the measured trajectory point onto the average fireball trajectory. Its distance from station S, $\bar{r}_n$, is given by (20), where $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$, (X_S, Y_S, Z_S) are written instead of (X_n, Y_n, Z_n), (X_A, Y_A, Z_A), respectively. Using equations (22) for station S, we can convert $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$ into the corrected direction from station S to the average-fireball-trajectory point. The projection of $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$ onto the Earth's surface is described at the end of Section 6, and equations (23) and (24) can be used in complete analogy as in case of the solution for fireball trajectory given by two stations only. The resulting values for station S and the average fireball trajectory are $\bar{h}_n$, $\bar{r}_n$, $\bar{\varphi}_n$, $\bar{\lambda}_n$, $\bar{l}_n$ for each point measured on the station $-$ S record of the trajectory.

9. Time Marks

In general the time marks are given by any timed occultation of the moving fireball image. The simplest case is a rotating shutter in front of a classical camera objective, giving a time-mark each $1/n_B$ second, where n_B is the number of occultations of the objective per second. In our case of fish-eye objective, we placed our rotating shutter very close to the focal plane, and the progressive motion of the shutter combines with the motion of the fireball image. Thus a correction for this effect is necessary.

If we define the relative time as zero at the first measurable time-mark ($t = 0$ for $l = l_1$), then the relative time, t_n, of time mark number n is

$$(31) \qquad t_n = (l_n - l_1 + n_{SR}\,\Delta\varphi_n/(2\pi))/f\,,$$

where n_{SR} is the number of rotating shutter arms ($n_{SR} = 2$ in our case), f is the number of rotations of the shutter motor per second and $\Delta\varphi_n$ is an angle between the position of the shutter at the occultation instant of time mark 1 and time mark n and is given by

$$(32) \qquad \Delta\varphi_n = \arctan\left((x_1 - x_c)/(y_1 - y_y)\right) - \arctan\left((x_n - x_c)/(y_n - y_c)\right),$$

where (x_c, y_c) are the rectangular coordinates of the axis of the rotating shutter motor, (x_1, y_1) are the rectangular coordinates of the time mark 1 and (x_n, y_n) are rectangular coordinates of the time mark n, all these rectangular coordinates in the original system of the Ascorecord measured coordinates of the image. In our case, x_c, is nearly 0 and $y_c \approx 52$ or -52 mm, depending on the two opposite orientations we exclusively use for our fixed cameras.

10. Length, Velocity, Deceleration as Function of Time

Having l_n and $\bar{h}_n$ for each t_n, we can use different methods to compute the best fit of them as functions of time: different sorts of interpolation formulae; numerical differentiation of l_n and some sort of smoothing the resulting velocities; the exact solution of the single-body motion equations of meteor physics (Pecina and Ceplecha, 1983, 1984). The last method is preferable in all cases, where enough change of velocity is inherent in the measured l_n.

One of the parameters of the problem of smooth fitting of l_n to t_n is always v_∞, the initial (no-atmosphere) velocity. This value corresponds to velocity very high in the atmosphere before the ablation of the meteoroid starts and before the deceleration overweights the Earth's gravity. The initial velocity, v_∞, and the radiant $(\bar{\xi}_R, \bar{\eta}_R, \bar{\zeta}_R) \equiv (\bar{\alpha}_R, \bar{\delta}_R)$ define the initial velocity vector, which is used for the meteoroid orbit computations together with the average velocity, $\bar{v}$, at the average point somewhere in the middle of the average trajectory (the same direction $\bar{\alpha}_R, \bar{\delta}_R$ is used for $\bar{v}$).

11. The Fireball Orbit

We start with the average observed values v_∞, $\bar{v}$ and $\bar{\alpha}_R$, $\bar{\delta}_R$. First, we correct the observed velocity vector, $\bar{v}$, $\bar{\alpha}_R$, $\bar{\delta}_R$ for the Earth's rotation. The Earth's rotation velocity, v_E, is given by

$$(33) \qquad v_E = 2\pi(\bar{R}_n + \bar{h}_n)\cos\varphi'_n/86\,164{\cdot}09 \quad \text{in km/s}\,,$$

if $(\bar{R}_n + \bar{h}_n)$, the radius vector to the average point $(\bar{X}_n, \bar{Y}_n, \bar{Z}_n)$ on the average fireball trajectory, where $\bar{v}$ was derived, is expressed in kilometers. Here φ'_n is the geocentric latitude of this average point. If the geocentric coordinates of the observed radiant are given by (9), we have the corrected velocity vector $\bar{v}_c(v_{xc}, v_{yc}, v_{zc})$ given by the observed average velocity vector $\bar{v}(\bar{v}_x, \bar{v}_y, \bar{v}_z)$, where $\bar{v}_x = |\bar{v}|\,\bar{\xi}_R$, $\bar{v}_y = |\bar{v}|\,\bar{\eta}_R$, $\bar{v}_z = |\bar{v}|\,\bar{\zeta}_R$ and by the corresponding geocentric representation of (33):

$$(34) \qquad \begin{aligned} v_{xc} &= \bar{v}_x - v_E\cos\alpha_E \\ v_{yc} &= \bar{v}_y - v_E\sin\alpha_E \\ v_{zc} &= \bar{v}_z \end{aligned}$$

where α_E is the right ascension of the east point corresponding to latitude $\bar{\varphi}_n$ and longitude $\bar{\lambda}_n$ of the average point of the average fireball trajectory,

The next step is the correction of $\bar{v}_c$ to the Earth's gravity, which will give the geocentric velocity vector v_G (v_{Gx}, v_{Gy}, v_{Gz}). First, we correct v_c for the no-atmosphere value (not changing the direction of the vector) by adding the difference of the initial velocity minus the average velocity to the absolute value of vector $\bar{v}_c$. The no-atmosphere value of $\bar{v}_c$ is then $v_{\infty c}$:

$$(35) \qquad v_{\infty c} = \bar{v}_c + v_\infty - \bar{v}$$

and the absolute value v_G of the geocentric velocity vector is then

$$(36) \qquad v_G = (v_{\infty c}^2 - 79\,7201{\cdot}0/(\bar{R}_n + \bar{h}_n))^{1/2} \quad \text{in km}\,, \quad s \text{ units}\,.$$

The coordinates of (34) can be transformed by (9) into α_c, δ_c, the right ascension and declination of the radiant corrected for the Earth's rotation. Then z_c is computed from

$$(37) \qquad \cos z_c = \sin\delta_c\sin\bar{\varphi}'_n + \cos\delta_c\cos\bar{\varphi}'_n\cos(\vartheta_n - \alpha_c)\,.$$

In (37), the geocentric latitude, $\bar{\varphi}'_n$, of the average point on the average trajectory is used and thus z_c is the zenith distance from the "geocentric zenith". This is done because the gravity acts gradually upon the change of direction of the meteoroid approaching the Earth and the average point of the gravity action lies $2R$ from the Earth's center. Thus the meteoroid motion is governed mostly by the whole Earth's body and not only by the particular part over which it was observed. We correct now z_c by $\Delta z_c > 0$ as given by

(38)
$$\Delta z_c = 2 \arctan \left((v_{\infty c} - v_G) \tan (z_c/2)/(v_{\infty c} + v_G) \right)$$

and

(39)
$$z_G = z_c + \Delta z_c$$

is the zenith distance of the geocentric radiant. Its azimuth did not change from the $\bar{v}_c$-value and can be computed from α_c, δ_c (again, geocentric latitudes are used).

(40)
$$a_G = a_c$$

We can transform a_G, z_G into the right ascension and declination of the geocentric radiant, α_G, δ_G, by $\bar{\varphi}'_n$, ϑ'_n. Because we used apparent coordinates of stars to convert x, y coordinates measured on the photographic record into the apparent azimuths and zenith-distances, a, z, the resulting α_G, δ_G are also in the apparent coordinate system. It is usual to convert them into coordinates of some standard epoch (1950·0 or J 2000·0), which is a standard procedure described in "The Astronomical Almanac", and depends on precessional and nutational constants and the time elapsed from or to the closest standard epoch.

Having v_G, α_G, δ_G, we can compute the heliocentric velocity vector, v_H, L_H, B_H, of the meteoroid in orbit at collision with the Earth (L, B are the ecliptical longitude and latitude). First we convert the apparent α_G, δ_G into ecliptical longitude and latitude, L_G, B_G for the epoch of the closest beginning of the year or the middle of the year. The heliocentric ecliptical system of rectangular coordinates we define as

(41)
$$X = r \cos L \cos B$$
$$Y = r \sin L \cos B$$
$$Z = r \sin B$$

where r is the distance from the Sun (radius vector). The position of the Earth in this system can be found either directly in "The Astronomical Almanac" or can be computed from the solar longitude, L_{SUN}, converted to the coordinate system of the closest beginning of the year) minus 180°, ($L_{SUN} - 180°$), and from the radius vector of the Earth, r, using definition (41). The velocity vector of the Earth in orbit can be computed from the time change of the solar longitude, L_{SUN}, and from the time change of the radius vector, r. If V_{AP} is the velocity of the Earth in AU per solar day, and t the time in solar days, then

(42)
$$V_{AP} = \left[\left(\frac{dr}{dt} \right)^2 + \left(r \frac{dL_{SUN}}{dt} \right)^2 \right]^{1/2} .$$

The direction of V_{AP} is given by the ecliptical longitude of the Earth's apex, L_{AP}:

(43)
$$L_{AP} = L_{SUN} - \pi/2 - \left(\frac{dr}{dt} \right) \Big/ \left(r \frac{dL_{SUN}}{dt} \right) .$$

(All angles in (43) are in angular measure of arc)

The rectangular coordinates of the heliocentric velocity of the meteoroid, $v_H (v_{Hx}, v_{Hy}, v_{Hz})$, can be computed from

(44)
$$v_{Hx} = -v_G \cos L_G \cos B_G + V_{AP} \cos L_{AP}$$
$$v_{Hy} = -v_G \sin L_G \cos B_G + V_{AP} \sin L_{AP}$$
$$v_{Hz} = -v_G \sin B_G .$$

Equations (41) can also be written for velocities

(45)
$$v_{Hx} = v_H \cos L_H \cos B_H$$
$$v_{Hy} = v_H \sin L_H \cos B_H$$
$$v_{Hz} = v_H \sin B_H .$$

Thus from (44) we can compute all rectangular components of the heliocentric velocity vector and substituting it with opposite sign into (45), we can also find the heliocentric radiant of the fireball, L_H, B_H, and the heliocentric velocity, v_H. Because in (44), the velocity V_{AP} is in AU per solar day and v_G from our photographic records was computed in km/s, we need a conversion factor. It holds (system of the IAU (1976) astronomical constants):

(46)
$$v_G \, [\text{km/s}] = 1731{\cdot}456829 v_G \, [\text{AU/solar day}] .$$

All velocities in (44) must be in the same system of units. (In many papers on meteor orbits only rough values of conversion factor were used, mainly based on wrong content of 1 AU in km.)

Now, if the velocities are in units of AU/solar day, we have the semimajor axis, a, of the orbit from

(47)
$$a = k^2 r / (2k^2 - r v_H^2) ,$$

where k is the Gaussian gravitational constant in AU-solar day-solar mass units ($k = 0{\cdot}01720209895$). The longitude of the ascending node, Ω, depends on the sign of B_H:

(48)
$$\text{for} \quad B_H > 0 , \quad \Omega = L_{SUN} ;$$
$$\text{for} \quad B_H < 0 , \quad \Omega = L_{SUN} - \pi .$$

The inclination of the orbit, i, is given by its cosine and sine:

(49)
$$\sqrt{p} \cos i = (r v_{Hx} \sin L_{SUN} - r v_{Hy} \cos L_{SUN})/k$$
$$\sqrt{p} \sin i = -r v_{Hz} \sin L_{SUN} / (k \sin \Omega) \quad \text{or}$$
$$\sqrt{p} \sin i = -r v_{Hz} \cos L_{SUN} / (k \cos \Omega) .$$

Eccentricity of the orbit, e, and the true anomaly, v, can be computed from

$$(50)\quad e \sin v = -\sqrt{(p)}\,(v_{Hx} \cos L_{SUN} + v_{Hy} \sin L_{SUN})/k$$
$$e \cos v = p/r - 1\,,$$

where p is given by equations (49).

The perihelion argument, ω, depends on the sign of B_H:

$$(51)\quad \text{for } B_H > 0\,,\quad \omega = \pi - v\,;$$
$$\text{for } B_H < 0\,,\quad \omega = -v\,.$$

If the orbit is eliptical ($a > 0$), the perihelion distance, q, and the aphelion distance, Q, can be computed from geometrical relations

$$(52)\quad q = a(1 - e)$$
$$Q = a(1 + e)$$

If the true anomaly, v, is converted into the mean anomaly, μ, we can compute the time DT elapsed from the last perihelion passage of the meteoroid:

$$(53)\quad \mathrm{D}T = (\mu a^{3/2})/k\,.$$

All the angular orbital elements now hold for the closest beginning of year (middle of the year) and have to be converted into values holding for some standard epoch (1950·0 or J 2000·0) using the formula and numerical parameters from "The Astronomical Almanac".

12. The Photometry

The Zeiss-Distagon ($f/3{\cdot}5$, $f = 30$ mm) fish-eye photographs can also be used for the fireball photometry. With enough comparison stars (star-trails), the precision of photometry of $\pm 0{\cdot}1$ to $\pm 0{\cdot}2$ stellar magnitudes can be achieved in the entire field from zenith (as center projection) down to zenith distance of 70°. The last 20° to the horizon are not so good, but can still be used photometrically with limited precision of $\pm$ several tenths of stellar magnitudes. The biggest trouble in photometry of a fireball trail lies in the fact that usually a good part of the trail is the brightest object on the whole photographic record and thus the extrapolation of the characteristic density curve sometimes yields results with standard deviations exceeding 1 stellar magnitude.

The definition of the diameter of a stellar image as function of the star brightness is so good in the interval $0° < z < 70°$ that measurements of the diameter of the stellar image (thickness of the star trail) can be used for photometric purposes.

Using 10 to 15 comparison stars for photometry, we can construct the characteristic density curve. We transform the catalogue magnitudes in the international 5-color *UBVRI* system to our "panchromatic" system of magnitudes, V_p, (emulsion ORWO NP-27):

$$(54)\quad V_p = V + 0{\cdot}62(B - V) - 0{\cdot}52(V - R)\,,$$

and we use V_p for the definition of the characteristic density curve. The apparent V_p is a function of zenith distance, z, and the velocity v_t (mm/s) of the image (trailing velocity, fixed camera: daily motion): $V_p = V_p(z, v_t)$. We use the standard trailing velocity $v_t = 0{\cdot}001$ mm/s for our focal length of 30 mm. The values of V_p computed from (54) are $V_p = V_p(0, v_t)$. For any z and v_t, we can then correct them to a given z and the standard trailing velocity by

$$(55)\quad V_p(z, 0{\cdot}001) = V_p(0, v_t) + K(1/\cos z - 1) + 2{\cdot}5 \log (v_t/0{\cdot}001)$$

The coefficient K is a combination of the extinction coefficient and of the coefficient describing the weakening of the image with distance from the center of projection (the center of projection is approximately in zenith). Zeiss Distagon $f/3{\cdot}5$, $f = 30$ mm, at an average night transparency in Europe gives about $K = 0{\cdot}35$. Using now V_p's from (55) and the measured widths, w, of star trails, we can construct the characteristic density curve. Measuring then w for the points on the fireball trail, we can ascribe V_p in system of (55) to each measured fireball point. We transform then these V_p's into the absolute (100 km distance) panchromatic fireball magnitudes, M_p, by taking into account the difference from system (55) in z and v_t and the difference of r from 100 km and finally the difference of occultation by the rotating shutter (star images are occulted by the shutter, but the fireball image is not). Then the resulting absolute panchromatic magnitude of any fireball point can be computed from

$$(56)\quad M_p = V_p(z,/0{\cdot}001) - K(1/\cos z - 1) - 2{\cdot}5 \log (v_f/0{\cdot}001) - 5 \log (r/100) + 0{\cdot}75\,,$$

where v_f is the trailing velocity in mm/s of the fireball image (it is easy determined at any point from the measured coordinates x, y of the time marks), r is the distance of the fireball point from the station and the correction 0·75 magnitudes originates from the rotating shutter occultations of the comparison stars.

13. The Dark Flight and the Impact Point

If a fireball penetrates very deep into the atmosphere, the computed velocities and decelerations at the end of the luminous trajectory may yield a computed mass of the body as high as a hundred grams and more. In these cases, a meteorite fall may follow the fireball and we are interested in predicting the meteorite impact point from the data of the luminous trajectory of the fireball. After terminating the luminous trajectory, the body continues in its flight in a "dark-flight trajectory" without emitting the light. The last measured velocity and deceleration at the terminal point, the position of the terminal point and the direction of flight define completely the solution of this problem. One of the complications in the computations of the dark flight lies in the poorly-known wind field and the main uncertainity originates from the unknown shape of the body, which we have to assume to be symmetrical. The equations of motion of a non-ablating body can be written as

(57)
$$(\mathrm{d}v_l/\mathrm{d}h) = (-\Gamma S\varrho v(V_1 + v_1) - 2\omega(v_x \sin\varphi + v_h \cos\varphi \sin a_R))/v_h$$
$$(\mathrm{d}v_h/\mathrm{d}h) = (-\Gamma S\varrho v v_h - g + 2\omega \cos\varphi \,.\, (v_l \sin a_R + v_x \cos a_R))/v_h$$
$$(\mathrm{d}v_x/\mathrm{d}h) = (\Gamma S\varrho v(V_x + v_x) + 2\omega(v_l \sin\varphi - v_h \cos\varphi \cos a_R))/v_h\,,$$

where v, the velocity of the meteoroid is composed of 3 perpendicular components: the vertical plane containing the fireball trajectory contains the horizontal component of the velocity, v_l; the same vertical plane contains the vertical velocity component, $v_h = \mathrm{d}h/\mathrm{d}t$; the horizontal direction perpendicular to the vertical plane containing the fireball trajectory contains the other horizontal component of velocity, v_x (perpendicular to the fireball trajectory in horizontal direction). The signs of the velocity components are chosen so that $v_l > 0$ is in the direction of the meteoroid flight, so that $v_h > 0$ is up (in the real problem of the meteoroid motion $v_h < 0$ always), and so that $v_x > 0$ to the right hand side viewing along the meteoroid motion. The other symbols of equations (57) are: V_l, V_x the wind velocity components, $V_l > 0$ *against* the meteoroid motion, $V_x > 0$ *against* the positive direction of v_x; Γ the drag coefficient as function of Mach number $\Gamma = \Gamma(M)$ (and thus function of v and the air temperature); $S = m/s$ the ratio of meteoroid mass, m, and meteoroid head crossection, s; ϱ the air density; φ the geographic latitude; a_R the astronomical azimuth of the direction of the meteoroid flight (south: $a_R = 0°$; west: $a_R = 90°$); ω the angular velocity of the Earth's rotation ($2\pi/86\,164$). The Coriolis-force terms in (57) can be omitted; they are only a small correction factor in the majority of practical cases. The solution of differential equations (57) can be performed numerically and we use the Runge-Kuta method of integration in this case. We usually chose the integration step $\mathrm{d}h = 0{\cdot}01$ km, but higher up in the atmosphere even bigger steps do not decrease the accuracy of computations. At each integration step, the velocity of the meteoroid is computed from

$$v^2 = v_h^2 + (v_l + V_l)^2 + (v_x + V_x)^2\,. \tag{58}$$

The initial values of v_l, v_h, v_x for the integration of system (57) with (58) are given by

$$\begin{aligned} v_l &= v_T \sin z_R \\ v_h &= -v_T \cos z_R \\ v_x &= 0\,, \end{aligned} \tag{59}$$

where v_T is the velocity at the terminal point of the luminous trajectory of the fireball computed from the measured lengths along the fireball trajectory (see section 10) and z_R is the zenith distance against the direction of flight (zenith distance of the radiant). The initial value of ΓS is given by the drag equation written for the terminal point of the luminous trajectory. This initial value of ΓS is completely given by the observations: implicitly it contains the unknown mass, shape, density and drag coefficient of the meteoroid. Thus the *relative change of ΓS* during the decreasing velocity is the only necessary assumption for solving (57). We change Γ as function of Mach number, M, for a symetrical shape and assume S constant (it means no change in the shape and orientation of the flight position takes place).

$$(\Gamma S)_T = -(\mathrm{d}v/\mathrm{d}t)_T/(\varrho_T v_T^2)\,, \tag{60}$$

where v_T and $(\mathrm{d}v/\mathrm{d}t)_T$ are the velocity and deceleration observed at the terminal point of the fireball luminous trajectory and ϱ_T is the air density at the terminal height h_T. The air densities for low penetrating fireballs are usually available from meteorological aeronomical measurements, or higher up (between 30 to 40 km) they can be extrapolated from them with the standard relative change taken from CIRA 1972 atmopshere (month and geographic latitude averages). The wind direction and velocity is given from aeronomic data and we usually take the closest values in time and location of the fireball and extrapolate (if the locations are not separated by singularities in meteorological situation). From aerological measurements, we can determine at each integration step for the particular height:

$$T = T(°\mathrm{C}) + 273{\cdot}15$$

$$\varrho = (3{\cdot}483676P/T) \times 10^{-4}\ \mathrm{g\,cm^{-3}}$$
$$c = 0{\cdot}0200468T^{1/2}\ \mathrm{km\,s^{-1}}\,,$$

where temperature, T, is the absolute temperature, P the air pressure in millibars (hectopascals), ϱ the air density in $\mathrm{g\,cm^{-3}}$ ($\mathrm{Mg\,m^{-3}}$). We are also interested in the total length of the dark flight from the terminal point of the luminous trajectory to the impact point: L denotes the component of this length in direction of the flight and L_x denotes the component perpendicular to the flight (positive to the right hand side viewing along the fireball flight), both these components are located in the horizontal plane or more accurately, in the zero level height of geoid. If h_s is the height above sea level of the Earth's surface at the impact point and h_T is the height of the terminal point of the luminous trajectory, then

$$(62)\qquad L = \int_{h_s}^{h_T} (v_l/v_h)\,\mathrm{d}h$$
$$L_x = \int_{h_s}^{h_T} (v_x/v_h)\,\mathrm{d}h\,.$$

We can compute the integrals (62) at each numerical step, $\mathrm{d}h$, of the Runge-Kuta solution replacing h_s by h. Thus we can compute the partial lengths $L = L(h)$, $L_x = L_x(h)$ as function of height from h_T down to h_s. The starting values of them are $L(h_T) = L_x(h_T) = 0$.

The last value from aerological measurements we need at each step of numerical solution of (57) with (58) is the direction of the wind. It is given by geodetic azimuth, a_V, from where the winds blows: nothern wind $a_V = 0°$, eastern wind $a_V = 90°$. Keeping this definition for the wind direction and keeping the astronomical azimuths for the meteoroid direction of flight, we can compute at each step of integration of (57) with (58) the following values:

$$(63)\qquad V_l = V\cos(a_V - a_R)$$
$$V_x = V\sin(a_V - a_R)\,,$$

where V is the total wind velocity. If the components of the geographical coordinates along the flight are denoted by suffix l, the components of them perpendicular to the flight are denoted by suffix x (positive to the right-hand side), then we have at each step:

$$a_x = a_R + 90°$$
$$\mathrm{d}\varphi_L = [\cos a_R/(R + h)]\,\mathrm{d}L$$
$$(64)\qquad \mathrm{d}\lambda_L = [\sin a_R/((R + h)\cos\varphi)]\,\mathrm{d}L$$
$$\mathrm{d}\varphi_x = [\cos a_x/(R + h)]\,\mathrm{d}L_x$$
$$\mathrm{d}\lambda_x = [\sin a_x/((R + h)\cos\varphi)]\,\mathrm{d}L_x$$

and the total change of geographical coordinates

$$(65)\qquad \mathrm{d}\varphi = \mathrm{d}\varphi_L + \mathrm{d}\varphi_x$$
$$\mathrm{d}\lambda = \mathrm{d}\lambda_L + \mathrm{d}\lambda_x\,.$$

The instantaneous azimuth and zenith distance against the direction of the meteoroid flight (of the radiant) is given by

$$(66)\qquad a = a_R + \arctan(v_x/v_l)$$
$$z = \arctan[(v_l^2 + v_x^2)/v_h^2]^{1/2}\,.$$

The numerical solution of (57) with (58) starts at the terminal height, h_T, of the luminous trajectory above the point given by geographical coordinates φ_T, λ_T, with v_l, v_h, v_x given by (59) and $(\Gamma S)_T$ given by (60). We then proceed step by step using the aeronomical data and equations (61), (62), (63), (64), (65), (66) and a tabulated function $\Gamma(M)$. ($\Gamma(4) = 0{\cdot}580$, $\Gamma(3) = 0{\cdot}618$, $\Gamma(2) = 0{\cdot}632$, $\Gamma(1{\cdot}5) = 0{\cdot}596$, $\Gamma(1{\cdot}2) = 0{\cdot}552$, $\Gamma(1) = 0{\cdot}504$, $\Gamma(0{\cdot}8) = 0{\cdot}441$, $\Gamma(0{\cdot}6) = 0{\cdot}389$, $\Gamma(0{\cdot}4) = 0{\cdot}351$, $\Gamma(0{\cdot}2) = 0{\cdot}328$). Thus for each height, h, we can compute v_l, v_h, v_x, v, L, L_x, φ, λ, a_R, z_R. We continue until the height is equal to the height of the Earth's surface, h_s, and we then have for the impact point: $v(h_s)$ (the impact velocity), $L(h_s)$, $L_x(h_s)$, $\varphi(h_s)$, $\lambda(h_s)$, $a_R(h_s)$, $z_R(h_s)$. The standard deviations of all measured values yield the standard deviations of these computed quantities and define so-called impact area, where the meteoritic body may have landed with the probability of being inside space of one standard deviation.

All these computations are independent of the meteoroid mass. However, we can estimate the terminal mass of the meteoroid using ΓS computed from (60) and substituting for $\Gamma = 0{\cdot}58$:

$$(67)\qquad m = (0{\cdot}05016/\Gamma S))^3 \quad \text{for the body density}$$
$$\varrho_m = 3{\cdot}7\ \mathrm{Mg/m^3}$$
$$m = (0{\cdot}07094/(\Gamma S))^3 \quad \text{for the body density}$$
$$\varrho_m = 2{\cdot}2\ \mathrm{Mg/m^3}\,.$$

14. The Problem of Standard Deviations. Computer Program FIRBAL

The values computed from the equations of this paper depend on the accuracy of observational values, on the accuracy of measurements of x, y coordinates on the photographic record and on the accuracy of the time data. The standard deviation of any of the computed values can be determined as well. It depends on the partial derivatives according to independent parameters defining the computed value. Mostly, these partial derivatives can be expressed in closed analytical form. If not, they can always be computed numerically by repeating the computations with a small change of the one parameter in question. Having n independent parameters, p_i $(i = 1, 2, \ldots, n)$, and their standard deviations, ε_i, we can compute the standard deviation of any function $\mathrm{f} = \mathrm{f}(p_i)$ from

$$\varepsilon_{\mathrm{f}}^2 = \sum_{i=1}^{n} \left(\frac{\partial \mathrm{f}}{\partial p_i}\right)^2 \varepsilon_i^2 . \tag{68}$$

All equations and procedures of this paper and procedures for the standard-deviation computations (and some more procedures) were incorporated in one computer program FIRBAL, ($\approx$4000 Fortran statements). This program is available at the Ondřejov Observatory. The majority of the computed and printed values are accompanied by their standard deviations. As a numerical example of all equations in this paper, you can use the values published for some of our photographic fireballs in several papers (the most suitable are: Ceplecha et al., 1976; Ceplecha, 1977; Ceplecha et al., 1979).

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