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OCCUPATION NUMBERS FOR REACTING PLASMAS: THE ROLE OF THE PLANCK-LARKIN PARTITION FUNCTION

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ABSTRACT

The physical basis of the Planck-Larkin partition function in equation of state (EOS) and occupation number calculations is examined. It is shown that a many-body activity expansion of the grand cannonical partition function in terms of effective composite particle activities, defined through Planck-Larkin (PL) bound state weight factors, has advantages over an activity expansion in terms of real activities defined through Boltzmann weight factors. The PL effective activity expansion gives the EOS and the real electron and nucleus activities (chemical potentials). Calculation of the EOS does not require a knowledge of actual occupation numbers, but calculation of radiative properties does. In view of this distinction, a complementary Boltzmann activity expansion is presented solely for the purpose of calculating occupation numbers. It is shown that the occupation number relations derived from this expansion can be directly evaluated using the electron and nucleus activities obtained in the EOS calculation.

Subject headings: equation of state — plasmas

I. INTRODUCTION

The divergence of the atomic partition function has been discussed in numerous papers (e.g., McChesney 1964). Even though a rigorous solution to this problem was developed more than twenty years ago (Larkin 1960), ad hoc techniques are still being used (Noels, Scuflaire, and Gabriel 1984). The rigorous solution involves the so-called Planck-Larkin partition function (PLPF), given by

$$\mathrm{PLPF} = \sum_{nl} (2l + 1)(e^{-\beta E_{nl}} - 1 + \beta E_{nl}) , \qquad (1)$$

Where n and l are quantum numbers, $\beta = 1/kT$ is the inverse temperature, k is Boltzmann's constant and E_{nl} is the binding energy. Equation (1) is always convergent, since the divergent terms are excluded. The PLPF has been applied to astrophysics in at least one case (Ulrich 1982).

The main purpose of this paper is to examine the physical basis of the PLPF and to present a consistent procedure for its use in equation of state (EOS) and occupation number calculations. In addition, I want to show that some recent criticisms of the PLPF by Rouse (1983, and references therein) are unfounded. Ebeling *et al.* (1985, and references therein) have already responded to the Rouse comments. Nevertheless, there is still reluctance within the astrophysics community to use the PLPF in EOS calculations. This is due to the fact that the occupation numbers predicted by the PLPF appear to be inconsistent with observation. This is discussed in detail by Däppen, Anderson, and Mihalas (1986). They also discuss methods for calculating occupation numbers from which they calculate hydrogen emissivities in good agreement with the experiments of Wiese, Kelleher, and Paquette (1972). A point to be established here is that the apparent shortcoming of the PLPF as a predictor of optical properties is the result of a misinterpretation. The PLPF is the most natural and accurate way to calculate the EOS, but in fact it does not give the actual occupation numbers. At low density, however, it will be shown that it is straightforward to obtain the actual populations from those given by the PLPF.

Much of what is discussed in this paper has appeared elsewhere in greater detail (Rogers 1981, and references therein). The purpose here is to present a clear explanation of why and how the PLPF should be used in astrophysical calculations of the EOS and occupation numbers of reacting plasmas. This is accomplished through a many-body activity expansion of the grand canonical partition function, where the activity is a physical quantity defined to have the units of density. It plays a similar role to density in a virial expansion and is proportional to the Gibbs free energy per particle divided by kT (see eq. [16]). A comprehensive discussion of activity (fugacity) expansions for reacting, nonideal, plasmas is given by Ebeling, Kraeft, and Kremp (1976). The initial activity expansion for the partially ionized plasma, which is developed in this approach, views the system in terms of its fundamental particles (electrons and nuclei). This fundamental particle activity expansion is then renormalized to account for the formation of ions and atoms. Since it started from a description of the system in terms of its fundamental particles interacting through the Coulomb potential, no ad hoc assumptions about the internal states of composite particles are required. It is for this reason that the activity expansion method is chosen in preference to the usal free-energy minimization method.

Section II of the paper discusses the analytic properties of the second cluster coefficient. It is demonstrated that the PLPF arises naturally for the Coulomb potential and effectively redefines the continuum as starting at $-kT$. As a result, all long-range divergences in the few-body problem are lumped together in the redefined continuum. Section III gives a discussion of a many-body effective activity expansion that embodies the PL weight factors and explaines why this does not produce true occupation numbers. Section IV presents an activity expansion in terms of Boltzmann composite particle activities. A low-density Saha-like equation for calculating occupation numbers is extracted from this expansion.

II. THEORETICAL BASIS OF THE PLANCK-LARKIN PARTITION FUNCTION

The PLPF first arose in a systematic many-body perturbation expansion of the logarithm of the grand partition function, ln $\Xi(\mu, V, T)$. Even though this procedure is rigorous, it is not very illuminating. A clear understanding of why the PLPF arises can be obtained from a study of the analytic properties of individual cluster coefficients (Rogers 1977, 1979). A rigorous discussion of the PLPF for hydrogenic plasmas can be found in Ebeling, Kraeft, and Kremp (1976). The many-body problem requires a simultaneous treatment of all cluster coefficients, but for the present discussion a study of the second cluster (virial) coefficient will suffice. The electron (e)–nucleus (α) second cluster coefficient is given by

$$b_{e\alpha} = \frac{1}{2V} \lambda_e^3 \lambda_\alpha^3 \frac{(Q_{e\alpha} - Q_e Q_\alpha)}{(2s+1)^2} , \tag{2}$$

where $\lambda_i = h/(2\pi m_i kT)^{1/2}$ is the thermal wavelength, $i = \{e, \alpha\}$, Q_e and Q_α are one-body partition functions, and $Q_{e\alpha}$ is the electron–nucleus two-body partition function. The physical significance of $b_{e\alpha}$ is apparent; it gives the correction to the EOS that results from switching on two-body interactions, i.e., the effect of distorting the ideal gas plane wave eigenstates due to two-body encounters. This process conserves the number of states but changes their character and may even transform some continuum states into bound states. The second cluster coefficient is proportional to the difference between the "sum over states" before and after switching on the interaction.

Beth and Uhlenbeck (1937) showed that $b_{e\alpha}$ could be described in terms of bound states E_{nl} and scattering state phase shifts δ_l, according to

$$b_{e\alpha} = b^b_{e\alpha} + b^c_{e\alpha} , \tag{3}$$

where

$$b^b_{e\alpha} = a_{e\alpha} \sum_{nl} (2l+1) e^{-\beta E_{nl}} , \tag{4}$$

$$b^c_{e\alpha} = \frac{a_{e\alpha}}{\pi} \int dp f_0 \exp\left(\frac{-p^2}{2\mu_{e\alpha} kT}\right) , \tag{5}$$

$$a_{e\alpha} = 4\pi^{3/2} \lambda^3_{e\alpha} ,$$

$$\lambda_{e\alpha} = h/(2\mu_{e\alpha} kT)^{1/2} ,$$

$$f_0 = \sum_l (2l+1) \frac{d\delta_l(p)}{dp} , \tag{6}$$

$\mu_{e\alpha}$ is the reduced mass, and p is the relative momentum.

The origin of the -1 subtraction in the PLPF is easy to identify. An integration by parts of $b^c_{e\alpha}$ gives

$$b^c_{e\alpha} = \frac{a_{e\alpha} f_1(0)}{\pi} + b^{c'}_{e\alpha} , \tag{7}$$

where

$$f_1(p) = \sum_l (2l+1) p \delta_l(p) , \tag{8}$$

$$b^{c'}_{e\alpha} = \frac{2a_{e\alpha}}{\mu_{e\alpha} kT} \int_0^\infty dp f_1 \exp\left(\frac{-p^2}{2\mu_{e\alpha} kT}\right) . \tag{9}$$

According to Levinson's theorem, $\delta_l(0) = n_l \pi$, with n_l the number of bound states having angular momentum l. The zero-energy term in $b^c_{e\alpha}$, therefore, exactly removes the leading term in the high-temperature expansion of $b^b_{e\alpha}$. This is a result of the fact that the ideal gas states are distorted by the introduction of a potential, but the total number of states is conserved. It is important to note that the integration by parts of $b^c_{e\alpha}$ has effectively redefined the continuum, i.e., $f_0(p)$ has been replaced by $f_1(p)$.

There exist higher order Levinson theorems (Rogers 1977; Bollé 1981, and references therein) such that it is possible to continue to integrate $b^{c'}_{e\alpha}$ by parts. One additional integration by parts and cancellation with $b^b_{e\alpha}$ gives

$$b_{e\alpha} = b^{b''}_{e\alpha} + b^{c''}_{e\alpha} , \tag{10}$$

where

$$b^{b''}_{e\alpha} = a_{e\alpha} \text{ PLPF} , \tag{11}$$

$$b^{c''}_{e\alpha} = \frac{4a_{e\alpha}}{(\mu_{e\alpha} kT)^2} \int_0^\infty dp f_2(p) \exp\left(\frac{-p^2}{\mu_{e\alpha} kT}\right) , \tag{12}$$

and

$$f_2(p) = \sum_l (2l+1) \int dp p^2 \delta_l(p) . \tag{13}$$

Equation (11) shows that the PLPF is part of the second cluster coefficient and is not a true partition function. This is the origin of the misunderstanding concerning the physical interpretation of the PLPF.

It is possible to continue this process of integration by parts, but there is no particular reason to do so, since the first two integrations by parts (specializing to the Coulomb potential) have accomplished the following:

1. They have removed all divergences in $b^b_{e\alpha}$ by compensation with scattering state contributions. There are additional divergences in $b^{c''}_{e\alpha}$, but they do not compensate with bound-state contributions. These divergences are removed through many-body resummations involving all cluster coefficients.
2. They have redefined the continuum such that weak bound states lying above $-kT$ are treated along with scattering states in a many-body perturbation expansion. States lying below $-kT$ are treated as strong bound states whose contribution to the EOS is dominant at low T.

These considerations are very useful in carrying out a systematic expansion of the EOS. As a consequence, certain artificial contributions to the EOS are canceled at the outset rather than carried through the entire analysis and canceled at the end. Furthermore, the number of basic constituents that have to be explicitly considered is greatly reduced, consisting only of electrons, nuclei, and electron–nucleus composites in strong bound states. It is important to note that the artificial divergences that are being compensated for apply to the EOS; the actual occupation of high-lying states is a different question. The separation of $b_{e\alpha}$ given by equation (10) applies explicitly to a Coulomb or screened Coulomb potential and is potential-dependent. Similar compensation is also present in all higher electron–ion cluster coefficients (Rogers 1979), and it is taken into account in the many-body resummations.

III. EQUATION OF STATE

Due to the long range of the Coulomb potential, the EOS of reacting plasmas is inherently a many-body problem. The natural way to handle this problem is through a many-body activity expansion of ln Ξ (Bartsch and Ebeling 1971; Rogers and DeWitt 1973). For a single element this expansion involves only the activities of electrons z_e and nuclei z_α. However, when bound states are appreciably occupied, it is necessary to define new activities that represent composite particles. Nevertheless, these new activities are proportional to products of the basic activities z_e and z_α. Next, the grand partition function is renormalized on the basis of the augmented set of activities. Since the analysis began from a description of the plasma in terms of its fundamental constituents, no loss of generality is introduced through this renormalization. Fortunately, the analytic properties of the cluster coefficients, which are discussed for the second cluster coefficient in § II, restrict the number of composites states that need to be explicitly assigned an activity. This produces an expansion in PL activities rather than Boltzmann activities and correspondingly does not yield true occupation numbers. It is shown in § IV that the z_e and z_α obtained in the EOS calculation can be used to evaluate terms in a complementary Boltzmann activity expansion that does give actual occupation numbers.

A detailed description of the PL activity expansion is given by Rogers (1981). Only the leading terms given in that work are required for the present discussion. Truncation of the many-body PL activity expansion after squared power terms gives, for the pressure P divided by kT,

$$\frac{P}{kT} = \sum_i \zeta_i + S_R(\bar{\lambda}_A) + \sum_i \zeta_i \left[\frac{\partial S_R(\bar{\lambda}_A)}{\partial \zeta_i}\right]^2 + \sum_i \sum_j \zeta_i \zeta_j s_{ij}(\bar{\lambda}_A) + \dots , \quad (14)$$

subject to

$$\bar{\rho}_i = \zeta_i \frac{\partial P/kT}{\partial \zeta_i} , \quad (15)$$

where the $\bar{\rho}_i$ are effective occupation numbers, optimized for the EOS. In the particular case of hydrogenic plasmas, i and j range over (e, α, H^{1s}, H^{2s}, ...], and H^{nl} signifies a hydrogenic ion in the state nl. The ζ_i for electrons and nuclei are real activities given by

$$\zeta_i \equiv z_i = (2s_i + 1)\lambda_i^{-3} e^{\mu_i/kT} . \quad (16)$$

where μ_i is the chemical potential.

For composite particles the ζ_i are effective activities defined by PL weight factors according to

$$\zeta_i = z_e z_\alpha a_{e\alpha} g_i (e^{-\beta E_i} - 1 + \beta E_i) , \quad (17)$$

where g_i is the statistical weight.

Due to the nonexponential form of equation (17), the composite particle activities are not the actual activities for state nl, This point is discussed in § IV. Guidelines for the values of nl to be included in the sums in equation (14) are also discussed in § IV. The terms in equation (14) involving S_R are related to the Debye-Hückel correction that arises in a density expansion and are given by (for nondegenerate plasmas)

$$S_R = \frac{1}{12\pi\bar{\lambda}_A^3} ,$$

$$\bar{\lambda}_A = \left[\frac{kT}{4\pi e^2 (\sum_i z_i^2 \zeta_i)}\right]^{1/2} ,$$

where Z_i is the charge on species i and $\bar{\lambda}_A$ is the activity-dependent screening length. In the low-density limit, $\zeta_i \rightarrow \bar{\rho}_i$, the number density; and $\bar{\lambda}_A \rightarrow \bar{\lambda}_D$, the effective Debye length; consequently, S_R approaches a Debye-Hückel–like result. The s_{ij} in equation (14) are closely related to the second cluster coefficients for a screened Coulomb potential having a screening length $\bar{\lambda}_A$; except that due to the many-body resummation, first- and second-order perturbation terms are excluded from the s_{ij}. Explicit definitions of the s_{ij} are given by Rogers (1981). It is important to point out that, whereas the E_i that enter the fundamental particle activity expansion are screened, the E_i that go in the PL activities (eq. [17]) are the isolated ion values. This is a consequence of the renormalization to create activities for composite particles described earlier. Due to the subtraction of the leading two terms at high temperature (for reasons discussed in § II), the ζ_i corresponding to $-\beta E_i < 1$ effectively do not contribute to P/kT. This is particularly important at high density, where many terms contribute to the pressure (eq. [14]).

It is possible to relate equations (14) and (15) to the more familiar free-energy minimization method. If only terms involving S_R are retained in equation (14), it can be shown that the free energy is given by

$$\frac{F}{kT} = -N_e \ln \left(\frac{2e}{\bar{\rho}_e \lambda_e^3}\right) - N_\alpha \ln \left(\frac{2e}{\bar{\rho}_\alpha \lambda_\alpha^3}\right) \times -N_{\mathrm{H}} \ln \left(\frac{e \, \mathrm{PLPF}}{\bar{\rho}_{\mathrm{H}} \lambda_{\mathrm{H}}^3}\right) - V S_R(\bar{\lambda}_D) , \quad (18)$$

where

$$\bar{\rho}_{\mathrm{H}} = \sum_k \bar{\rho}_k ,$$

$$k = \{\mathrm{H}^{nl}\} ,$$

and

$$\bar{\lambda}_{\mathrm{D}} = \left(kT/4\pi e^2 \sum_i z_i^2 \bar{\rho}_i\right)^{1/2} .$$

The resulting Saha-like equation that gives the effective number densities for EOS calculations is (Rogers 1984, 1985)

$$\frac{N_e N_\alpha}{N_H} = \frac{2V\lambda_{\mathrm{H}}^3 \exp(ze^2/kT\bar{\lambda}_D)}{\lambda_e^3 \lambda_\alpha^3 \, \mathrm{PLPF}} . \quad (19)$$

Higher order terms in equation (14) can also be included in the analysis (Ebeling *et al.* 1985), but this is not required for the present discussion.

IV. OCCUPATION NUMBERS

Section III gave the rigorous low-density Saha-like equation that should be used for EOS calculations. It is clear from § II that this does not produce actual occupation numbers, i.e., EOS contributions from high-lying states have been canceled with the continuum contributions. This procedure has effectively redefined the continuum. As a result the N_e in equation (19) includes electrons in weak bound states as well as electrons in continuum states. This was done for compelling procedural reasons, but in principle it is not necessary. In fact, we could carry along the compensating terms throughout the entire analysis. For reasons already enumerated, this would greatly complicate the problem. The results of § III produce the correct EOS and unique values of z_e and z_α (see eq. [16]). However, the definition of effective composite particle activities is somewhat arbitary, i.e., slightly different definitions for the effective composite particle activities will yield identical results for P, E, z_e, and z_i, provided a sufficient number of terms are used in the different versions of equation (14).

If instead of using PL weight factors to define composite particle activities we use the true continuum, the activities are given directly in terms of Boltzmann factors according to

$$z_k = z_e z_\alpha a_{e\alpha} g_i e^{-\beta E_k} ,$$
$$k = \{\mathrm{H}^{nl}\} . \tag{20}$$

Comparison with equation (17) shows that $z_k = h_k \zeta_k$, where $h_k = e^{-\beta E_k}/(e^{-\beta E_k} - 1 + \beta E_k)$.

The z_k defined in equation (20) are real activities. Now a renormalization similar to the one leading to equation (14) gives

$$\frac{P}{kT} = \sum_i z_i + S_R(\lambda_A) + \sum_i \frac{z_i}{2} \left[\frac{\partial S_R(\lambda_A)}{\partial z_i}\right]^2 + \sum_i \sum_j z_i z_j s_{ij}(\lambda_A) + \dots , \tag{21}$$

subject to

$$\rho_i = z_i \frac{\partial(P/kT)}{\partial z_i} , \tag{22}$$

where i ranges over (e, α, $\{\mathrm{H}^{nl}\}$), $i = k$ for composites (see eq. [20]),

$$S_R = \frac{1}{12\pi\lambda_D^3} ,$$
$$\lambda_A = \left(\frac{kT}{4\pi e^2 \sum_i Z_i^2 z_i}\right)^{1/2} , \tag{23}$$

and the s_{ij} are now defined directly in terms similar to $b_{e\alpha}^c$ for a screened Coulomb potential. i.e., equation (5) instead of equation (12). Equation (21) is an expansion in Boltzmann activities and can be used to obtain occupation numbers. All the divergences present in the few-body problem have been removed through many-body correlations. If properly carried out, equations (14) and (15) and equations (21) and (22) will give exactly the same P, E, z_e, and z_α. Consequently, the results of a self-consistent solution of equations (14) and (15) can be used directly to evaluate equations (21) and (22).

As an explicit example of the method, consider the occupation number of hydrogenic composites in quantum state nl. Direct differentiation of equation (21) gives

$$\rho_\nu = z_\nu + \frac{3}{2}(Z - 1)^2 z_\nu \left[\frac{S_R}{U} + \left(\frac{\partial S_R}{\partial U}\right)^2 + \left(\frac{\partial S_R}{\partial U}\right)\left(\frac{\partial^2 S_R}{\partial U^2}\right)\right] + 2z_e z_\nu s_{e\nu} + 2z_\alpha z_\nu s_{\alpha\nu} + 2z_\nu^2 s_{\nu\nu} + \sum_i \sum_j z_\nu \frac{\partial S_{ij}(\lambda_A)}{\partial Z_\nu} + \dots , \tag{24}$$

where $\nu \equiv \mathrm{H}^{nl}$, $U = \sum_i Z_i^2 z_i$, i and s range over (e, α, $\{\mathrm{H}^{nl}\}$), and z_ν is obtained from the solution of equations (14)–(15) according to $z_\nu = h_\nu \zeta_\nu$. Explicit definitions of the s_{ij} can be found in Rogers (1981). It is important to note that: (1) The Debye-Hückel–like terms in equation (24) do not depend on the quantum numbers nl, i.e., they change the total number density but not the relative occupation numbers; and (2) The terms involving two-particle and higher scattering contributions (not considered herein) explicitly depend on the quantum states of the composite particles involved in the scattering. This effect increases with increasing n. Consequently, the relative distribution of states falls off more rapidly than predicted by a simple Boltzmann distribution.

To relate equations (21) and (22) to the free-energy minimization method, these equations are truncated after terms linear in S_R. Since most particles are in low-lying states which are not much affected by the PL compensation, $\bar{\lambda}_A$ and λ_A are not very different, so that

$$\rho_k \approx h_k \bar{\rho}_k . \tag{25}$$

This relation does not hold for states for which higher order terms are important; nevertheless, the occupation numbers can be systematically calculated from the procedure presented here. In the low-density limit the occupation number version of equation (19) now takes the form

$$\frac{N_e N_\alpha}{N_\mathrm{H}} = \frac{2\lambda_H^3 V}{\lambda_e^3 \lambda_\alpha^3} \exp \frac{(Ze^2/kT\lambda_\mathrm{D})}{\phi} , \tag{26}$$

where

$$\phi = \sum_k g_k e^{-\beta E_k} \tag{27}$$

is the usual internal partition function; except, for reasons discussed further on, equation (27) include only a finite number of bound states. In the low-densities limit, the occupation of individual internal states is given by

$$\frac{N_k}{N_\mathrm{H}} = \frac{g_k e^{-\beta E_k}}{\phi} . \tag{28}$$

The sole advantage of equation (26) is that it estimates actual occupation numbers. However, due to the slower convergence properties of equation (21), described earlier, percentage errors in the occupation numbers given by equation (26) are larger than those for the EOS obtained through equation (19). This disparity can be reduced by inclusion of higher order terms.

Equation (26) shows that the Debye-like terms which come from a solution of the many-body problem have the effect of lowering the continuum, so that weak bound states are "swallowed" by the lowered continuum. At low density there are many states that lie between the Planck-Larkin cutoff at $-kT$ and the lowered continuum. A proper interpretation of the renormalized activity expansion given by equation (21) requires a careful study of how these states should be treated.

Figure 1 shows the 1s energy level in the exponentially screened Coulomb (Debye-like) potential that arises in the fundamental particle activity expansion when λ_e/λ_A is small. For large values of λ_A, the energy can be expanded in inverse powers of λ_A. Near the zero-energy cutoff, c_{1s}, the energies can be expanded in powers of $\lambda_A - c_{1s}$. The leading terms in these expansions are also plotted in Figure 1. In the region where inverse power expansions can be used, it is possible to systematically define an activity for a composite particle in state nl, such that the renormalized activity expansion can be expressed according to equation (21). As a result, unscreened energy levels appear in the definition of z_k. When a level is close to the continuum this is not possible. Consequently, a new activity should not be introduced for particles in these states, i.e., the effect of these states is left in the electron–nucleus s_{ij} (see eq. [21]). At some intermediate value of λ_A, as the density increases, it is necessary to quit defining a new activity for the state nl, so that i and j range over one less activity. The state nl is present in both forms of the expansion, but it is manifested in

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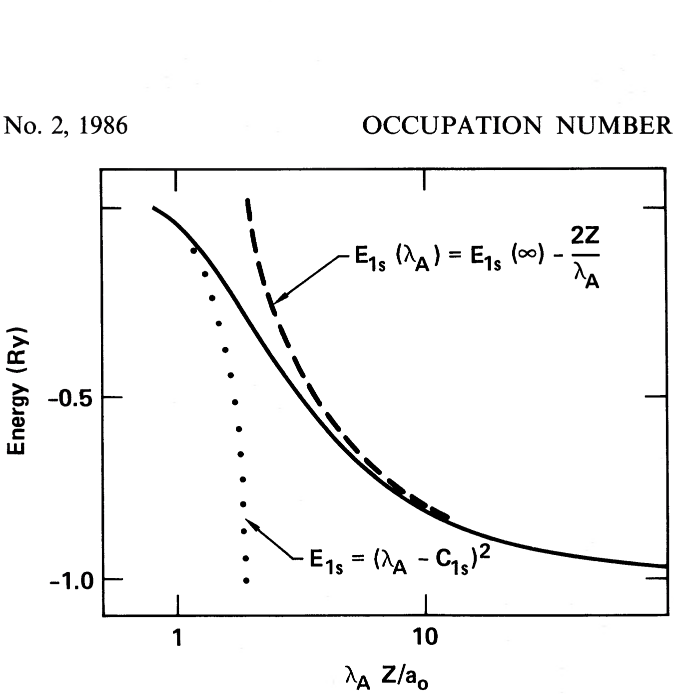

FIG. 1.—Energy vs. screening length for an exponentially screened Coulomb potential (*solid curve*). Leading terms in the weak and strong screening expansions are also plotted. In the weak screening limit (*dashed line*) the energy is expandable in inverse powers of λ_A. Second-order terms in this expansion depend explicitly on nl. A completely different expansion (*dotted line*) applies in the strong screening limit.

different ways. Since the term under consideration is small compared to the dominant terms, the calculation should be insensitive to the switching procedure. A convenient choice for the value of λ_A at which the treatment of the nl state should be changed is given by the condition

$$E_{nl}(\infty) + \frac{Ze^2}{\lambda_A} = 0 . \tag{29}$$

Figure 1 shows that there is some arbitrariness in condition (29).

It is apparent that both equations (14) and (21) have discontinuities when the switch determined by equation (29) is carried out. This is due to the fact that the activity expansion is truncated after a few terms. Since equation (14) involves the PL weight factors, which largely removes the effect of bound states lying above $-kT$, it is quite insensitive to these discontinuities until the continuum is lowered to $-kT$. Low-order derivatives of the EOS obtained from equation (14) are also not much affected by the switching procedure. It is for these reasons that complementary activity expansions are introduced for the calculation of the EOS and occupation numbers. Formally the EOS and number densities are continuous, and the discontinuities arise from a practical need to truncate the calculation at some point. At very low densities, the merging due to Stark broading leads to an optical lowering of the continuum, which for hydrogenic ions is somewhat larger than the true continuum lowering; consequently, the fine details concerning continuity occur in a region that is not observable as line radiation. The number of observed lines should be consistent with the Inglis-Teller (1939) formula.

A specific example of the foregoing statements is displayed in Figure 2. It shows the location of the various continua in the solar photosphere. The temperature and electron density used by Ebeling *et al.* (1985) have been assumed in this example, i.e., $T = 5785$ K and $n_e = 10^{13}$ cm^{-3}. Figure 2 shows that continuum lowering in the plasma causes scattering state energies to overlap bound state energies for $n > 106$. This overlap, even disregarding line-broadening mechanisms, diminishes the observability of states up to $n = 167$, the maximum allowed principal quantum number in the Debye-Hückel potential for the conditions that exist in the photosphere. According to the arguments of § IV, occupation numbers can be calculated for states having $n < 106$, but not for states having $106 < n < 167$ (see the discussion of Fig. 1). The presence of bound states in the continuum could lead to resonance phenomena.

Figure 2 shows that optical lowering of the photospheric continuum due to line merging, as qualitatively predicted by the Inglis-Teller formula, prevents the observation of any line having $n > 21$. This is in reasonable agreement with the observations of Moore, Minneart, and Houtgast (1966), who observed members of the Balmer series up to $n = 17$. Since continuum lowering varies more rapidly than optical lowering with increasing Z, it can be larger than optical lowering for highly ionized high-Z ions. When this occurs, continuum lowering, rather than optical lowering, determines the observability of a given level. In this latter case, even the last few observable line profiles may not show appreciable overlap. This phenomenon probably cannot be observed in radiation from astrophysical objects, but it is being studied using laser-driven gas microballoon implosions (Goldsmith, Griem, and Cohen 1984; Delamater *et al.* 1985).

Figure 2 also shows that Planck-Larkin compensation places the effective continuum between $n = 5$ and 6. This means that states having $n > 5$ make small individual contributions and should, for EOS calculations, be treated together

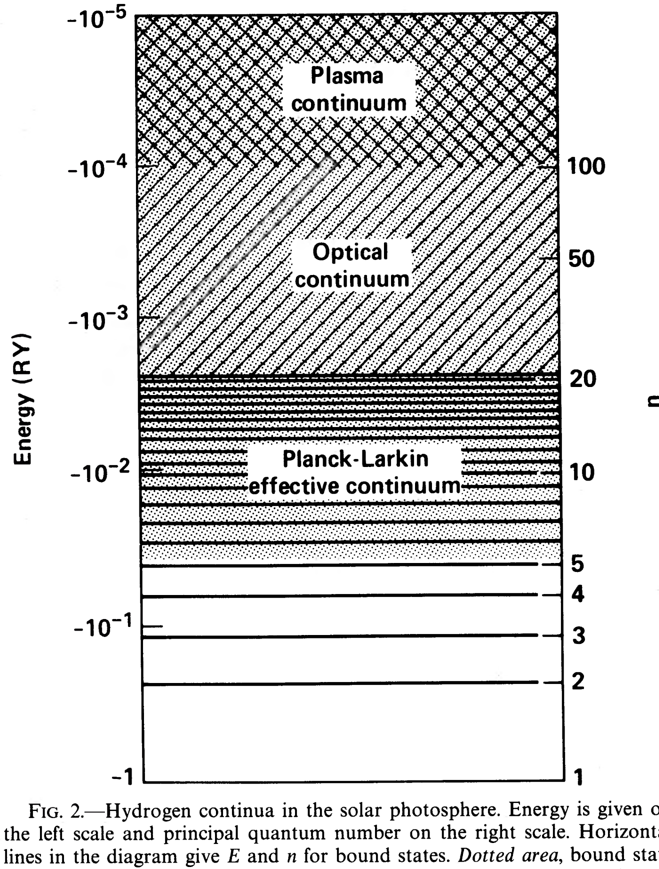

FIG. 2.—Hydrogen continua in the solar photosphere. Energy is given on the left scale and principal quantum number on the right scale. Horizontal lines in the diagram give E and n for bound states. *Dotted area*, bound state range that is part of the Planck-Larkin continuum. *Singly hatched area*, bound state range subject to line merging. *Cross-hatched area*, location of the plasma continuum.

with scattering states in a systematic perturbation expansion. At low density, the location of the Planck-Larkin effective continuum is independent of density but, according to the above discussion, it will move down at high density.

V. CONCLUSION

A discussion of how the PLPF arises in a systematic many-body analysis of the EOS has been given. Complementary activity expansions for the calculation of EOS and occupation numbers were presented. It was shown that the PLPF is important for the calculation of both the EOS and occupation numbers. Due to the wide use of free-energy minimization methods in astrophysical applications, it was demonstrated that, in the low-density limit, the Saha-like equation that gives effective occupation numbers for the EOS is different from the Saha-like equation that gives occupation numbers for radiation. This latter equation is somewhat similar to what is already used, except that in current astrophysical applications the same equation is also used to calculate the EOS. The point is that this is incorrect. Likewise, it is incorrect to use effective occupation numbers derived directly from the PLPF in applications that require actual occupation numbers.

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