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GUST 86. An analytical ephemeris of the Uranian satellites

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Summary. The General Uranus Satellite Theory GUST (Laskar, 1986) is used for the construction of an analytical ephemeris for the Uranian satellites. The theory is fitted against Earth-based observations from 1911 to 1986 and all radio and optical data obtained during Voyager encounter with Uranus. Earth-based observations alone allow the determination of masses which are within 15% of the values determined by the Uranus flyby. The analysis of all the observations confirm the values of the masses obtained during the encounter (Stone and Miner, 1986) and give a complete set of dynamical parameters for the analytical theory. We obtain an analytical ephemeris, GUST86, with an estimated precision of about 100 km with respect to Uranus.

Key words: astrometry – celestial mechanics – perturbation theory – ephemerides – Uranus – satellites.

1. Introduction

In a previous paper (Laskar, 1986) a general analytical theory (GUST) was developed for the motion of the Uranian satellites including for the first time short period terms as well as secular terms. The importance of the short period terms in the determination of the satellites' masses through Voyager optical navigation data was emphasized. The internal accuracy of the theory was obtained by comparison with a numerical integration and estimated to a few tens of kilometers, which is compatible with the precision of the data obtained during Voyager flyby.

The theory is now fit to all avaible observations (Earth-based from 1911 to 1986, Voyager optical navigation data and radio data). A complete set of dynamical and physical constants is derived and an analytical ephemeris is given for the satellites with an accuracy of a few tens of kilometers.

This paper is the direct outgrowth of the first paper (Laskar, 1986) which describes the construction of the theory and which will be referred as (P1) in the present work. In the first part, we give some information about the significance of the dynamical parameters of the theory. The second part describes the shape of the theory, the form of the different terms and the accuracy of the theory. The last part discusses the estimation of the parameters using first Earth-based observations only, and then all the observations including radio and optical data from the spacecraft navigation.

In this paper, we will give two different sets of parameters, respectively obtained with Earth based observations only and with all observations. All general information on the theory is made with the latest parameters, and unless explicitly noted, all computations are made with these parameters which come from the most complete set of data.

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2. The parameters of the theory

Construction of the theory GUST and its internal precision is discussed in (P1) as well as the definition of all variables. In this first part we shall discuss more thoroughly the parameters of GUST. When one uses a simplified model for the representation of the orbits (Keplerian motion) the definition of the parameters seems to be canonical; it is not the case in the general problem and the dynamical parameters depend on the form of the solution. They still have an important physical meaning, but their utilization implies a good understanding of their definition which can slightly vary from one theory to another. Actually, in GUST, we tried to keep the parameters as natural as possible.

2.1. Constants of the Lagrange solution

The Laplace-Lagrange equations giving the secular terms of the solution are (P1, Eq. 11)

$$\frac{d}{dt}\begin{bmatrix} z \\ \zeta \end{bmatrix} = \sqrt{-1}\mathbf{A}\begin{bmatrix} z \\ \zeta \end{bmatrix} \tag{1}$$

with

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{2}$$

This system is diagonalizable, and the solution is given by

$$\begin{bmatrix} z \\ \zeta \end{bmatrix} = \mathbf{S}\boldsymbol{\beta} \tag{3}$$

where

$$\mathbf{S} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \tag{4}$$

Table 1. Matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ of the eigenvectors of Laplace-Lagrange secular system (Eq. 4) normed by the euclidian norm

S_1	E_1	E_2	E_3	E_4	E_5
Miranda	0.999997	0.059274	0.017385	0.005428	0.003471
Ariel	−0.002553	0.980346	0.214697	0.057494	0.030783
Umbriel	−0.000160	−0.188166	0.972995	0.248610	0.122884
Titania	−0.000013	−0.001061	−0.080862	0.750105	0.620678
Oberon	−0.000003	−0.000286	0.018571	−0.610082	0.773756
S_2	I_1	I_2	I_3	I_4	I_5
Miranda	0.999995	0.073321	0.026516	0.012439	0.008948
Ariel	−0.003215	0.972453	0.250062	0.099890	0.056638
Umbriel	−0.000287	−0.221247	0.959772	0.357707	0.177577
Titania	−0.000038	−0.002866	−0.120800	0.700544	0.630819
Oberon	−0.000012	−0.000837	0.031797	−0.609221	0.753157

is the matrix of the eigenvectors of $\mathbf{A}$ normed by the euclidian norm (Table 1) and $\boldsymbol{\beta}$ is the column vector of the solution of the diagonalized system. That is

$$\beta_i = \rho_i \exp \sqrt{-1}(c_i t + \phi_i) \qquad (i = 1, 10) \tag{5}$$

The constants $(c_i)_{i=1,10}$ (Table 2a) are the eigenvalues of the matrix $\mathbf{A}$. They depend only on the masses m_i of the satellites, on the harmonics J_2 and J_4 of Uranus and on the mean mean motions which were used to develop the disturbing function. The secular system (1) is a first approximation of the complete secular system (Laskar, 1985), and terms of higher order can contribute to a small change in the values of the frequencies c_i. To take this into account, we used some corrections obtained by comparison to a numerical integration with similar values of the masses (P1). The change of the values of the masses from the preencounter values is large, and in the present case we computed again these corrections using the latest numerical integration made after the encounter. The ratio of the corrected values over the computed values is then fixed during the fit to the observations (this is why this correcting ratio is the same in Table 2a and 3a). The comparison with the values obtained in (P1) shows a consistent result with the previous determination, especially for the inner satellites which are the only ones where these corrections are significant.

The 20 parameters ρ_i and ϕ_i are integration constants which are determined by comparison to the observations. To avoid any problem of non-linearity during the fit to the observations, we preferred to use the parameters $\rho_i \sin\phi_i$ and $\rho_i \cos\phi_i$. Their values are determined at the date JD2444239.5 (1/1/1980 0^h) (Table 2a). The derived values of ρ_i and ϕ_i are given in Table 4.

2.2. *Mean longitude at the origin and mean mean motion*

The mean mean motion N_i is defined as the coefficient of t in the mean longitude λ_i. The remaining parameter is the value of the mean longitude at the origin. Their values are given in Table 2b. We have

$$\lambda = Nt - \sqrt{-1}q$$
$$n = N(1 + p) \tag{6}$$

where q and p are auxiliary variables actually used in our equations (P1, Eq. 2) (q is imaginary and p is real, both depend on the time t). The initial value for n_i, or in an equivalent way for P_i is then computed from (P1, Eq. 2) using the fact that the constant term in dq/dt is equal to zero.

It should be noted that the osculating mean motion n is different from the mean mean motion N. And even more, the mean value of the osculating mean motion n is not equal to the mean mean motion N (that is the constant part p_0 of p is not equal to zero). The semi-major axis is obtained from the osculating mean motion through Kepler's Law

$$n^2 a^3 = GM_U(1 + m/M_U) \tag{7}$$

where G is the gravitational constant, M_U the mass of Uranus, and m the satellite mass. The osculating mean motion expression is of the form

$$n = n_0 + \Delta n \tag{8}$$

where n_0 is a constant and Δn denotes the periodic terms in the expression of n. The mean value of the semi-major axis is then given by

$$a_0 = (G(m + M_U))^{1/3} n_0^{-2/3} \left(1 + \frac{5}{9}\left\langle\left(\frac{\Delta n}{n_0}\right)^2\right\rangle + \cdots\right) \tag{9}$$

where $\langle x \rangle$ is the averaged part of x. The computed values of the average mean motions and semi-major axis are given in Table 5.

2.3. *Other parameters*

Apart from Ariel and Umbriel, the masses of the satellites were poorly determined before the encounter of Voyager with Uranus. The optical navigation data of Voyager, however, permitted a very good determination of the masses during the encounter (Taylor et al., 1986). The determination of the masses with GUST uses the same data as (Taylor et al., 1986), the only difference being the use of the theory instead of the numerical integration for the representation of the satellite orbits. This allows us also to use an extended arc of Earth-based observations (1911–1986), but as far as the masses are concerned it does not really improve their determinations which are strongly dominated by Voyager data. This is even more true for the determination of the system mass which is completely determined by the Doppler data on the spacecraft during the encounter.

Nevertheless the use of the theory GUST with Earth-based data alone shows surprisingly good results, as compared to pre-encounter values of (Veillet, 1983b), (Table 12).

The last parameters, which were not estimated, are the harmonics J_2 and J_4 of the potential of Uranus, and the directions α, δ, of the pole of Uranus. Their adopted values (Table 2b) are the values from (French et al., 1986), computed from the ring observations. Although these parameters were not estimated, their uncertainty was considered in the computation of the standard errors on the other parameters.

3. The theory

The full theory GUST86 is given in Tables 6–10 for each satellite. It gives the elliptic elements for all satellites in the reference frame UME50* of the mean equator of Uranus 1950 with origin at the ascending node of Earth mean equation 1950 over Uranus mean equator 1950 (not to be confused with UME50 which origin is the ascending node of Uranus mean equator over the Earth mean equator). The transformation giving the coordinates in the

Table 2a. Solution GUST86 from Voyager and Earth-based data. Computation of the frequencies c_i and determination of the constant of integration $\rho_i \cos \phi_i$, $\rho_i \sin \phi_i$ of the Laplace-Lagrange system (Eq. 5). The computed value of c_i (Eq. 5) is given in the column "comp. c_i". The value of c_i used in the ephemeris (column c_i) is the product of this value by the correcting ratio (column "ratio"). The uncertainty on the parameters is given in parenthesis and estimated at 3 times the formal standard error σ

i	c_i	comp. c_i $(deg/year)$	ratio	c_i $(deg/year)$	$\rho_i \cos \phi_i$ $\times 10^6$	$\rho_i \sin \phi_i$ $\times 10^6$
1	E_1	19.440	1.033	20.082	1075 (27)	753 (24)
2	E_2	5.944	1.046	6.217	−900 (22)	810 (34)
3	E_3	2.731	1.049	2.865	−1913 (28)	3527 (40)
4	E_4	1.735	1.198	2.078	922 (25)	834 (46)
5	E_5	0.383	1.009	0.386	1639 (46)	747 (27)
6	I_1	−19.453	1.044	−20.309	31660 (230)	−20782 (300)
7	I_2	−5.999	1.048	−6.288	340 (46)	142 (102)
8	I_3	−2.706	1.048	−2.836	964 (80)	645 (87)
9	I_4	−1.828	1.008	−1.843	−170 (52)	964 (66)
10	I_5	−0.259	1.000	−0.259	−290 (44)	−524 (60)

Table 2b. Other parameters determined by Voyager and Earth based data

i	satellite	Gm_i $(km^3 s^{-2})$	$\lambda_{0i} \times 10^6$ (rad)	$N_i \times 10^6$ (rad/day)
1	Miranda	4.4 (1.0)	−238051 (980)	4445190.550 (0.372)
2	Ariel	86.1 (7.0)	3098046 (450)	2492952.519 (0.180)
3	Umbriel	84.0 (7.5)	2285402 (285)	1516148.111 (0.120)
4	Titania	230.0 (6.2)	856359 (97)	721718.509 (0.042)
5	Oberon	200.0 (6.0)	−915592 (85)	466692.120 (0.040)

$GM_{SU} = 5794554.5\ (10.0)\ km^3 s^{-2}$

considered parameters adopted from (French *et al.*, 1986)

$J_2 = 3.3461(0.0030) \times 10^{-3}$
$J_4 = -3.21(0.37) \times 10^{-5}$
α (1950.0)= 76.6067 (0.0192) (deg)
δ (1950.0)= 15.0322 (0.0358) (deg)

Table 3a. Solution from Earth-based data only. Computation of the frequencies c_i and determination of the constant of integration $\rho_i \cos \phi_i$, $\rho_i \sin \phi_i$ of the Laplace-Lagrange system (Eq. 5). The computed value of c_i (Eq. 5) is given in the column "comp. c_i". The value of c_i used in the ephemeris (column c_i) is the product of this value by the correcting ratio (column "ratio"). The uncertainty on the parameters is given in parenthesis and estimated at 3 times the formal standard error σ

i	c_i	comp. c_i $(deg/year)$	ratio	c_i $(deg/year)$	$\rho_i \cos \phi_i$ $\times 10^6$	$\rho_i \sin \phi_i$ $\times 10^6$
1	E_1	19.513	1.033	20.151	526 (753)	−556 (762)
2	E_2	5.805	1.046	6.073	−887 (453)	1400 (450)
3	E_3	2.753	1.049	2.889	−2113 (375)	3984 (414)
4	E_4	1.785	1.198	2.136	881 (141)	905 (200)
5	E_5	0.353	1.009	0.356	1526 (216)	873 (195)
6	I_1	−19.514	1.044	−20.378	32326 (2187)	−23794 (1833)
7	I_2	−5.860	1.048	−6.143	625 (1284)	−560 (1053)
8	I_3	−2.723	1.048	−2.855	666 (828)	73 (720)
9	I_4	−1.876	1.008	−1.892	−470 (351)	808 (300)
10	I_5	−0.236	1.000	−0.236	−442 (312)	−701 (308)

Table 3b. Other parameters determined by Earth-based data only

i	satellite	Gm_i $(km^3 s^{-2})$	$\lambda_{0i} \times 10^6$ (rad)	$N_i \times 10^6$ (rad/day)
1	Miranda	4.8 (2.4)	−237924 (1171)	4445190.857 (0.441)
2	Ariel	94.4 (27.3)	3098578 (711)	2492952.726 (0.216)
3	Umbriel	71.2 (20.4)	2285286 (474)	1516148.050 (0.150)
4	Titania	201.3 (20.4)	856441 (267)	721718.519 (0.057)
5	Oberon	233.6 (40.2)	−915620 (126)	466692.130 (0.051)

$GM_{SU} = 5791856.9\ (1903.5)\ km^3 s^{-2}$

considered parameters adopted from (French *et al.*, 1986)

$J_2 = 3.3461(0.0030) \times 10^{-3}$
$J_4 = -3.21(0.37) \times 10^{-5}$
α (1950.0)= 76.6067 (0.0192) (deg)
δ (1950.0)= 15.0322 (0.0358) (deg)

Table 4. Derived parameters from solution GUST86 (Voyager and Earthbased data). The constants ρ_i, ϕ_i of the Laplace-Lagrange system (Eq. 5) are directly computed from the results of Table 2

i	c_i	ρ_i $\times 10^6$	ϕ_i (rad)
1	E_1	1312.38	0.611392
2	E_2	1211.43	2.408974
3	E_3	4013.06	2.067774
4	E_4	1243.57	0.735131
5	E_5	1805.91	0.426767
6	I_1	37871.91	5.702313
7	I_2	368.39	0.395757
8	I_3	1160.02	0.589326
9	I_4	978.84	1.746237
10	I_5	599.73	4.206896

EME50 reference frame (Earth mean equator and equinox 1950.0) is then given by

$$X_{\mathrm{EME50}} = \mathbf{R} X_{\mathrm{UME50*}} \qquad (10)$$

with

$$\mathbf{R} = \begin{bmatrix} \sin\alpha & \cos\alpha\sin\delta & \cos\alpha\cos\delta \\ -\cos\alpha & \sin\alpha\sin\delta & \sin\alpha\cos\delta \\ 0 & -\cos\delta & \sin\delta \end{bmatrix} \qquad (11)$$

where α and δ are the right ascension and declination of the pole of Uranus in the EME50 frame, given in Table 2b.

3.1. Presentation of GUST86

For each satellite, the terms of the theory are of the following form:

Mean motion: $A\cos(\omega t + \phi)$
Mean longitude: $A\sin(\omega t + \phi)$
Variables z and ζ: $A\exp\sqrt{-1}(\omega t + \phi)$

The amplitude A of each term is given in column 2 of the tables. The period and phase which are given in the Tables 6–10 are only intended to be used for a quick reference. For a more accurate use of the theory one should compute the frequency ω and phase ϕ from the column "Argument" and the values given in Tables 2 and 4. The frequency ω of a term is computed directly from the expression given in the column "argument". The corresponding phase ϕ is obtained by replacing N_i by λ_{0i} and E_i or I_i by the corresponding phase ϕ_i (Table 4).

3.2. Internal accuracy of GUST86

The development of the theory was made using Veillet's determination of the satellites masses (P1). With these values, we retained only the short period terms of amplitude greater than $2\,10^{-5}$ (about 10 km for Oberon), although we actually computed or estimated all the short period terms up to the degree 3 in the variables eccentricity and inclination z and ζ. In fact, the estimated amplitude of the terms of degree 2 and 3 fell beneath our truncature level and the only terms which remained were of degree 0 or 1.

Once a term of degree 1 was selected, we kept all terms of amplitude greater than 10^{-6} resulting from the substitution of the solution of the long period system in these terms of degree 1 in eccentricity. The same policy was used for the second order terms coming from the harmonics of the Laplacian inequality. All the terms from the secular Laplace-Lagrange system were kept, even if their amplitude was not significant, mainly to keep this information in the Tables 6–10 of the solution. For practical ephemeris computations, the terms of very small amplitude can be discarded.

As several values of the masses decreased in their final estimation the final precision is in fact of greater accuracy (the amplitudes of the short period terms are roughly proportional to the values of the masses).

The global internal accuracy of the solution is obtained by direct comparison with a numerical integration of reference (P1). The maximum absolute discrepancies goes from about 10 km for Miranda to about 80 km for Oberon over 12 years.

3.3. Terms of the theory

We can distinguish different kinds of terms in the theory (Table 11):

a) Short period terms of degree 0 and order 1 (Example $(N_4 - N_5)_{\mathrm{Oberon}}$).

These terms have a very simple dependence with respect to the parameters. The amplitude is proportional to the mass of the disturbing satellite while frequency and phase are simple combinations of the mean mean motions and longitude at the origin. These terms cannot be damped through non-gravitational effects. They are sometimes called the forced terms. The only change which could affect these terms would be a secular evolution in the semimajor axis which would then change the integration factor $N_4 - N_5$. Except in the case of a resonance ($N_4 - N_5 \simeq 0$), this will not change the amplitude of the terms very significantly.

b) Short period terms of degree 0 and order 2 (Example $(N_1 - 3N_2 + 2N_3)_{\mathrm{Miranda}}$).

These terms are very similar to the previous ones; their amplitude is now proportional to a product of two masses.

c) Long period terms (argument E_i or I_i).

These terms come from the resolution of the secular system of Laplace-Lagrange. Their period is of several years and they

Table 5. Other derived quantities of GUST86 (Voyager and Earth based data)

i	satellite	$m_i/M_U \times 10^5$	$N_i \times 10^6$ (rad/day)	$n_{0i} \times 10^6$ (rad/day)	a_{0i} (km)
1	Miranda	0.075	4445190.550	4443522.67	129872
2	Ariel	1.49	2492952.519	2492542.57	190945
3	Umbriel	1.45	1516148.111	1515954.90	265998
4	Titania	3.97	721718.509	721663.16	436298
5	Oberon	3.45	466692.120	466580.54	583519

Table 6

MIRANDA			
Argument	Amplitude $\times 10^6$	Period (days)	Phase (rad)
Mean motion (n)			
constant	4443522.67		
$N_1 - 3N_2 + 2N_3$	−34.92	−4583.63	1.32
$2N_1 - 6N_2 + 4N_3$	8.47	−2291.81	2.64
$3N_1 - 9N_2 + 6N_3$	1.31	−1527.88	3.97
$N_1 - N_2$	−52.28	3.22	2.95
$2N_1 - 2N_2$	−136.65	1.61	5.89
Mean longitude (λ)			
constant	−238051.58		
T	4445190.55		
$N_1 - 3N_2 + 2N_3$	25472.17	−4583.63	1.32
$2N_1 - 6N_2 + 4N_3$	−3088.31	−2291.81	2.64
$3N_1 - 9N_2 + 6N_3$	−318.10	−1527.88	3.97
$4N_1 - 12N_2 + 8N_3$	−37.49	−1145.91	5.29
$N_1 - N_2$	−57.85	3.22	2.95
$2N_1 - 2N_2$	−62.32	1.61	5.89
$3N_1 - 3N_2$	−27.95	1.07	2.56
$z = k + \sqrt{-1}h = e \exp \sqrt{-1}\varpi$			
E_1	1312.38	6547.58	0.61
E_2	71.81	21151.38	2.41
E_3	69.77	45889.84	2.07
E_4	6.75	63289.04	0.74
E_5	6.27	340563.87	0.43
$-N_1 + 2N_2$	−123.31	11.62	0.15
$-2N_1 + 3N_2$	39.52	−4.45	3.48
N_1	194.10	1.41	6.04
$\zeta = q + \sqrt{-1}p = \sin i/2 \exp \sqrt{-1}\Omega$			
I_1	37871.71	−6474.62	5.70
I_2	27.01	−20909.68	0.40
I_3	30.76	−46368.68	0.59
I_4	12.18	−71358.57	1.75
I_5	5.37	−507255.84	4.21

Table 7

ARIEL			
Argument	Amplitude $\times 10^6$	Period (days)	Phase (rad)
Mean motion (n)			
constant	2492542.57		
$N_1 - 3N_2 + 2N_3$	2.55	−4583.63	1.32
$N_2 - N_3$	−42.16	6.43	0.81
$2N_2 - 2N_3$	−102.56	3.22	1.63
Mean longitude (λ)			
constant	3098046.41		
T	2492952.52		
$N_1 - 3N_2 + 2N_3$	−1860.50	−4583.63	1.32
$2N_1 - 6N_2 + 4N_3$	219.99	−2291.81	2.64
$3N_1 - 9N_2 + 6N_3$	23.10	−1527.88	3.97
$4N_1 - 12N_2 + 8N_3$	4.30	−1145.91	5.29
$N_2 - N_3$	−90.11	6.43	0.81
$2N_2 - 2N_3$	−91.07	3.22	1.63
$3N_2 - 3N_3$	−42.75	2.14	2.44
$2N_2 - 2N_4$	−16.49	1.77	4.48
$z = k + \sqrt{-1}h = e \exp \sqrt{-1}\varpi$			
E_1	−3.35	6547.58	0.61
E_2	1187.63	21151.38	2.41
E_3	861.59	45889.84	2.07
E_4	71.50	63289.04	0.74
E_5	55.59	340563.87	0.43
$-N_2 + 2N_3$	−84.60	11.65	1.47
$-2N_2 + 3N_3$	91.81	−14.36	0.66
$-N_2 + 2N_4$	20.03	−5.99	4.89
N_2	89.77	2.52	3.10
$\zeta = q + \sqrt{-1}p = \sin i/2 \exp \sqrt{-1}\Omega$			
I_1	−121.75	−6474.62	5.70
I_2	358.25	−20909.68	0.40
I_3	290.08	−46368.68	0.59
I_4	97.78	−71358.57	1.75
I_5	33.97	−507255.84	4.21

represent the precessing motion of the orbits. The dependence of these terms with respect to the parameters is more involved. The frequency is a function of the masses and J_2, J_4. The amplitude is proportional to an integration constant ρ_i and to a function of the masses and J_2, J_4. The phase ϕ_i is an integration constant. Through non-Newtonian effects (tide, collisions, . . .) there can be a circularization of the orbits. This will affect only the values ρ_i and thus the amplitude of all long period terms, while the amplitude of the short period terms will not be much changed. The amplitude of the term E_i of satellite S_i is thus sometimes called "free eccentricity". We must notice that the forced terms in the solution of Titania and Oberon are of about the same amplitude as the "free eccentricity" (Tables 6–10).

d) Short period terms of degree 1 (Example $(2N_4 - 3N_5 + E_5)_{\mathrm{Oberon}}$)

These term are products of short period terms of degree 0 and long period terms. Their dependence with respect to the parameters is then more complicated than in the other cases. An interesting feature is the information they give on the constants ρ_i and ϕ_i through short period terms. Unfortunately, none of them (at the precision of the theory) give any information on the constants in inclination which are then only determined by the long period precessing terms.

4. Description of the data

4.1. Earth-based data

The Earth-based observations consist of 4122 pairs of right ascension and declination or separation and position angle from 1911 to 1986. All the photographic measures are refered to one of the satellites which allows a much better accuracy than a measure from the planet center, not very well defined. The average improvement by doing this is a gain of a factor 2 to 3. All the observations are listed in Table 13 and the residual with the final solution GUST86 in right ascension and declination are plotted in Figs. 1 and 2. Most of the observations are photographic plates except the early ones which are micrometric observations. Extensive bibliography can be founded in (Jacobson, 1986). Discussion of the observations can be found in previous works, (Harris, 1949, Dunham, 1971, Veillet, 1983b). Some sets of observations which were not of sufficient accuracy were discarded. The remaining observations are of very good accuracy (Table 13) and we must mention particularly the remarkable sets of observations of Harrington and especially Veillet for their number and high quality. The observations were weighted by observer and instrument based on an assessment of the presumed observation accuracy given by their residual. Weights range from 0″058 to 0″348

Table 8

Argument	Amplitude $\times 10^6$	Period (days)	Phase (rad)
UMBRIEL			
Mean motion (n)			
constant	1515954.90		
$N_3 - 2N_4 + E_3$	9.74	86.25	2.64
$N_2 - N_3$	−106.00	6.43	0.81
$2N_2 - 2N_3$	54.16	3.22	1.63
$N_3 - N_4$	−23.59	7.91	1.43
$2N_3 - 2N_4$	−70.70	3.95	2.86
$3N_3 - 3N_4$	−36.28	2.64	4.29
Mean longitude (λ)			
constant	2285401.69		
T	1516148.11		
$N_1 - 3N_2 + 2N_3$	660.57	−4583.63	1.32
$2N_1 - 6N_2 + 4N_3$	−76.51	−2291.81	2.64
$3N_1 - 9N_2 + 6N_3$	−8.96	−1527.88	3.97
$4N_1 - 12N_2 + 8N_3$	−2.53	−1145.91	5.29
$N_3 - 4N_4 + 3N_5$	−52.91	214.07	2.40
$N_3 - 2N_4 + E_5$	−7.34	86.39	1.00
$N_3 - 2N_4 + E_4$	−1.83	86.30	1.31
$N_3 - 2N_4 + E_3$	147.91	86.25	2.64
$N_3 - 2N_4 + E_2$	−7.77	86.06	2.98
$N_2 - N_3$	97.76	6.43	0.81
$2N_2 - 2N_3$	73.13	3.22	1.63
$3N_2 - 3N_3$	34.71	2.14	2.44
$4N_2 - 4N_3$	18.89	1.61	3.25
$N_3 - N_4$	−67.89	7.91	1.43
$2N_3 - 2N_4$	−82.86	3.95	2.86
$3N_3 - 3N_4$	−33.81	2.64	4.29
$4N_3 - 4N_4$	−15.79	1.98	5.72
$N_3 - N_5$	−10.21	5.99	3.20
$2N_3 - 2N_5$	−17.08	2.99	0.12
$z = k + \sqrt{-1}h = e \exp \sqrt{-1}\varpi$			
E_1	−0.21	6547.58	0.61
E_2	−227.95	21151.38	2.41
E_3	3904.69	45889.84	2.07
E_4	309.17	63289.04	0.74
E_5	221.92	340563.87	0.43
N_2	29.34	2.52	3.10
N_3	26.20	4.14	2.28
$-N_2 + 2N_3$	51.19	11.65	1.47
$-2N_2 + 3N_3$	−103.86	−14.36	0.66
$-3N_2 + 4N_3$	−27.16	−4.44	6.13
N_4	−16.22	8.71	0.85
$-N_3 + 2N_4$	549.23	−86.41	5.71
$-2N_3 + 3N_4$	34.70	−7.25	4.28
$-3N_3 + 4N_4$	12.81	−3.78	2.85
$-N_3 + 2N_5$	21.81	−10.78	2.16
N_3	46.25	4.14	2.28
$\varsigma = q + \sqrt{-1}p = \sin i/2 \exp \sqrt{-1}\Omega$			
I_1	−10.86	−6474.62	5.70
I_2	−81.51	−20909.68	0.40
I_3	1113.36	−46368.68	0.59
I_4	350.14	−71358.57	1.75
I_5	106.50	−507255.84	4.21

Table 9

Argument	Amplitude $\times 10^6$	Period (days)	Phase (rad)
TITANIA			
Mean motion (n)			
constant	721663.16		
$N_3 - 2N_4 + E_3$	−2.64	86.25	2.64
$2N_4 - 3N_5 + E_5$	−2.16	144.84	4.89
$2N_4 - 3N_5 + E_4$	6.45	144.57	5.19
$2N_4 - 3N_5 + E_3$	−1.11	144.45	0.24
$N_2 - N_4$	−62.23	3.55	2.24
$N_3 - N_4$	−56.13	7.91	1.43
$N_4 - N_5$	−39.94	24.64	1.77
$2N_4 - 2N_5$	−91.85	12.32	3.54
$3N_4 - 3N_5$	−58.31	8.21	5.32
$4N_4 - 4N_5$	−38.60	6.16	0.80
$5N_4 - 5N_5$	−26.18	4.93	2.58
$6N_4 - 6N_5$	−18.06	4.11	4.35
Mean longitude (λ)			
constant	856358.79		
T	721718.51		
$N_3 - 4N_4 + 3N_5$	20.61	214.07	2.40
$N_3 - 2N_4 + E_5$	−2.07	86.39	1.00
$N_3 - 2N_4 + E_4$	−2.88	86.30	1.31
$N_3 - 2N_4 + E_3$	−40.79	86.25	2.64
$N_3 - 2N_4 + E_2$	2.11	86.06	2.98
$2N_4 - 3N_5 + E_5$	−51.83	144.84	4.89
$2N_4 - 3N_5 + E_4$	159.87	144.57	5.19
$2N_4 - 3N_5 + E_3$	−35.05	144.45	0.24
$3N_4 - 4N_5 + E_5$	−1.56	21.06	0.38
$N_2 - N_4$	40.54	3.55	2.24
$N_3 - N_4$	46.17	7.91	1.43
$N_4 - N_5$	−317.76	24.64	1.77
$2N_4 - 2N_5$	−305.59	12.32	3.54
$3N_4 - 3N_5$	−148.36	8.21	5.32
$4N_4 - 4N_5$	−82.92	6.16	0.80
$5N_4 - 5N_5$	−49.98	4.93	2.58
$6N_4 - 6N_5$	−31.56	4.11	4.35
$7N_4 - 7N_5$	−20.56	3.52	6.12
$8N_4 - 8N_5$	−13.69	3.08	1.61
$z = k + \sqrt{-1}h = e \exp \sqrt{-1}\varpi$			
E_1	−0.02	6547.58	0.61
E_2	−1.29	21151.38	2.41
E_3	−324.51	45889.84	2.07
E_4	932.81	63289.04	0.74
E_5	1120.89	340563.87	0.43
N_2	33.86	2.52	3.10
N_4	17.46	8.71	0.85
$-N_2 + 2N_4$	16.58	−5.99	4.89
N_3	28.89	4.14	2.28
$-N_3 + 2N_4$	−35.86	−86.41	5.71
N_4	−17.86	8.71	0.85
N_5	−32.10	13.46	5.36
$-N_4 + 2N_5$	−177.83	29.68	3.59
$-2N_4 + 3N_5$	793.43	−144.91	1.82
$-3N_4 + 4N_5$	99.48	−21.06	0.05
$-4N_4 + 5N_5$	44.83	−11.35	4.56
$-5N_4 + 6N_5$	25.13	−7.77	2.79
$-6N_4 + 7N_5$	15.43	−5.91	1.02
$\varsigma = q + \sqrt{-1}p = \sin i/2 \exp \sqrt{-1}\Omega$			
I_1	−1.43	−6474.62	5.70
I_2	−1.06	−20909.68	0.40
I_3	−140.13	−46368.68	0.59
I_4	685.72	−71358.57	1.75
I_5	378.32	−507255.84	4.21

Table 10

OBERON			
Argument	Amplitude $\times 10^6$	Period (days)	Phase (rad)
Mean motion (n)			
constant	466580.54		
$2N_4 - 3N_5 + E_5$	2.08	144.84	4.89
$2N_4 - 3N_5 + E_4$	−6.22	144.57	5.19
$2N_4 - 3N_5 + E_3$	1.07	144.45	0.24
$N_2 - N_5$	−43.10	3.10	4.01
$N_3 - N_5$	−38.94	5.99	3.20
$N_4 - N_5$	−80.11	24.64	1.77
$2N_4 - 2N_5$	59.06	12.32	3.54
$3N_4 - 3N_5$	37.49	8.21	5.32
$4N_4 - 4N_5$	24.82	6.16	0.80
$5N_4 - 5N_5$	16.84	4.93	2.58
Mean longitude (λ)			
constant	−915591.80		
T	466692.12		
$N_3 - 4N_4 + 3N_5$	−7.82	214.07	2.40
$2N_4 - 3N_5 + E_5$	51.29	144.84	4.89
$2N_4 - 3N_5 + E_4$	−158.24	144.57	5.19
$2N_4 - 3N_5 + E_3$	34.51	144.45	0.24
$N_2 - N_5$	47.51	3.10	4.01
$N_3 - N_5$	38.96	5.99	3.20
$N_4 - N_5$	359.73	24.64	1.77
$2N_4 - 2N_5$	282.78	12.32	3.54
$3N_4 - 3N_5$	138.60	8.21	5.32
$4N_4 - 4N_5$	78.03	6.16	0.80
$5N_4 - 5N_5$	47.29	4.93	2.58
$6N_4 - 6N_5$	30.00	4.11	4.35
$7N_4 - 7N_5$	19.62	3.52	6.12
$8N_4 - 8N_5$	13.11	3.08	1.61
$z = k + \sqrt{-1}h = e \exp \sqrt{-1}\varpi$			
E_1	0.00	6547.58	0.61
E_2	−0.35	21151.38	2.41
E_3	74.53	45889.84	2.07
E_4	−758.68	63289.04	0.74
E_5	1397.34	340563.87	0.43
N_2	39.00	2.52	3.10
$-N_2 + 2N_5$	17.66	−4.03	1.35
N_3	32.42	4.14	2.28
N_4	79.75	8.71	0.85
N_5	75.66	13.46	5.36
$-N_4 + 2N_5$	134.04	29.68	3.59
$-2N_4 + 3N_5$	−987.26	−144.91	1.82
$-3N_4 + 4N_5$	−126.09	−21.06	0.05
$-4N_4 + 5N_5$	−57.42	−11.35	4.56
$-5N_4 + 6N_5$	−32.41	−7.77	2.79
$-6N_4 + 7N_5$	−19.99	−5.91	1.02
$-7N_4 + 8N_5$	−12.94	−4.77	5.53
$\zeta = q + \sqrt{-1}p = \sin i/2 \exp \sqrt{-1}\Omega$			
I_1	−0.44	−6474.62	5.70
I_2	−0.31	−20909.68	0.40
I_3	36.89	−46368.68	0.59
I_4	−596.33	−71358.57	1.75
I_5	451.69	−507255.84	4.21

and lead to weighted standard error of about 1 for all sets of observations (Table 13).

4.2. Spacecraft data

The Voyager data set contained all the navigation data used in the reconstruction of the spacecraft trajectory for the encounter period (Taylor et al., 1986). The included data were:

1) Spacecraft imaging of the satellites in the form of 311 camera line and pixel pairs.

2) Spacecraft imaging of the reference stars in the form of 554 camera line and pixel pairs.

3) Spacecraft imaging of the planet in the form of 5 camera line and pixel pairs.

4) Earth-based radiometric observations of the spacecraft in the form of 4430 Doppler and 779 range measurements.

Among the Doppler measurements were those acquired for the radio science determination of the Miranda mass (Tyler et al., 1986). The data time span covered the period from the start of the observatory phase of the encounter (1st November 1985) through closest approach (24th January 1986) to a time just prior to the outgoing trajectory correction maneuver (14th February 1986).

Observations of the satellites were weighted at 0.25 pixel ($\simeq 2.5\mu$ rad) except for two sets of observations made with long exposure times and the set of post encounter observations. The long exposure observations were deweighted to 10 pixels because of their lower quality due to image smear. The post encounter observations were deweighted to 1.75 pixels because of their lower quality due to difficulties in determining image centers (the satellites appeared only as crescents). The stellar observations were weighted at 0.5 pixels, a lower weight (higher sigma) than the satellites, to account for more poorly identified star image locations (Synnott et al., 1986). The planet observation weights were 1.75 and 1.0 in pixel and lines, respectively, which were lower than the satellite weight due to the difficulty in determining the center of the extended planet image.

In the Doppler data set were measurements taken at either X band or S band depending upon data quality and acquired in either the two-way or three-way tracking mode at the measurement time. Doppler count times varied from 10 minutes when the spacecraft was far from the planet to 5 seconds during the Miranda closest approach. The Doppler weights were set according to band and count time except for some periods were poor quality dictated the use of lower weights. The range measurements, made only at X band, were weighted at 5 km, considerably less than their inherent accuracy. This lower weight was used because of the sensitivity of long arcs of range data to the unmodelled spacecraft accelerations (Campbell, 1980).

The residual in lines and pixels after the fit of the solution GUST86 are given in Table 14 and Figs. 3 and 4.

5. Estimation of the parameters

During the estimation of the parameters, we used Earth-based data and then Voyager optical navigation and radio data. Each type of data gives different information which combines very well to provide the final solution. In Fig. 5 we show for each parameter the relative change in the sigma resulting from the suppression of a given set of data. A 100% change shows for example that the parameter is entirely determined by the given set of data while 0% change shows that the data is not sensible at all to the considered parameter. This diagram is sometimes difficult to

Table 11. Different kinds of terms in the theory GUST86. In the solution of a given satellite (Sat.), (deg.) and (ord.) are respectively the degree and order of the considered term. These examples give the analytical dependence of the terms with respect to the parameters of the theory

	deg.	ord.	Sat.	Amplitude	Frequency	Phase
Short period	0	1	5	$C \times m_4$	$N_4 - N_5$	$\lambda_{04} - \lambda_{05}$
Short period	0	2	1	$C \times m_2 m_3$	$N_1 - 3N_2 + 2N_3$	$\lambda_{01} - 3\lambda_{02} + 2\lambda_{03}$
Long period	1	1	5	$C(J_2, J_4, m_i) \times \rho_5$	$E_5(J_2, J_4, m_i)$	ϕ_5
Short period	1	1	5	$C(J_2, J_4, m_i) \times \rho_4 \times m_4$	$2N_4 - 3N_5 + E_5(J_2, J_4, m_i)$	$2\lambda_{04} - 3\lambda_{05} + \phi_5$

Table 12. Determination of the Uranian satellites masses from Earth based observation (Ebo). The starting values are pre-encounter values of the parameters (Veillet, 1984 for the masses of the satellites masses and Jacobson, 1985 for the system mass). The fit is then made with the analytical theory GUST against Earth-based observations only. The results obtained after the first iteration are given, as well as the final results obtained after 4 iterations (no changes between the 3rd iteration and the 4th iteration). Comparison is made with the results of GUST86 which use the full set of data including Voyager navigation observations (Table 2). In the last column we compare our results with the results obtained with numerical integration in the post-encounter reconstruction (Taylor et al., 1986). Unit for the constant of gravitation Gm_i is $km^3 s^{-2}$

	(Veillet) pre enc.	GUST (Ebo) 1 iter.	GUST (Ebo) 4 iter.	GUST86	Num. int.
Ariel	104.0	100.3	94.4	86.1	87.4
Umbriel	69.3	67.2	71.2	84.0	83.5
Titania	393.0	203.2	201.3	230.0	230.6
Oberon	398.8	233.3	233.6	200.0	202.9
Miranda	11.6	5.3	4.8	4.4	4.7
Uranus (sys.)	5780694.4	5791437.7	5791856.9	5794554.5	5794557.

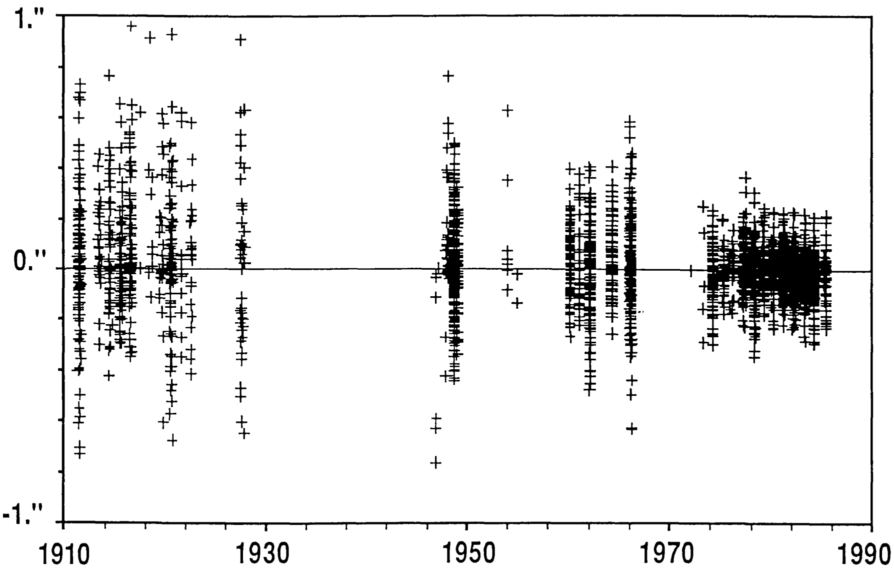

Fig. 1. Right ascension residuals of GUST86 against Earth-based observations from 1910 to 1986 listed in Table 13

Table 13. Earth-based observations. N_α and N_δ denotes the number of observations in right ascension and declination. Observations in position angle (P) and separation angle (S) are converted in rectangular coordinates ($X = S\sin P$, $Y = S\cos P$) and denoted by N_X and N_Y. Micrometric observations are noted (^{M}). Location with (*) means that a few observations where taken in other location. The mean, standard error σ and weighted standard error σ_w are post fit statistics

Observer	Period	Location	$N_{Obs.}$	mean ('')	σ ('')	σ_w
Aitken^M	1911-14	Lick	94_X	−0.010	0.253	0.892
			94_Y	−0.049	0.247	0.759
Barnard^M	1911-22	Yerkes	323_X	0.103	0.238	1.055
			323_Y	0.042	0.228	0.938
Hall^M	1920-22	Washington	32_X	−0.161	0.259	0.739
			32_Y	0.080	0.348	0.602
Harris	1913-48	Lowell*	62_α	−0.029	0.169	0.634
			61_δ	−0.055	0.198	0.710
Harrington	1979-86	Lick*	360_α	−0.008	0.063	0.970
			360_δ	−0.004	0.061	0.940
Ianna	1983-84	McCormick	124_α	−0.010	0.079	1.047
			124_δ	−0.001	0.059	0.787
Mulholand	1974-83	Mc Donald	216_α	−0.008	0.094	0.958
			216_δ	0.017	0.097	0.990
Pascu	1981-85	Washington	76_α	−0.008	0.113	1.126
			76_δ	0.056	0.080	0.796
Steavenson^M	1947-49	Cambridge	30_X	−0.013	0.334	1.115
			30_Y	0.016	0.205	1.120
Struve^M	1927-28	Berlin-Babelsberg	41_X	0.020	0.346	1.385
			41_Y	0.160	0.215	0.450
van Biesbroeck	1948-66	Mc Donald, Catalina*	918_α	0.004	0.161	0.946
			918_δ	0.016	0.145	0.837
Veillet	1977-84	Pic du Midi, St Michel	485_α	0.004	0.094	0.772
			485_δ	−0.007	0.112	0.855
Veillet	1977-84	ESO, CFH	1162_α	−0.007	0.063	0.954
			1162_δ	−0.006	0.063	0.974
Walker	1974-77	Flagstaff	73_α	−0.009	0.067	0.953
			73_δ	0.001	0.058	0.828
Whitaker	1948-73	Mc Donald*	126_X	0.092	0.155	0.713
			126_Y	−0.012	0.106	0.827
All	1911-86		4122_α	0.006	0.142	0.961
			4121_δ	0.006	0.131	0.910

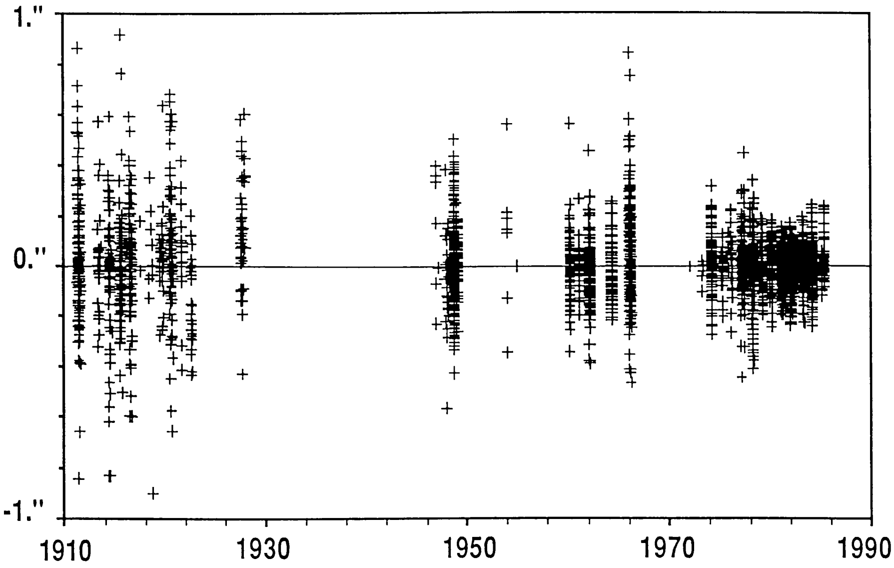

Fig. 2. Declination residuals of GUST86 against Earth-based observations from 1910 to 1986 listed in Table 13

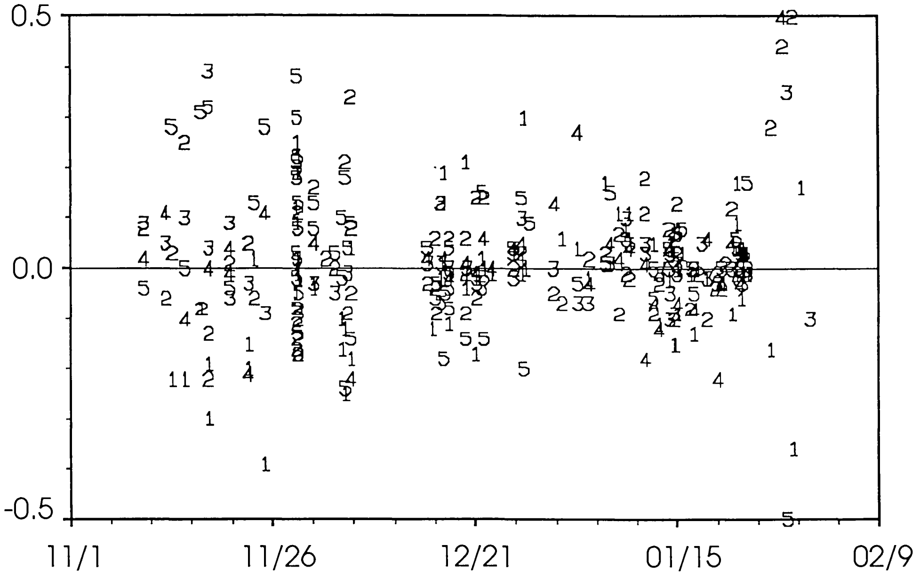

Fig. 3. Pixel residuals of GUST86 against Voyager optical navigation observations from November 1985 to February 1986. The number refers to the observed satellite with the conventions adopted during the encounter (1: Ariel, 2: Umbriel, 3: Titania, 4: Oberon, 5: Miranda). The x-axis gives the calendar date

Table 14. Voyager pixel and line data and residuals from optical navigation observations. The mean, standard error σ and weighted standard error σ_w are post fit statistics

Observations	$N_{Obs.}$	mean	σ	σ_w
pixels	311	0.0021	0.149	0.912
lines	311	0.0243	0.163	0.812

analyse but gives a rough idea of the contribution of each type of data in the final solution. Actually, sometimes when one adds one set of data, the determination of a parameter can be improved by the indirect effect of the improvement of all the other parameters.

5.1. Data processing

The Earth-based observations were processed to obtain an augmented square root information array. This array, which is equivalent to the normal equations, was produced by using Householder transformations to pack the matrix of weighted observation partial derivatives and the weighted residual vector (actual minus observations) into an upper triangular square root information matrix and associated residual vector (Lawson and Hawson, 1974). The matrix and vector constitute the array. Each row and column of the matrix and each element of the vector were associated with a particular GUST parameter.

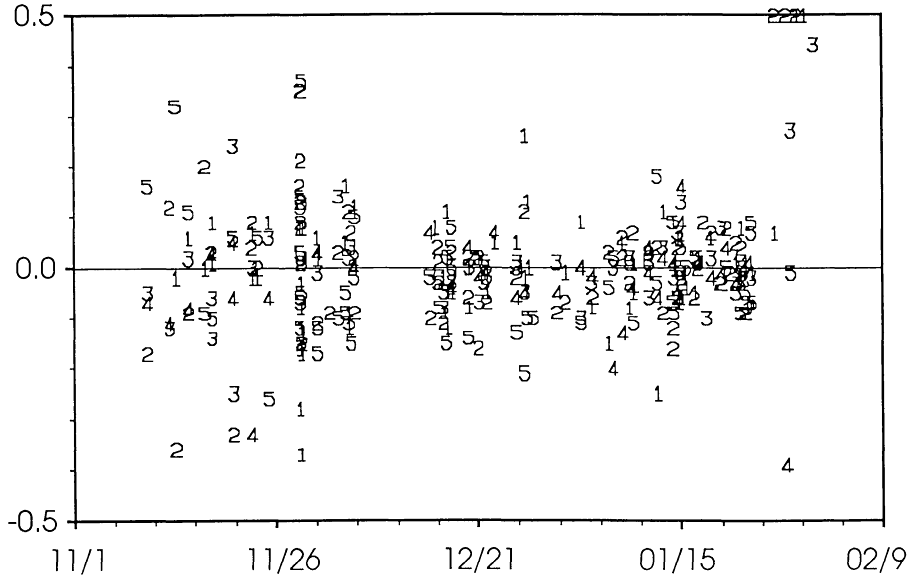

Fig. 4. Line residuals of GUST86 against Voyager optical navigation observations from November 1985 to February 1986. The number refers to the observed satellite with the conventions adopted during the encounter (1: Ariel, 2: Umbriel, 3: Titania, 4: Oberon, 5: Miranda). The x-axis gives the calendar date

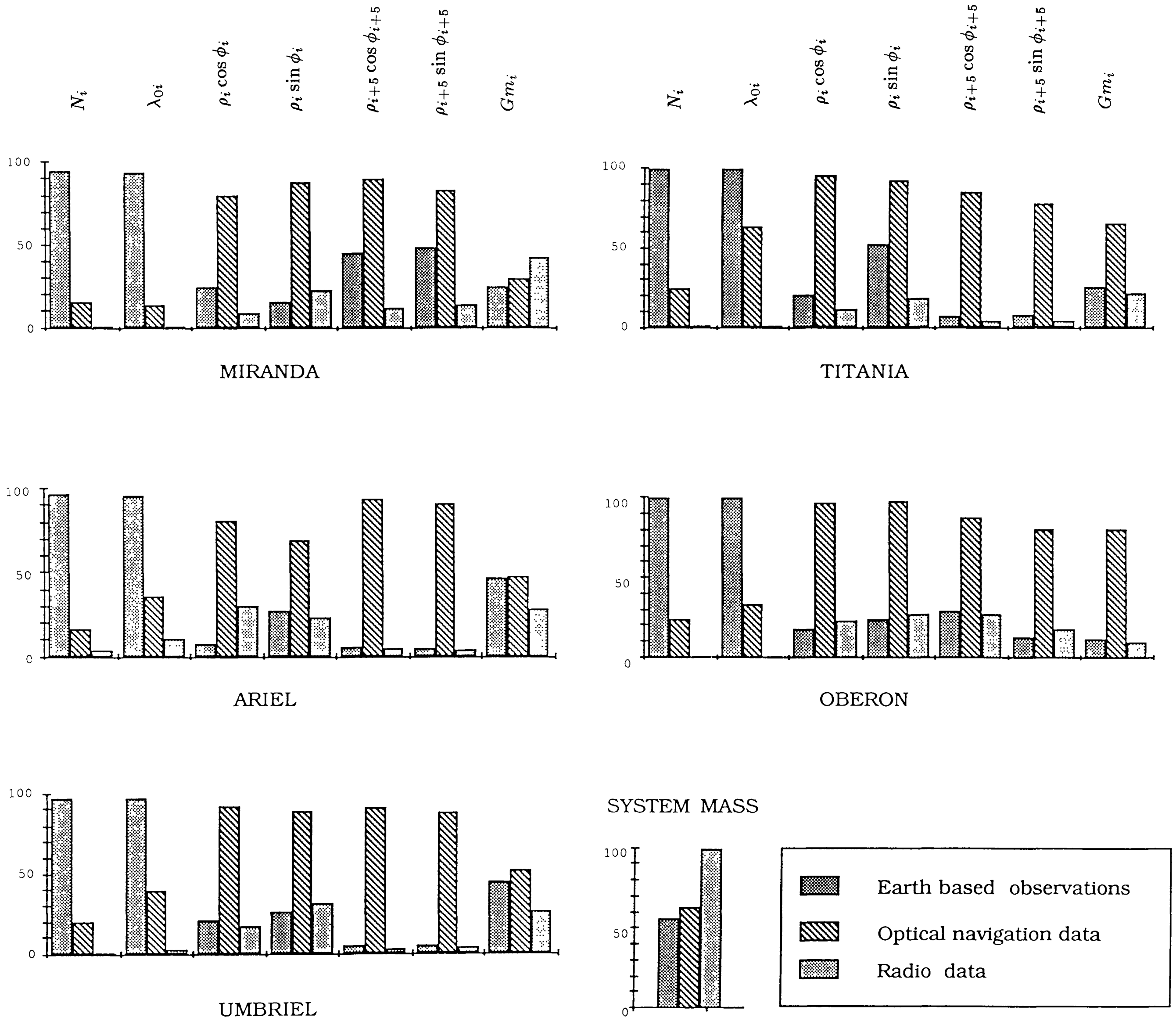

Fig. 5. Contribution of the different kinds of observations (Earth-based observations, Voyager optical navigation data and radio data) in the determination of each parameter of GUST86. In each case, the column represents the relative change in the standard error when the corresponding kind of observations is supressed in the determination of the solution

The Voyager data were processed to obtain an augmented square root information array. This processing was done with the Jet Propulsion Laboratory Orbit Determination Program (Ekelund, 1979) and led to an array containing information for not only the GUST parameters but also Uranus planetary ephemeris parameters and a number of spacecraft navigation related parameters. Prior to use the array was modified removing the rows and columns associated with the navigation parameters and retaining those associated with the planetary ephemeris and GUST parameters. This modification was performed in a way which preserved all information relative to the retained parameters (Curkendall, 1973).

The square root information array for the composite data set was formed by using Householder transformations to combine the Earth-based and Voyager arrays.

The solutions for the parameters using the Earth-based and complete data sets were generated and analysed by means of singular value decomposition techniques (Lawson and Hawson, 1974) applied to the appropriate square root information arrays.

5.2. *Earth-based data results*

As was somewhat expected (Dermott and Nicholson, 1986), the masses of Titania and Oberon determined during Voyager encounter (Smith et al., 1986, Tyler et al., 1986) were revealed to be much smaller than what was predicted by Veillet (1983b) using Earth based observations and a precessing ellipse model. The value of Miranda's mass was also divided by two after the encounter. On the other hand the masses of Ariel and Umbriel agreed well with the previous estimation of Veillet, their determination being mostly based on the Laplacian term $\lambda_1 - 3\lambda_2 + 2\lambda_3$ which has an amplitude of about 3000 km in the longitude of Miranda (Veillet, 1983b; Lazzaro et al., (1984)) (it is the largest perturbation in the whole system). The general understanding was then that the Earth-based observations were not accurate enough to allow any determination of the masses of Miranda, Titania and Oberon.

Although the general theory GUST was completed before the encounter (P1) which then introduced the possibility of good determination of the outer satellites masses through the optical navigation data, the fit of the theory to real data was not achieved until several months after the encounter. The Voyager optical and radio data provide now a very high precision set of observations which need to be included in the final determination of the parameters of the theory. Even after the encounter, it is still interesting to see what results can be extracted from Earth-based observations only, using the complete dynamical theory GUST.

To have a realistic idea of the efficiency of the theory, we started the fit of the theory using the pre-encounter values of all parameters which were used in (P1). The summary of the results of the fit is given in Table 12. After the first estimation, the masses of the satellites already come much closer to the final values determined by Voyager. The central mass also has a large change of about 9000 $km^3 s^{-2}$ which brings it very close to the Voyager value. At the first iteration, the other parameters also changed significantly, and three more iterations were made to ensure a good convergence of the solution (the last iteration did not give any significant change to the parameters). The final values of all parameters determined by Earth-based data are given in Table 3 with their uncertainty estimated to 3 times their formal standard error after several different estimations.

The results on the determination of the masses are surprisingly consistent with the Voyager results. All masses are within 15% of Voyager masses, and Miranda's mass is even within 5% of Voyager's value. The system mass is within 0.05% of Voyager determination.

With the Earth-based data, the theory, although less accurate than the numerical integration, is in fact a much easier tool to use.

The dependence of the theory with respect to its parameters is nearly linear, which is not the case in a numerical integration. This is of no importance when one starts with initial values close to the solution, but when all parameters are not well determined it is very difficult to converge with a numerical integration until one reaches the linear range of the parameters.

The other main advantage of the theory in this case is that it can be used over a larger span of time than a numerical integration especially in computation of the partial derivatives with respect to the parameters. This allows us to use observations from 1911 to 1986 which constrain the mean mean motions N_i and mean longitudes at the origin λ_{0i}.

Looking to Table 3b, we can see that apart from the masses already discussed, the Earth-based data give very good values of the mean mean motions N_i and of the mean longitudes at the origin λ_{0i}. These values will not be changed much when Voyager data are included. The values of the mean mean motions N_i are especially importannt; they give the long term evolution of the system and the possibility of resonances.

The constants of the secular system in eccentricity (ρ_i, ϕ_i, ($i = 1, 5$)) are also fairly well determined by Earth-based data, except for E_5 (related to Oberon) which has a very long period (about 100 years) and small amplitude. Apart from I_1 (related to Miranda), the constants in the inclinations are poorly determined due to the orientation of Uranus system during the period of the most accurate observations (for a discussion see Veillet, 1983).

We do not give here the residuals of the Earth-based observations against this solution. They are in fact nearly the same as the residuals of the complete solution which are given in Table 13 and Figs 1 and 2.

5.3. Voyager and Earth-based data results

Voyager radio data (range and Doppler) and optical navigation data combined extremely well. The radio data is essentially sensitive to Uranus system mass and to the mass of Miranda during the close approach (Tyler et al., 1986). The final determination of the mass M_{SU} of Uranus system depends in fact essentially on these data.

On the other hand, optical navigation data do not allow the determination of Miranda's mass, while the other parameters are fairly determined. During the relatively short period of the encounter, the optical navigation data are essentially sensitive to the amplitude of all the short period terms in the mutual perturbations of the satellites. This gives a direct way to measure the masses, and also the constants ρ_i through the term of degree 1 in eccentricity (Table 11). The optical navigation data were also sensitive to the geometry of the orbits and improved very much the constants relative to the inclinations, which were very poorly determined using Earth-based data alone (Fig. 5). Miranda's mass was determined using Earth-based and Voyager radio data only, and then kept to this value in the final determination.

6. Conclusion

The theory GUST86 presented here provides a complete ephemeris of the Uranian Satellite System using all Earth-based and Voyager data available. The accuracy of GUST86 is estimated to be of about 100 km to 200 km over about 10 years from 1986, and somewhat more over 100 years. The form of the ephemeris is sufficiently condensed to allow anyone to program it on a small computer. The full theory is analytical and includes all the computations of the partial derivatives which allow to fit the parameters to the observations.

GUST86 is the first analytical ephemeris of the Uranian satellites including the short period terms which were decisive for the determination of the satellites masses using Voyager data. Although its precision is consistent with the level of accuracy of the actual observations, further improvements can be made in several directions to obtain a theory with an internal accuracy of about 1 km:

1) We must increase the number of terms by using a lower level of truncation in the first order computations. Doing this, we will probably have to consider more terms of non-zero degree in the solution, and probably terms of degree 2 or 3 in eccentricity.

2) We will have to consider more terms of the second order with respect to the masses which are usually difficult to compute.

3) We need to consider the solar perturbations. These terms have been estimated in (Dunham, 1971) but were not included in the present theory. The main effect should be a term of long period (relative to Uranus motion) and of amplitude of the order

of 100 km to 200 km. In the present determination, this effect has been absorbed in the determination of the constants, but the accuracy of the solution will probably be degraded within the amount of the perturbation after 20 to 30 years.

4) We would like also to include the second order (and eventually terms of higher degree) in the computation of the secular system to avoid any correction obtained by fitting a preliminary numerical integration.

A program is started in order to make these improvements ready for the reduction of Space Telescope data. In any case, the partial derivatives will not have to be updated, their actual accuracy being far sufficient for any fit to observations. The physical and dynamical parameters determined in this study will also provide a strong basis for any further investigation.

Voyager encounter with Uranus provided an important set of data which allow the determination of all the parameters of the theory. Earth-based data are also very important and give some consistent results with Voyager determinations. Improvement in the accuracy of the determination of the masses (of Miranda, for example) is still needed. It can be achieved through the acquisition of more high quality Earth-based observations over an extended span of time, and probably with Space Telescope data which will then arrive several years after Voyager encounter and could be linked accurately with the present data using the improved version of GUST86.

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