ON THE MINIMUM DISTANCE BETWEEN TWO KEPLERIAN ORBITS WITH A COMMON FOCUS

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Abstract. The methods used so far for determination of the closest approach between two orbits are discussed, and corrected versions of two of them are presented.

1. Introduction

The problem of the minimum distance between two orbits may be encountered for example in:

- determination of close approaches of comets and planetoids to the Solar System planets,
- calculation of the comentary radiants,
- investigation of the evolution of meteor streams.

The problem of the minimum distance has been considered by several authors, for example by Dubyago (1949), Kramer (1953), Lazovič (1967, 1981), Sitarski (1968), Babadjanov *et al.* (1980), Murray *et al.* (1980), and Hoots *et al.* (1984). However, methods described by these authors have some disadvantages from the viewpoint of practical computer calculations. These disadvantages result from the necessity of having suitable initial values (Lazovič (1967, 1981), Murray *et al.* (1980), Hoots *et al.* (1984)) or from the limited applicability of the methods proposed (Dubyago (1949), Sitarski (1968), Babadjanov *et al.* (1980)).

In this paper we propose a solution of the problem of determination of the minimum distance between two arbitrary common focus orbits which does not have these disadvantages.

2. Preliminaries

Let Σ_1 and Σ_2 be arbitrary common focus Keplerian orbits (ellipses, parabolas or hyperbolas) defined by the perihelion distance q_k, the eccentricity e_k, the longitude of the ascending node Ω_k, the argument of the perihelion ω_k and the inclination i_k, for $k = 1, 2$, respectively.

The problem of determination of the closest approach between orbits Σ_1 and Σ_2 is

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reduced to minimization of the function:

$$D(f_1, f_2) = (\bar{r}_2 - \hat{S}\bar{r}_1)^T \cdot (\bar{r}_2 - \hat{S}\bar{r}_1), \tag{1}$$

where f_1, f_2 are the true anomalies,

$$\bar{r}_k = r_k(\cos f_k, \sin f_k, 0)^T,$$

$$r_k = \frac{q_k(1 + e_k)}{(1 + e_k \cos f_k)}, \tag{2}$$

$$k = 1, 2,$$

and $\hat{S} = \{s_{ij}\}$, $i, j = 1, 2, 3$ is the orthogonal matrix of the transformation between the Σ_1 orbital coordinate system and the Σ_2 one. Elements of $\hat{S}$ are functions of Ω_k, ω_k, i_k $(k = 1, 2)$ only.

From definition (1) it follows that:

$$\underset{(f_1, f_2) \in U}{\forall} D(f_1, f_2) \geqslant 0, \tag{3}$$

thus there obviously exists such a value D^* that:

$$D^* = \inf\{D(f_1, f_2) | (f_1, f_2) \in U\}, \tag{4}$$

where U is the domain of the function D.

Every pair $(f_1^*, f_2^*) \in U$ satisfying the equation:

$$D(f_1^*, f_2^*) = D^*,$$

are solutions to our problem. The set of all such pairs we denote by U^*. The existence of at least one pair $(f_1^*, f_2^*) \in U^*$ is ensured when the domain U and the function D satisfy assumptions of the Weierstrass theorem. Namely, if U is a compact set and D is a continuous function over U then D takes a minimum value somewhere in U.

3. Discussion of the Previously Used Methods

In all the papers quoted in the Introduction, the necessary and sufficient conditions of the local minimum existence were used, i.e.:

$$\frac{\partial D}{\partial f_1} = 0, \qquad \frac{\partial D}{\partial f_2} = 0, \tag{5.1}$$

$$\frac{\partial^2 D}{\partial f_1^2} > 0 \qquad \left(\frac{\partial^2 D}{\partial f_2^2} > 0\right), \tag{5.2}$$

$$\frac{\partial^2 D}{\partial f_1^2} \frac{\partial^2 D}{\partial f_2^2} - \left(\frac{\partial^2 D}{\partial f_1 \partial f_2}\right)^2 > 0. \tag{5.3}$$

When all solutions of the system (5.1) are known, then the conditions (5.2) and (5.3) are examined and finally, the smallest of the local minima may be taken as the solution of the problem.

According to definitions (1) and (2), equations (5.1) may be rewritten in the form:

$$\frac{\partial D}{\partial f_1} = A_1 \cos^2 f_1 + A_2 \cos f_1 + + A_3 \sin f_1 \cos f_1 + A_4 \sin \mathrm{f}_1 + A_5 = 0, \tag{6.1}$$

$$\frac{\partial D}{\partial f_2} = B_1 \cos^2 f_2 + B_2 \cos f_2 + + B_3 \sin f_2 \cos f_2 + B_4 \sin f_2 + B_5 = 0, \tag{6.2}$$

where $A_i = A_i(f_2)$, $B_i = B_i(f_1)$, $i = 1, 2, 3, 4, 5$, or in a different manner, i.e.:

$$\frac{\partial D}{\partial f_1} = A'_1 + A'_2 \sin f_2 + A'_3 \cos f_2 = 0, \tag{7.1}$$

$$\frac{\partial D}{\partial f_2} = B'_1 + B'_2 \sin f_1 + B'_3 \cos f_1 = 0, \tag{7.2}$$

where $A'_i = A'_i(f_1)$, $B'_i = B'_i(f_2)$, $i = 1, 2, 3$.

The equation set (7) has been worked out by Dubyago (1949), Sitarski (1968) and Babadjanov *et al.* (1980) by finding the value $f_2^{(0)}$ from equation (7.1) with a fixed value $f_1^{(0)}$ and next by testing how the obtained pair $(f_1^{(0)}, f_2^{(0)})$ would satisfy equation (7.2) to correct the assumed value of $f_1^{(0)}$.

This procedure may lead to failure because in some cases it is impossible to find the appropriate value of $f_2^{(0)}$ from equation (7.1). This may happen even in the close neighbourhood of the solution (see Figure 1). The procedure described above is equivalent to finding the points of intersection of the plane Π perpendicular to Σ_1 at $P_1(f_1^{(0)})$ with the orbit Σ_2.

In Figure 1a it may be seen that:

$$\Pi(P_1^{(0)}) \cap \Sigma_2 = \phi$$

A similar situation for both orbits Σ_1 and Σ_2 is shown in Figure 1b.

Kramer (1953) described the iterative method for solving equation set (7). With $f_1^{(0)}$ fixed in equation (7.1) the value $f_2^{(0)}$ is obtained. Next from equation (7.2), now with fixed $f_2 = f_2^{(0)}$ a new value $f_1^{(1)}$ is calculated and so on. Apart from the difficulties mentioned earlier such a defined iterative procedure never converges to the minimum. Namely, if $P_1^{(0)} \in \Sigma_1$ is fixed by $f_1^{(0)}$ and $P_2^{(0)} \in \Sigma_2$ described by $f_2^{(0)}$ is obtained from (7.1) then (see Figure 2) $P_1^{(0)}$ is the point of Σ_1 which is the closest to $P_2^{(0)}$. The new point $P_1^{(1)} \neq P_1^{(0)}$ obtained from equation (7.2) with $f_2^{(0)}$ fixed is more

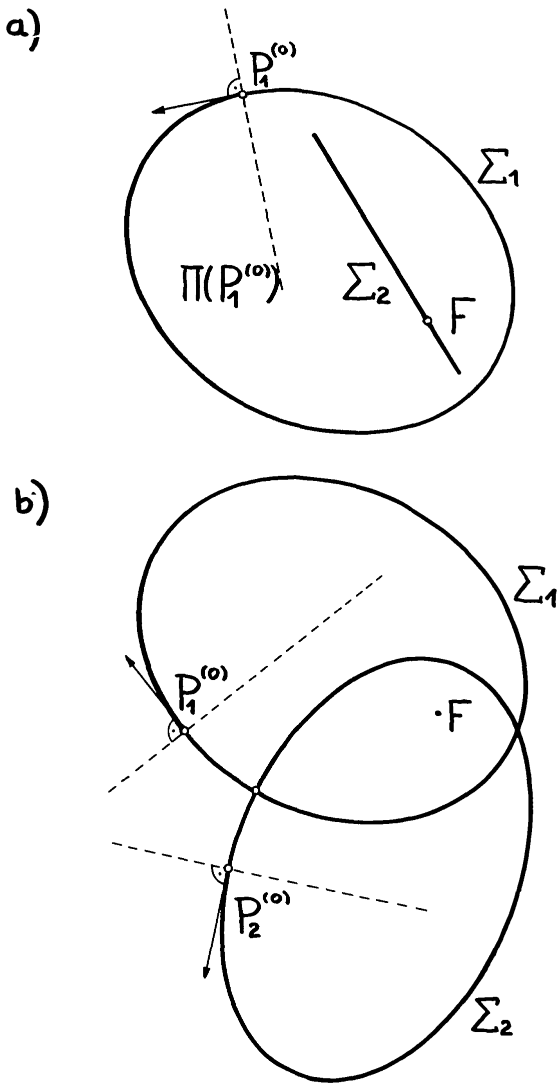

Fig. 1. Examples of the situation when there is no solution of the equation (7.1) with respect to f_2. (a) Σ_1 plane $\perp\Sigma_2$, (b) Σ_1 plane $||\Sigma_2$.

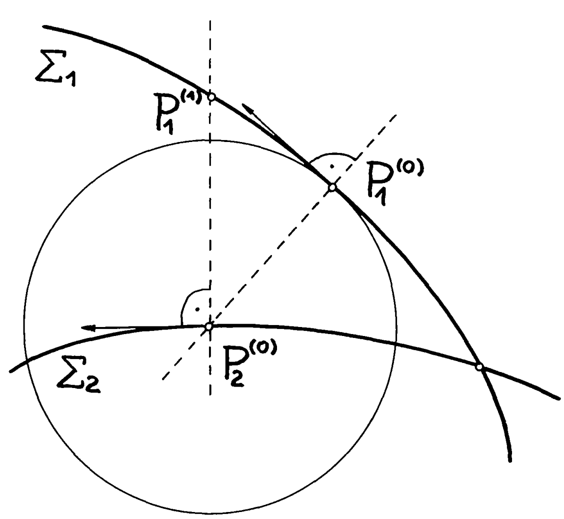

Fig. 2. See text.

distant from $P_2^{(0)}$ then $P_1^{(0)}$, so we have:

$$D(f_1^{(0)}, f_2^{(0)}) < D(f_1^{(1)}, f_2^{(0)}).$$

Consequently, after each step of such an iterative procedure we always have:

$$D(f_1^{(n)}, f_2^{(n)}) \leq D(f_1^{(n+1)}, f_2^{(n)}), \qquad n \in N.$$

The difficulties described thus far result from improper solving of the set (5.1). The first equation of set (5.1) should be solved with respect to f_1 instead of f_2. The second one with respect to f_2 instead of f_1, but the formulae (6) implied by such a procedure are more complicated.

Another group of methods (Lazovič, 1967, 1981), (Hoots *et al.*, 1984), for finding the minimum distance between two orbits (only elliptical) is based on the iterative calculation of corrections Δf_1, Δf_2 to the given starting values $f_1^{(0)}$, $f_2^{(0)}$. These corrections are obtained by Newton's method applied to the equation set equivalent to (6). However such a procedure may be ineffective (see section 5), when the initial values $f_1^{(0)}$ and $f_2^{(0)}$ do not ensure convergence to a local minimum. Such a convergence has not been proved for the starting values determined according to Lazovič (1974, 1976, 1978, 1980, 1981) or to Hoots *et al.* (1984).

Murray *et al.* (1980) solved the system (6) with the numerical method due to Powell, for elliptical orbits only, requiring 'suitable starting values'. The method of choosing the initial values was not presented.

4. Methods Proposed

As was mentioned in the Section 2, the set of solutions may be defined as:

$$U^* = \{(f_1^*, f_2^*) \mid (f_1^*, f_2^*) \in U \wedge D(f_1^*, f_2^*) = D^*\}.$$

To be sure that $U^* \neq \emptyset$ it is sufficient in our case to restrict the domain U to its compact subset $\tilde{U}$ (such a restriction is fully justified from the viewpoint of practical applications). Before that, we replace the true anomaly f_k by a variable γ_k defined by the well-known relations:

(i) when $e_k < 1$ then $\gamma_k = E_k$

$$\mathrm{tg}(E_k/2) = ((1 - e_k)/(1 + e_k))^{1/2} \, \mathrm{tg}(f_k/2), \tag{8.1}$$

(ii) when $e_k = 1$ then $\gamma_k = \sigma_k$

$$\sigma_k = \mathrm{tg}(f_k/2), \tag{8.2}$$

(iii) when $e_k > 1$ then $\gamma_k = H_k$

$$\mathrm{tg}(H_k/2) = ((e_k - 1)/(e_k + 1))^{1/2} \mathrm{tg}(f_k/2), \tag{8.3}$$

for $k = 1, 2$.

Using these variables, equations (6) may be rewritten in a simpler form, with the simple geometrical interpretation of the coefficients, which simplifies the solution of the problem. Now we define the new compact domain as the rectangle:

$$\tilde{U} = \tilde{I}_1 \times \tilde{I}_2$$

where $\gamma_1 \in \tilde{I}_1$, $\gamma_2 \in \tilde{I}_2$ and

$$\tilde{I}_k = [-\pi, \pi] \quad \text{when } e_k < 1, \tag{9.1}$$

$$\tilde{I}_k = [-M, M] \quad \text{when } e_k = 1, \tag{9.2}$$

$$\tilde{I}_k = [-\pi/2 + \varepsilon, \pi/2 - \varepsilon] \quad \text{when } e_k > 1. \tag{9.3}$$

Here $k = 1, 2$, $0 < M < \infty$ arbitrarily large and $\varepsilon > 0$ arbitrarily small.

The methods of determination of the minimum distance between two orbits, presented below, are based, like those discussed earlier, on the solution of the set (5.1). However, this solution is obtained in a different manner, free from the previously presented difficulties and with condition (5.2) included.

Let us rewrite the conditions (5.1) in the form:

$$\frac{\partial D}{\partial \gamma_1} = F_1(\gamma_1, \gamma_2) = 0 \tag{10.1}$$

$$\frac{\partial D}{\partial \gamma_2} = F_2(\gamma_1, \gamma_2) = 0 \tag{10.2}$$

where γ_1, γ_2 are defined by (8). The methods proposed consist in successively solving the equations of set (10) with various fixed values of one of the variables γ_k. Thus, we solve equation (10.1) with respect to γ_1, while equation (10.2) is solved with respect to γ_2. In both cases there are two to four solutions in general; but the detailed geometric analysis makes it possible to obtain the isolation interval containing the only solution satisfying the condition (5.2). In this way we obtain the point of one of the two orbits that is the closest to the fixed point on the other one.

First of all we show how to obtain the solution of equation (10.1) with γ_2 fixed for different types of orbit Σ_1. Equation (10.2) with γ_1 fixed is solved analogously.

Let us consider the following cases:

(a) $e_1 < 1$

We rewrite equation (10.1) using a new variable $E_1 \in \tilde{I}_1$, according to (8.1) and (9.1) in the form:

$$G_1 \sin E_1 - G_2 \sin E_1 \cos E_1 - G_3 \cos E_1 = 0 \tag{11}$$

where:

$$G_1 = a_1(c_1 + \xi_2),$$
$$G_2 = c_1^2,$$
$$G_3 = b_1 \eta_2,$$

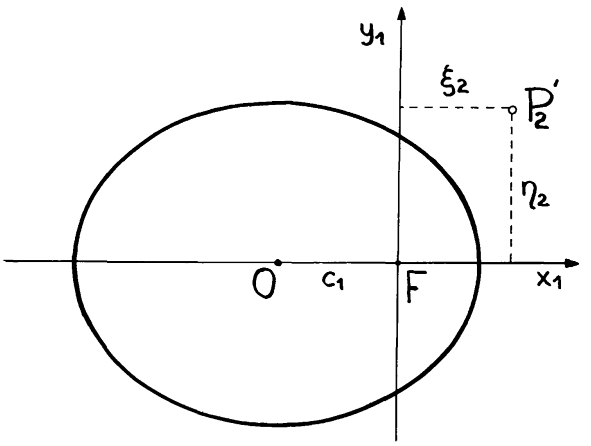

Fig. 3. See text.

and

$$a_1 = q_1/(1 - e_1),$$
$$b_1 = a_1(1 - e_1^2)^{1/2},$$
$$c_1 = a_1 e_1,$$
$$\xi_2 = s_{11} x_2 + s_{21} y_2,$$
$$\eta_2 = s_{12} x_2 + s_{22} y_2.$$

The values ξ_2, η_2 are the coordinates of the point P_2' being the rectangular projection of the point $P_2(\bar{r}_2) \in \Sigma_2$ on to the Σ_1 orbital plane. The values x_2 and y_2 are the coordinates of $\bar{r}_2(E_2)$ in the Σ_2 orbital reference system. The elements of $\hat{S}$ matrix are defined as follows:

$$s_{11} = \bar{P}_1 \cdot \bar{P}_2,$$
$$s_{12} = \bar{Q}_1 \cdot \bar{P}_2,$$
$$s_{21} = \bar{P}_1 \cdot \bar{Q}_2,$$
$$s_{22} = \bar{Q}_1 \cdot \bar{Q}_2,$$

where $\bar{P}_k$ and $\bar{Q}_k$, $(k = 1, 2)$, are the unit vectors of the orbital reference system of the orbit Σ_k. The vector $\bar{P}_k$ is directed towards the perihelion point, the vector $\bar{Q}_k$ towards the point described by the true anomaly $f_k = \pi/2$.

It is easy to see that the value E_1' describing the point P_1 on Σ_1 which is the closest to the fixed point P_2 on Σ_2 may be obtained as follows:

(i) if $G_1 = 0$, $G_2 \geqslant 0$, $G_3 \neq 0$ then $E_1' = \pi/2$ for $G_3 > 0$
$E_1' = -\pi/2$ for $G_3 < 0$

(ii) if $G_1 \neq 0$, $G_2 \geqslant 0$, $G_3 = 0$ then $E_1' = 0$ for $G_1 \geqslant G_2$
$E_1' = \pi$ for $G_1 \leqslant -G_2$
$\cos E_1' = G_1/G_2$ for $G_2 \neq 0$
and $-G_2 < G_1 < G_2$,

(iii) if $G_1 = G_3 = 0,\ G_2 \neq 0$ then $\cos E'_1 = 0$,

(iv) if $G_1 = G_2 = G_3 = 0$ then E'_1 is an arbitrary value from $[-\pi, \pi]$,

(v) if $G_1 \neq 0,\ G_2 = 0,\ G_3 \neq 0$ then $\operatorname{tg} E'_1 = G_3/G_1$ and E'_1 belongs to the interval given in (vi),

(vi) if $G_1 \cdot G_2 \cdot G_3 \neq 0$ then equation (11) may be solved with the arbitrary numerical method (bisections, iteration, etc.) in the following isolation intervals:

if $G_1 > 0$ and $G_3 > 0$ then $E'_1 \in]0, \pi/2[$
if $G_1 < 0$ and $G_3 > 0$ then $E'_1 \in]\pi/2, \pi[$
if $G_1 < 0$ and $G_3 < 0$ then $E'_1 \in]-\pi, -\pi/2[$
if $G_1 > 0$ and $G_3 < 0$ then $E'_1 \in]-\pi/2, 0[$

Notice, that in cases (iii) and (ii) when $-G_2 < G_1 < G_2$ we obtain two equivalent solutions and in the case (iv) an infinite number of solutions.

(b) $e_1 = 1$

With the variable $\sigma_1 \in \tilde{I}_1$ according to (8.2) and (9.2) equation (10.1) takes the form:

$$\sigma_1^3 + G_1\sigma_1 - G_3 = 0 \tag{12}$$

where

$$G_1 = (\xi_2 + q_1)/q_1$$
$$G_3 = \eta_2 q_1$$

and ξ_2, η_2 are defined as previously. The value σ'_1 satisfying equation (12) and the respective condition (5.2) may be obtained as follows:

(i) if $G_1 = G_3 = 0$ then $\sigma'_1 = 0$
(ii) if $G_1 = 0,\ G_3 \neq 0$ then $\sigma'_1 = (G_3)^{1/3}$
(iii) if $G_1 \neq 0,\ G_3 = 0$ then $\sigma'_1 = 0$ for $G_1 > 0$, or $\sigma'_1 = \pm(-G_1)^{1/2}$ for $G_1 < 0$
(iv) if $G_1 \cdot G_3 \neq 0$ then the σ'_1 may be obtained analytically, taking into account that:
if $G_3 > 0$ then $\sigma'_1 \in]0, M]$
if $G_3 < 0$ then $\sigma'_1 \in [-M, 0[$.

Note that in the case (iii), when $G_1 < 0$ we have two equivalent solutions.

(c) $e_1 > 1$

Choosing $H_1 \in \tilde{I}_1$ as a variable according to (8.3) and (9.3) we rewrite equation (10.1) in the form:

$$G_1 \sin H_1 + G_2 \operatorname{tg} H_1 - G_3 = 0 \tag{13}$$

where

$$G_1 = a_1(\xi_2 - c_1)$$
$$G_2 = c_1^2$$
$$G_3 = b_1 \eta_2$$
$$a_1 = q_1/(e_1 - 1)$$
$$b_1 = a_1(e_1^2 - 1)^{1/2}$$
$$c_1 = a_1 e_1$$

and ξ_2, η_2 are defined as in the case of $e_1 < 1$.

The value of H_1' may be obtained as follows:

(i) if $G_1 = 0$ then $\operatorname{tg} H_1' = G_3/G_2$

(ii) if $G_3 = 0$ then $H_1' = 0$ for $G_1 \geqslant -G_2$, or $\cos H_1' = -G_2/G_1$ for $G_1 < -G_2$,

(iii) if $G_1 \cdot G_3 \neq 0$ then we have the following isolation intervals for the solution of (13):
if $G_3 > 0$ then $H_1' \in]0, \pi/2 - \varepsilon]$
if $G_3 < 0$ then $H_1' \in [-\pi/2 + \varepsilon, 0[$.

Note that in the case (ii) when $G_1 < -G_2$ there are two equivalent solutions.

When each equation of the set (10) is solved as shown above it is easy to find the minimal distance between orbits Σ_1 and Σ_2 in every case. To this aim we apply two different methods.

(*a*) *Alternating iterative method*

This method is analogous to the cyclic-coordinate ascent method described, for example by Zangwill (1974). Namely, starting with an arbitrary initial value, for example $\gamma_2^{(0)} \in \tilde{I}_2$, we find (in the way described previously) the value $\gamma_1^{(0)} \in \tilde{I}_1$ satisfying equation (10.1) and additionally condition (5.2). Next having $\gamma_1^{(0)}$ we solve in an analogous manner equation (10.2) obtaining a new value $\gamma_2^{(1)} \in \tilde{I}_2$. A successive repeating of such steps gives the pair $(\gamma_1^{(\alpha)}, \gamma_2^{(\alpha)})$ which approximate the solution of set (10) with required accuracy. The proof of the convergence of such an iterative procedure may be found in Zangwill (1974).

If the obtained pair $(\gamma_1^{(\alpha)}, \gamma_2^{(\alpha)})$ satisfies condition (5.3) then it is the local minimum point of the function D.

Generally, the function $D(\gamma_1, \gamma_2)$ may have more than one local minimum and we cannot find the global minimum in this problem at once. So, the iteration should be repeated for several initial values of $\gamma_2^{(0)}$. The practical calculations have shown that to obtain the global minimum of the function D, it is sufficient to start the iteration from an arbitrary value $\gamma_2^{(0)}$ and next to repeat it for the following starting points: $\gamma_2^{(0)} + \pi/2$, $\gamma_2^{(0)} + \pi$, $\gamma_2^{(0)} + 3\pi/2$ in the elliptical case. When one or both orbits are open curves it is sufficient to make three iterative procedures only.

(*b*) *Scanning method*

The main idea of this method comes from Sitarski (1968). We have changed here only the method of solving the equations of set (10).

Let us choose the sequence of points $\gamma_2^{(i)}$ from the interval $[a, b] = \tilde{I}_2$, $i = 0, 1, \ldots, n$, such that:

$$a = \gamma_2^{(0)} < \gamma_2^{(1)} < \gamma_2^{(2)} < \cdots < \gamma_2^{(n)} = b$$

and

$$\gamma_2^{(m+1)} - \gamma_2^{(m)} = \Delta\gamma = (b - a)/n, \; m = 0, 1, 2, \ldots, n - 1.$$

Putting values $\gamma_2^{(i)}$ into equation (10.1) we obtain (in the manner described previously) the sequence of values $\gamma_1^{(i)}$, $i = 0, 1, \ldots, n$, satisfying the equation:

$$F_1(\gamma_1^{(i)}, \gamma_2^{(i)}) = 0.$$

Next from equation (10.2) we determine the sequence:

$$\gamma_i = F_2(\gamma_1^{(i)}, \gamma_2^{(i)}).$$

By consequent N bisections ($N = (\ln \Delta\gamma - \ln \varepsilon)/\ln 2$, where ε is the required accuracy) of every subinterval $[\gamma_2^{(m)}, \gamma_2^{(m+1)}]$ such that:

$$v_m \cdot v_{m+1} \leqslant 0,$$

we obtain a set of pairs $(\gamma_1^{(\alpha)}, \gamma_2^{(\alpha)})$. Among them there are all local minima of the function D (if the appropriate step has been assumed). For finding the pair describing the global minimum it is sufficient – instead of examining conditions (5) – to calculate the values of function D for all obtained pairs and choose the smallest of them.

5. Numerical testing

The numerical testing of the methods described by Sitarski (1968), Babadjanov *et al.* (1980), Dubyago (1949), and Kramer (1953) seems to be irrelevant, because their disadvantages were shown clearly in Section 3. For testing we chose the following methods:

NL–Newton's method by Lazovič (1967),
NH–Newton's method by Hoots *et al.* (1984),
SM–scanning method described in this paper,
AM–alternating iterative method also described in this paper.

We applied all these methods to find the closest approaches between the Earth' orbit and all elliptical cometary orbits taken from Marsden cataloque (1982). The results obtained show that Newton's method is the fastest one (formulae given by Lazovič (1967) and by Hoots *et al.* (1984) are almost equivalent).

However, the NL and NH methods sometimes do not give the minimum distance between the orbits considered. Examples of such situations are shown in Table I.

Similar cases have occurred several times. In Table II the comparison of reliability of the NH, NL, AM and SM methods on the whole sample of cometary orbits is given.

The NH and NL methods consist of two consequent iterations from two different pairs of starting data. In the NL method we used formulae for starting data proposed by Lazovič (1980). They give values of true anomalies describing the relative nodes of Σ_1 and Σ_2 orbits. In the case of the NH method the first pair was

TABLE I

Minimum distances between Earth's mean orbit (Seidelmann *et al.*, 1974) and selected cometary orbits. For NH and NL methods we stopped calculations when corrections to f_E and f_c were smaller than 1×10^{-5} rad.

Comet	Method	D-min [AU]	f_E	f_c
1882 II	NH	0.598	346°.726	191°.354
	NL	0.516	13.008	168.858
	AM	0.516	13.008	168.858
	SM	0.516	13.008	168.858
1969 I	NH	2.700	342°.300	352°.390
	NL	1407.3	172.981	180.000
	AM	2.700	342.300	352.390
	SM	2.700	342.300	352.390
1980 XI	NH	0.652	63°.266	7°.094
	NL	0.652	63.266	7.093
	AM	0.178	175.542	116.900
	SM	0.178	175.542	116.900

TABLE II

Comparison of four methods of numerical testing

Number of solutions	Method			
	NH	NL	AM	SM
correct	269	273	290	290
incorrect	20[(a)]	17	0	0

(a) In the case of the comet 1889 III it was impossible to obtain the solution by NH method within a sensible machine time.

the same but according to Hoots *et al.* (1984) the second pair of starting anomalies is obtained by:

$$f_1^{(0)} = f_1^* + \pi$$

$$f_2^{(0)} = f_2^* + \pi$$

where f_1^*, f_2^* are solutions obtained from the first iteration. In our opinion this is the reason for the different number of incorrect solutions for NH and NL methods given in Table II.

All incorrect solutions follow from the fact that starting values were out of the convergence region for the Newton's method.

6. Conclusions

It should be stressed that the aim of this paper was to find the method which gives the minimum distance between two orbits of any type and of any configuration. Both methods proposed in Section 4 meet this requirement.

The numerical testing shows that in many cases the NH and NL methods may be used as well as AM and SM ones. In those cases NH and NL methods are superior because they are extremely fast.

However, in our opinion it is rather dangerous to use them because sometimes they give wrong solutions.

The methods proposed by Sitarski (1968), Babadjanov et al. (1980) and Dubyago (1949) may be applied only in the limited number of problems, when the difficulties mentioned in Section 3 do not appear.

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