

THEORY OF AXISYMMETRIC MAGNETOHYDRODYNAMIC FLOWS: DISKS

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ABSTRACT

A general theory is developed for relativistic, steady, axisymmetric, ideal magnetohydrodynamic flows around a black hole or a rotating magnetized star. The theory leads to an autonomous second-order partial differential equation—a Grad-Shafranov equation—for the magnetic flux function $\Psi(r, z)$. One limit of this equation gives the familiar Grad-Shafranov equation which describes the equilibrium of axisymmetric fusion plasmas. Another limit gives the equation describing general nonmagnetic flows of matter with angular momentum. A further limit gives the “pulsar equation” of Scharlemann, Wagoner, and Michel for relativistic plasma flows around an aligned, rotating, magnetized neutron star.

Applications of the theory are made to thin, magnetized disks around a Schwarzschild black hole and around an aligned, rotating, magnetized star. The present work does not consider field-aligned currents or flows, so that the oppositely directed electromagnetic jets discussed in our earlier work are not included. Nevertheless, the magnetic forces may have significant dynamical effects on the disks. For the case of disks around a Schwarzschild black hole, a virial equation is used to show that the ratio ξ of the magnetic energy of the disk to its gravitational binding energy is less than unity. An analysis of the axisymmetric stability of magnetized disks leads to a simple, local stability condition which is a generalization of Rayleigh’s criterion for Couette flows. Using numerical methods, a family of axisymmetrically stable, self-consistent disk equilibria have been generated, which have a single O -point in the equatorial plane inside the disk. For this family we find $\max(\xi) \approx 0.36$. The magnetic force on the disk matter is radially outward (inward) for radial distances less (greater) than that of the O -point. Consequently, the azimuthal velocity of the disk matter inside the O -point may be much less than the Keplerian value. Also, the magnetic field acts to compress the disk vertically.

For the case of disks around a rotating magnetized star, we have generated numerically a family of self-consistent disk equilibria having, with increasing r , an X -point and an O -point in the equatorial plane. Inside the X -point, which must lie within the speed-of-light radius of the star, the disk matter corotates with the star. Outside the X -point, the angular rotation of the disk matter is a function of the flux function Ψ .

Subject headings: black holes — hydromagnetics — relativity — stars: accretion — stars: magnetic

I. INTRODUCTION AND SUMMARY

A disklike distribution of gas arises naturally from the radiative cooling of a rotating gas cloud owing to the slow process of angular-momentum removal. Of particular interest is the nature of the disks subsequent to the formation of a star or a black hole. The theory of thin Keplerian accretion disks around stars or black holes has been extensively developed (Shapiro and Teukolsky 1983, and references therein). The key ingredient of the theory is the turbulent viscosity which allows the gradual outflow of angular momentum and the slow inflow of matter. The infalling matter loses its excess energy to radiation.

The influence of small-scale ordered (Coroniti 1984, and references therein) and fine-scale turbulent magnetic fields (Eardley and Lightman 1975; Galeev, Rosner, and Vaiana 1979; Coroniti 1981; and Sakimoto and Coroniti 1981) has been studied in detail. Less attention has been given to the nature of disks under conditions where there is a highly ordered magnetic field.

Ghosh and Lamb (1979*a, b*) and Lamb (1984) have developed a detailed physical theory of accretion disks surrounding an aligned, rotating neutron star with an intrinsic magnetic dipole field. This theory provides a good fit to the main observational parameters for pulsating X-ray sources. However, the theory does not attempt to give a self-consistent derivation of the magnetic field structure.

Recently, Michel and Dessler (1981) have discussed a heuristic theory of the thin, low-mass disks envisioned to exist around rotating magnetic neutron stars which are seen as radio pulsars. The work of Michel and Dessler was stimulated in part by the observations of a thin plasma disk around Jupiter (see, e.g., Dessler 1983).

The present work develops a general theory for the axisymmetric flows of an ideal relativistic magnetohydrodynamic (MHD) fluid around a star or a black hole. The theory leads to a basic, second-order, nonlinear partial differential equation for the flux function $\Psi(r, z)$ [in a cylindrical (r, ϕ, z) coordinate system]. This equation for Ψ is derived from the basic equations of general relativity in the limit of weak gravitational fields. The function Ψ is such that the poloidal (r, z) projection of a magnetic field line

is described by $\Psi(r, z) = \text{constant}$. The equation for Ψ involves five essentially arbitrary functions of Ψ which arise from (1) the conservation of mass (F); (2) the perfect electrical conductivity (G); (3) the conservation of angular momentum (H); (4) conservation of energy (J , which is Bernoulli's constant); and (5) the conservation of entropy (S). The identification of these functions follows from the works of Lüst and Schlüter (1954), Chandrasekhar (1956*a, b*), Woltjer (1959), Mestel (1961, 1968*a, b*), Heinemann and Olbert (1978), and others (see, in particular, Zehrfeld and Green 1972; Chan and Henriksen 1980; Tsinganos 1981; Hameiri 1983; Aly 1984).

In the absence of flow ($F, G = 0$), the equation for Ψ becomes the Grad-Shafranov equation, which is the basic equation for the equilibrium of axisymmetric fusion plasmas such as tokamaks (Grad and Rubin 1958; Shafranov 1966; Miyamoto 1976). For this reason we refer to our basic equation for Ψ as the generalized Grad-Shafranov equation. Numerical techniques are highly developed for solving the Grad-Shafranov equation for laboratory plasma configurations such as tokamaks (see, e.g., Grimm, Dewar, and Manickan 1983). These techniques appear promising for adaptation to the calculation of astrophysical MHD flows (Coppi and Pegoraro 1979; Mestel, Phillips, and Wang 1979; Mestel and Wang 1979; Sulkanen and Lovelace 1986). Alternatives to the present essentially analytical approach are the numerical simulation of the time-dependent axisymmetric MHD equations (Kuo-Petravic, Petravic, and Roberts 1974, 1975; Sloan and Smarr 1985), and the relativistic particle simulation of the full axisymmetric collisionless electromagnetic equations (Gisler, Norman, and Lovelace 1986).

In the absence of a magnetic field, the equation for Ψ becomes the equation for the Stokes stream function of a general axisymmetric flow (Scott and Lovelace 1982). In a further limit, the equation for Ψ gives, except for an additional term, the "pulsar equation" (Scharlemann and Wagoner 1973; Michel 1973; Cohen, Coppi, and Treves 1973) for relativistic plasma flows around aligned, rotating, magnetized neutron stars.

Detailed applications of the theory are made in the calculation of thin, self-consistent magnetized disk equilibria around a Schwarzschild black hole and around an aligned, rotating, magnetized star. For simplicity, these applications exclude the possibility of field-aligned currents and flows ($F = 0, H = 0$). Thus, the oppositely directed electromagnetic beams proposed earlier as a model for extragalactic double radio sources (Lovelace 1976; Blandford 1976; Lovelace and Ruchti 1983) are not included. Nevertheless, the magnetic forces may have significant, nonperturbative dynamical effects. In particular, the disk equilibria show appreciable departures from Keplerian motion.

For the case of thin disks around a Schwarzschild black hole, simple, nonrelativistic radial and vertical force balance laws are derived from the generalized Grad-Shafranov equation. Radial and vertical virial equations are derived, combined, and used to show that the ratio ξ of the magnetic energy of the disk to its gravitational binding energy is less than unity. An analysis of the axisymmetric stability of the magnetized disks leads to a generalization of Rayleigh's criterion for Couette flows. Using numerical methods, a family of axisymmetrically stable, self-consistent disk equilibria have been generated which have a single O -point of the magnetic field in the equatorial plane. The O -point occurs at a distance larger than the inner radius of the disk. An important aspect of these disk equilibria—which follows from the Grad-Shafranov equation—is that the angular rotation rate of the disk matter is a constant on each flux surface, $\Psi(r, z) = \text{constant}$. Thus, a field line which arches from one radius, r_1 , of the disk to another, r_2 , acts to enforce the equality of the rotation rates of the matter at r_1 and r_2 . Consequently, the angular rotation rate has a nonmonotonic dependence on r : it decreases with increasing distance from the O -point. For this family of equilibria we find $\max(\xi) \approx 0.36$. The magnetic force on the disk matter is radially outward (inward) interior (exterior) to the O -point, and the magnetic field acts to vertically compress the disk.

For the case of thin disks around a rotating star with an intrinsic dipole magnetic field, simple relativistic radial and vertical force balance laws are derived from the generalized Grad-Shafranov equation. Three basic length scales enter: the radius of the star, the "centrifugal radius" (where the gravitational force of the star balances the centrifugal force on a mass element corotating with the star), and the radius of the "speed-of-light cylinder" (where a particle corotating with the star would travel at the speed of light).

We consider disks which extend from the radius of the star to distances well beyond the speed-of-light radius. We then argue that there must be at least one singularity of the magnetic field in the equatorial plane—an X -point—within the speed-of-light radius but outside the star. A simplification of the full relativistic Grad-Shafranov equation allows us to generate numerically a family of self-consistent disk equilibria having, with increasing r , an X -point, and an O -point in the equatorial plane. The disk matter corotates with the star out to the X -point. Beyond this point the angular rotation rate is a function of Ψ , and it decreases going away from the O -point.

In addition to the above-mentioned three length scales, this family of disk equilibria is characterized by two main parameters: the self-magnetic flux of the disk, Φ_d , and a length scale, λ . The radius of the O -point is controlled essentially by λ , and the radius of the X -point by λ and Φ_d . The theory relates Φ_d to the mass of the disk and to the star's mass, magnetic moment, and angular rotation rate.

The disk equilibria of the present work do not include the influence of dissipative processes such as those arising from a turbulent viscosity, an anomalous electrical conductivity, and radiative energy losses. These processes are, of course, crucial in that they determine the rate of matter accretion and the rate of energy losses (Ghosh and Lamb 1979*a, b*). Nevertheless, the forces associated with these processes are expected to be small compared with the equilibrium forces. Thus, a perturbative method can in principle be used to calculate the slow dissipative evolution of a disk. Clearly, the influence of the magnetic field on the turbulent viscosity would need to be included. Also, notice that a simple turbulent viscosity tends to cause the radial inflow of angular momentum for radial distances interior to the O -point, but exterior to the X -point or the inner radius of the disk. Over this range

of distances, however, the above-mentioned axisymmetric instability may occur. This instability and others may have a role in the outflow of angular momentum and the inflow of matter over this range of distances. However, the calculation of the dissipative evolution of the disks is beyond the scope of the present work.

In § II we give a derivation of the Grad-Shafranov equation for axisymmetric magnetohydrodynamic equilibria with gravity and arbitrary nonrelativistic flow. In § III we apply the results of § II to thin magnetized disks around a Schwarzschild black hole. In § IV, we derive the generalized Grad-Shafranov equation for MHD equilibria with weak gravity and arbitrary relativistic flow from the basic equations of general relativity. Finally, in § V, we apply the results of § IV to thin magnetized disks around an aligned, rotating magnetized star.

II. AXISYMMETRIC MHD EQUILIBRIA WITH GRAVITY AND ARBITRARY NONRELATIVISTIC FLOW

This section gives a derivation of the basic equation governing axisymmetric MHD equilibria with gravity and arbitrary nonrelativistic flow. This “generalized Grad-Shafranov equation” has applications to both laboratory and astrophysical plasmas. The work of this section is based mainly on the earlier works of Chandrasekhar (1956*a*, *b*), Woltjer (1959), Mestel (1961, 1968*a*, *b*), and Heinemann and Olbert (1978).

The equilibrium ($\partial/\partial t = 0$) is axisymmetric, so that $\partial/\partial\phi = 0$ in a cylindrical (r, ϕ, z) coordinate system. The basic equations are

- (1) Mass conservation:

$$\nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

where ρ is the mass density and $\mathbf{v}$ is the plasma flow velocity;

- (2) Momentum conservation (Euler equation):

$$\rho(\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \rho \nabla \Phi_g + \frac{1}{c} \mathbf{J}_e \times \mathbf{B}, \quad (2)$$

where $\mathbf{J}_e$ is the plasma current density, Φ_g is the gravitational potential, and p is the plasma pressure, which is assumed isotropic (the case with an anisotropic pressure is discussed in Appendix A);

- (3) Ampere’s law:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_e; \quad (3)$$

- (4) Faraday’s law:

$$\nabla \times \mathbf{E} = 0; \quad (4)$$

- (5) Perfect conductivity equation:

$$\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} = 0; \quad (5)$$

- (6) Poisson’s equation for the gravitational potential:

$$\nabla^2 \Phi_g = 4\pi G [\rho + M\delta(\mathbf{r})], \quad (6)$$

where G is the universal gravitational constant, and M is the mass of a central object (a modified form of Φ_g is considered in § III so as to include the influence of general relativity); and

- (7) Equation of state:

$$S = S(p, \rho), \quad \mathbf{v} \cdot \nabla S = 0, \quad (7)$$

where S is the entropy per particle of the plasma (see, e.g., Landau and Lifshitz 1959, chap. 1). Equation (7) simply states that there is no heat flow into or out of a given mass element of fluid during its motion. Of course, we also have $\nabla \cdot \mathbf{B} = 0$.

In order for the Euler equation (eq. (2)) to be applicable, the Reynolds number $R = vL/\nu$ must be large compared with unity. Here, v denotes a typical flow speed, L a typical length scale, and ν the kinematic viscosity of the plasma. Of course, in a magnetized plasma, the viscosity is anisotropic (Braginskii 1965), so that the requirement $R \gg 1$ depends on the geometry of the

magnetic field and the flow. In order for the perfect conductivity equation (5) to be applicable, the magnetic Reynolds number $R_m = vL/\nu_m$ must be large compared with unity, where $\nu_m = c^2/(4\pi\sigma)$ is the magnetic diffusivity, and σ is the electrical conductivity. R_m is simply the ratio of the diffusion time scale for the magnetic field to the dynamical time scale for the fluid motion. For a fully ionized hydrogen plasma $\nu_m \approx r_e c (v_{te}/c)^{-3}$, where r_e is the classical electron radius, $v_{te} \equiv (k_B T/m_e)^{1/2}$ is the electron thermal speed, and m_e is the electron mass. Evidently, the magnetic Reynolds number is very large compared with unity owing to the large scale size L of the astrophysical disks.

In order for the isentropic equation (7) to be applicable, the “thermal Reynolds number” (or Péclet number) $R_t = vL/\chi_{\parallel}$ must be large compared with unity, where $\chi_{\parallel}$ is the thermal diffusivity parallel to the magnetic field. R_t is simply the ratio of the time scale for the diffusion of heat parallel to the magnetic field to the dynamical time scale for the fluid motion. For a fully ionized hydrogen plasma (with the possible contribution of radiation neglected), $\chi_{\parallel} \approx c(nr_e^2)^{-1}(v_{te}/c)^5$, where n is the plasma number density. (Notice that under fairly general conditions, $\chi_{\parallel} \gg \chi_{\perp}$.) The condition $R_t \gg 1$ corresponds to the number density being large compared with a critical value dependent on v , v_{te} , and L . If, on the other hand, the density is small compared with this critical value ($R_t \ll 1$, but $vL/\chi_{\perp} \gg 1$), then equation (7) can be replaced by the isothermal equation $(\mathbf{B} \cdot \nabla)T = 0$. The modifications needed to treat this limit are indicated at a number of points in this section.

The axisymmetry implies $E_{\phi} = 0$, and in turn equation (5) requires

$$\mathbf{v}_p = \kappa(r, z) \mathbf{B}_p, \quad (8)$$

where the subscript p denotes the poloidal (r, z) component of the vector. The continuity equation reduces to $\nabla \cdot (\rho \mathbf{v}_p) = 0$, or $\nabla \cdot (\rho \kappa \mathbf{B}_p) = 0$, so that

$$\mathbf{B}_p \cdot \nabla (\rho \kappa) = 0 \quad (9)$$

because $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{B}_p = 0$.

The poloidal magnetic field may be expressed in terms of the *flux function* $\Psi = rA_{\phi}(r, z)$ (with A_{ϕ} the azimuthal component of the vector potential),

$$B_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad B_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (10)$$

The three-dimensional surface $\Psi(r, z) = \text{constant}$ —a flux surface—is generated by rotating the poloidal projection of a field line about the z -axis. Equation (9) implies

$$4\pi\rho(r, z)\kappa(r, z) = F(\Psi), \quad (11)$$

where $F(\Psi)$ is an arbitrary function of Ψ .

From equation (8) we have

$$\mathbf{v} \times \mathbf{B} = \frac{1}{r} (v_{\phi} - \kappa B_{\phi}) \nabla \Psi.$$

Equations (4) and (5) require

$$\frac{1}{r} (v_{\phi} - \kappa B_{\phi}) = G(\Psi), \quad (12)$$

where $G(\Psi)$ is another arbitrary function of Ψ . The electrostatic potential Φ is simply related to G :

$$\Phi(r, z) = \Phi(\Psi) = \frac{1}{c} \int^{\Psi} d\Psi' G(\Psi'). \quad (13a)$$

Hence the electric field is

$$\mathbf{E} = -\frac{1}{c} G(\Psi) \nabla \Psi. \quad (13b)$$

The plasma flow velocity can now be expressed as

$$\mathbf{v} = \frac{F(\Psi)}{4\pi\rho(r, z)} \mathbf{B}_p(r, z) + \left[\frac{F(\Psi)}{4\pi\rho(r, z)} B_{\phi}(r, z) + rG(\Psi) \right] \hat{\phi}. \quad (14)$$

Consider the Euler equation (2). The toroidal component of this equation can be written as

$$\mathbf{B}_p \cdot \nabla (rB_\phi - Frv_\phi) = 0,$$

which implies

$$rB_\phi - Frv_\phi = H(\Psi), \quad (15)$$

where $H(\Psi)$ is a third arbitrary function of Ψ . This corresponds to the conservation of angular momentum of a particle moving on a given flux surface, $\Psi = \text{constant}$. Equations (11), (12), and (15) may be rewritten to give

$$rB_\phi = (H + r^2 FG) \left(1 - \frac{F^2}{4\pi\rho}\right)^{-1} \quad (16a)$$

and

$$rv_\phi = \left(r^2 G + \frac{FH}{4\pi\rho}\right) \left(1 - \frac{F^2}{4\pi\rho}\right)^{-1}. \quad (16b)$$

The ratio $F^2/(4\pi\rho) = 4\pi\rho v_p^2/B_p^2$ is the square of the poloidal Alfvén Mach number, M_a .

The apparent singularity of equations (16) is a manifestation of the shear Alfvén waves which exist in a magnetized plasma. In order to have a field line or streamline pass through the Alfvén surface (where $M_a = 1$), the numerators of equations (16a) and (16b) must vanish on this surface (Mestel 1968*a*, *b*; Aly 1984). Thus, $H = -r^2 FG$ and $r^2 G = -FH/(4\pi\rho)$ at the Alfvén surface. Furthermore, if s denotes the arc length along the poloidal projection of a field line which penetrates the Alfvén surface, then $d\Psi/ds = 0$, and from l'Hospital's rule we have

$$(rB_\phi)_A = -H \frac{d[\ln(r^2)]_A/ds}{d(\ln\rho)_A/ds}$$

and

$$\begin{aligned} (rv_\phi)_A &= r_A^2 G \left\{ 1 + \frac{d[\ln(r^2)]_A/ds}{d(\ln\rho)_A/ds} \right\} \\ &= r_A^2 G \left[1 - \frac{(rB_\phi)_A}{H} \right] \end{aligned}$$

at the Alfvén surface (Mestel 1961). For example, in the unipolar dynamo model of Lovelace (1976) and Blandford (1976) one has $G > 0$ and $F > 0$ for $z > 0$. In order for there to be an Alfvén surface, we must have $H < 0$. We then have $(B_\phi)_A < 0$ and $(v_\phi)_A < r_A G$ if $\{d(\ln\rho)/d[\ln(r^2)]\}_s < 0$ at the Alfvén surface.

The poloidal component of the Euler equation (2) parallel to B_p implies, after some algebraic manipulation,

$$\int (dp/\rho) \Big|_{\Psi=\text{const}} + v^2/2 + \Phi_g - rv_\phi G = J(\Psi), \quad (17)$$

where $J(\Psi)$ is the fourth arbitrary function of Ψ . Equation (17) is Bernoulli's equation, which expresses the conservation of the energy flow on a given flux surface. The unfamiliar term $-rv_\phi G$ is due to the electromagnetic energy transport by the Poynting flux (see § IV). Equation (17) utilizes the isentropic equation (7). As mentioned earlier, for a low-density plasma, equation (7) must be replaced by $(\mathbf{B} \cdot \nabla)T = 0$. In this limit the integral in equation (17) has the constraint $T = \text{constant}$ in place of $S = \text{constant}$.

The equation of state of the plasma is considered to be ideal. The first law of thermodynamics, $de = TdS - pd(1/n)$, and equation (7) imply that the entropy per particle,

$$S(p, \rho) = k_B(\gamma - 1)^{-1} \ln(p/\rho^\gamma) = S(\Psi), \quad (18a)$$

depends only on Ψ . Here, e is the internal energy per particle; $n = \rho/m$ is the plasma number density with m the mean particle mass; k_B is Boltzmann's constant; and γ is the adiabatic index. Equivalently, we have $p = p(\rho, S) = p_0(\rho/\rho_0)^\gamma \exp[(\gamma - 1)S/k_B]$, where p_0 and ρ_0 are constant reference values. Thus, in Bernoulli's equation we may use the relation $d\omega = TdS + dp/n$

to obtain

$$\int (dp/\rho) \Big|_{\Psi = \text{const}} = w(\rho, S)/m = C(\Psi) \rho^{\gamma-1}, \quad (18b)$$

where w is the enthalpy per particle, and $C(\Psi) = p_0 \rho_0^{-\gamma} \gamma (\gamma - 1)^{-1} \exp [S(\Psi)(\gamma - 1)/k_B]$. Evidently, $C(\Psi)$ or $S(\Psi)$ represents a fifth arbitrary function of Ψ . For a low-density plasma where $(\mathbf{B} \cdot \nabla)T = 0$, equation (18a) is replaced by $T = T(\Psi)$, and in place of $w(\rho, S)$ one has the Gibbs free energy $g(p, T) = k_B T \ln(\rho/\rho_0)$ in the definition of $J(\Psi)$, equation (17) (Hameiri 1983). Notice that $dg = -SdT + dp/n$, so that $S = -(\partial g/\partial T)_p$ in this case.

The poloidal component of the Euler equation (2) parallel to $\nabla\Psi$ now gives a “generalized, nonrelativistic Grad-Shafranov equation” for $\Psi(r, z)$:

$$\left(1 - \frac{F^2}{4\pi\rho}\right) \Delta^*\Psi - F\nabla\left(\frac{F}{4\pi\rho}\right) \cdot \nabla\Psi = -4\pi\rho r^2(J' + rv_\phi G') - (H + rv_\phi F)(H' + rv_\phi F') + 4\pi r^2 p(S'/k_B). \quad (19)$$

Here

$$\Delta^* \equiv r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2};$$

primes denote differentiation with respect to Ψ ; and rv_ϕ is given by equation (16b). Equation (19) requires the specification of the five arbitrary functions: F , G , H , J , and S . Somewhat less general forms of equation (19) have been derived and discussed previously (see, e.g., Zehrfeld and Green 1972 and Aly 1984). For a low-density plasma where $(\mathbf{B} \cdot \nabla)T = 0$, the term $4\pi r^2 p(S'/k_B)$ must be replaced by $-4\pi r^2(\rho/m)ST'$ in equation (19).

The mathematical nature of the operator on the left-hand side of equation (19) can be extracted by substituting for ρ from Bernoulli's equation (17). That is, equations (16b) and (17) can be manipulated to give $\nabla\rho$ in terms of $\nabla(\nabla\Psi)^2$. Retaining only the highest (second) order derivatives of Ψ , the left-hand side is

$$\left(1 - \frac{F^2}{4\pi\rho}\right) \left(A_{rr} \frac{\partial^2 \Psi}{\partial r^2} + A_{rz} \frac{\partial^2 \Psi}{\partial r \partial z} + A_{zz} \frac{\partial^2 \Psi}{\partial z^2} \right),$$

where

$$A_{rr} \equiv 1 - \frac{v_p^2 v_z^2}{R},$$

$$A_{rz} \equiv \frac{2v_p^2 v_r v_z}{R},$$

$$A_{zz} \equiv 1 - \frac{v_p^2 v_r^2}{R},$$

and

$$R \equiv c_s^2 v_a^2 - [c_s^2 + v_a^2] v_p^2 + v_p^4.$$

Here $c_s \equiv [(\partial p/\partial \rho)_S]^{1/2}$ is the adiabatic sound speed, $v_a \equiv [(B_p^2 + B_z^2)/(4\pi\rho)]^{1/2}$ is the Alfvén speed, $v_a \equiv |\mathbf{B}_p|/(4\pi\rho)^{-1/2}$ is the Alfvén speed projected into the poloidal plane, and $v_p = (v_r^2 + v_z^2)^{1/2}$ is the poloidal flow speed. We may write $R = (v_p^2 - v_1^2)(v_p^2 - v_2^2)$, where v_1 and v_2 are the speeds of the slow and fast magnetosonic waves (see, e.g., Miyamoto 1976).

Equation (19) is an elliptic or hyperbolic equation depending on whether the discriminant $D \equiv (A_{rz})^2 - 4A_{rr}A_{zz}$ is negative or positive. It follows that $D = [v_p^2(c_s^2 + v_a^2) - c_s^2 v_a^2]/R$. In general, with v_p increasing from zero, equation (19) is elliptic, hyperbolic, elliptic, and ultimately hyperbolic at high flow speeds (Zehrfeld and Green 1972; Heinemann and Olbert 1978; Aly 1984). The four flow regimes are separated by three critical points. Two of the critical points correspond to the flow speed equal to the slow and fast magnetosonic speeds ($R=0$), and one critical point corresponds to the vanishing of the numerator of D at $v_p = v_0 \equiv c_s v_a (c_s^2 + v_a^2)^{-1/2}$, where v_0 is a characteristic speed of the slow-wave wavefront propagation (Whitham 1974). We have in general

$$v_0 < v_1 < (v_a, c_s) < v_2.$$

Thus, the above-mentioned Alfvén singularity of equations (16) occurs for a poloidal flow speed between v_1 and v_2 .

Notice that under some conditions it is possible to have solutions to equation (19) with different values of F , H , and J between the upper and lower half-spaces ($z \geq 0$) on a given Ψ surface. For an illustration of this, consider a thin disk threaded by a poloidal magnetic field as in the unipolar dynamo model of Lovelace (1976) and Blandford (1976). The conservation of magnetic flux threading the disk area dS may be expressed as $[B_p \cdot dS] = 0$, where $dS = \hat{z} dx dy$, and where the square brackets denote the difference in the quantity between $z > 0$ and $z < 0$. The rate of mass outflow from the disk is $[F]B_p \cdot dS/4\pi$, with $[F] = F_+(\Psi) - F_-(\Psi)$. The rate of angular-momentum outflow is $[(\rho v_\phi v_p - rB_\phi B_p/4\pi) \cdot dS] = -[H]B_p \cdot dS/4\pi$. The rate of energy outflow is $[JF]B_p \cdot dS/4\pi$. The loss of mass, angular momentum, and energy from the disk has only a small influence if the fractional changes in the disk quantities are small compared with unity during a time on the order of the dynamical time scale of the disk. In the dynamo model, notice that $F_+ = -F_-$, $H_+ = -H_-$, and $J_+ = J_-$.

A subcase of equation (19) of importance to axisymmetric laboratory fusion plasmas such as tokamaks and spheromaks (see, e.g., Miyamoto 1976, p. 180) is the case where the flow is negligible [$F(\Psi) = 0$ and $G(\Psi) = 0$], and gravity is negligible [$\Phi_g = 0$]. Thus $H(\Psi) = rB_\phi$, and $J(\Psi) = C(\Psi)\rho^{\gamma-1}$, so that $\rho = \rho(\Psi)$ and $p = p(\Psi)$. Using the thermodynamic identity $p dS/k_B - n dw = -dp$, equation (19) simplifies to

$$\Delta^* \Psi = -4\pi r^2 p' - HH'. \quad (20)$$

This is the “standard” form of the Grad-Shafranov equation (see, e.g., Miyamoto 1976, p. 180). Equation (20) also applies in the low-density limit where $(\mathbf{B} \cdot \nabla)T = 0$.

It is naturally of interest to inquire as to what remains of equation (19) in the absence of a magnetic field. To obtain this limit, first notice that $v_p = [F(\Psi)/(4\pi\rho)](r^{-1}\nabla\Psi \times \hat{\phi})$ with $|B_p| = r^{-1}|\nabla\Psi|$. Thus, if we let $\Psi \rightarrow \tilde{\Psi}/Z$ and $F \rightarrow Z\tilde{F}$, then in the $Z \rightarrow \infty$ limit, F , Ψ , and $|v_p|$ remain finite, while the poloidal magnetic field vanishes. In addition, we need to have $H \rightarrow Z\tilde{H}$, $G \rightarrow \tilde{G}/Z$, $J \rightarrow \tilde{J}$, and $S \rightarrow \tilde{S}$. Applying these rescalings to equation (19) and letting Z become infinite yields

$$\left(\frac{F^2}{4\pi\rho}\right)\Delta^* \Psi + F\nabla\left(\frac{F}{4\pi\rho}\right) \cdot \nabla \Psi = 4\pi\rho r^2 J' - 4\pi\rho LL' + 4\pi r^2 p(S'/k_B), \quad (21)$$

where the tildes have been dropped, where

$$L(\Psi) \equiv -H(\Psi)/F(\Psi) = rv_\phi$$

is the angular momentum per unit mass, and where the Bernoulli constant $J(\Psi)$ is given by equation (17) with $G = 0$. With the choice $F=1$, Ψ becomes the Stokes stream function for the fluid motion (see, e.g., Scott and Lovelace 1982). The nature of the operator on the left-hand side of equation (21) follows from the earlier discussion of equation (19). With $v_A = 0$, the discriminant is $D = -[1 - (v_p/c_s)^2]^{-1}$, so that the equation is elliptic for $v_p < c_s$, whereas for $v_p > c_s$ it is hyperbolic.

A subcase of equation (19) of interest to us in the next section is that corresponding to no poloidal flow [$v_p = 0$ or $F(\Psi) = 0$], no toroidal magnetic field [$B_\phi = 0$ or $H(\Psi) = 0$], and a low-density plasma so that $(\mathbf{B} \cdot \nabla)T = 0$:

$$\Delta^* \Psi = -4\pi\rho r^2 (J' + r^2 GG') - 4\pi r^2 (\rho/m) ST'. \quad (22)$$

In this case, $v_\phi = rG$, and $J(\Psi) = [k_B T(\Psi)/m] \ln(\rho/\rho_0) - r^2 G^2/2 + \Phi_g$.

The boundary conditions on Ψ in equations (19)–(22) are clearly important. In subsequent sections we consider two limiting cases of equation (22): (1) plasma around a nonrotating Schwarzschild black hole (§ III) and (2) plasma around an aligned, rotating magnetized star (§ V). In case 1 we take $\Psi(r \rightarrow 0) = 0$, and we assume that the current density is concentrated within a finite distance of the central object. That is, we assume $\Psi(r \rightarrow \infty) = m_d r^2 (r^2 + z^2)^{-3/2}$, where m_d is the magnetic moment of the plasma. In case 2 we let $\Psi = \Psi_* + \Psi_d$, where Ψ_* is the flux function of the star (due to its internal currents) and Ψ_d is the self-flux function due to currents in the plasma. For simplicity, we take $\Psi_*(r, z) = m_* r^2 (r^2 + z^2)^{-3/2}$, where m_* is the magnetic moment of the star. At the star's surface $\Psi_*(|r| = r_*) = 0$, which corresponds to the plasma flux not penetrating the star. Also, in this case we assume $\Psi(r \rightarrow \infty) = \text{const} \times r^2 (r^2 + z^2)^{-3/2}$.

III. THIN DISKS

Here, we analyze magnetized disks with no poloidal flow, $F(\Psi) = 0$, and no toroidal magnetic field, $H(\Psi) = 0$. (The more complicated treatment including B_ϕ will be described in a subsequent paper.) We consider the disk vertical structure in § IIIa, the global field structure in § IIIb, the virial equations in § IIIc, the axisymmetric stability in § IIId, specific disk equilibria in § IIIe, and the Bennett pinch relation in § IIIf.

The disk thickness Δz is assumed to be small compared with r . The conditions under which $\Delta z \ll r$ are then established. A number of further assumptions are made so as to simplify the analysis:

1. The disk has reflection symmetry about the $z = 0$ plane.
2. The gravitational potential $\Phi_g(r, z)$ is predominantly due to the central object of mass M ; the requirement for this is $M \gg 4\pi\sigma r^3/\Delta z$, where σ is the surface mass density of the disk.

3. The equation of state of the plasma on any given flux surface is isothermal, $(\mathbf{B} \cdot \nabla)T = 0$, so that $p = \rho k_B T(\Psi)/m$, with m the mean plasma particle mass, and

$$\int (dp/\rho) \Big|_{\Psi = \text{const}} = [c_s(\Psi)]^2 \ln(\rho/\rho_0), \quad (23)$$

where $c_s = [k_B T(\Psi)/m]^{1/2}$ is the Newtonian sound speed, and ρ_0 is a constant reference density.

4. The variation of c_s with Ψ is gradual in comparison with that of $J(\Psi)$ and $G(\Psi)$, so that the term $-4\pi r^2(\rho/m)ST'$ is negligible in equation (22). This corresponds to the radial pressure force on the disk matter being negligible compared to the radial inertial and magnetic forces.

a) Vertical Structure of Disk

Consider first the interior of the disk. Because the disk is thin, we may neglect terms such as $|\Psi|/r^2$ or $|\partial^2 \Psi / \partial r^2|$ in comparison with $|\partial^2 \Psi / \partial z^2|$. Consequently, we may write $\Psi(r, z) = \Psi_0(r) + \Psi_1(r, z)$, where $\Psi_1(r=0, z=0) = 0$. The reflection symmetry of the disk implies that $\Psi_1(r, z) = \Psi_1(r, -z)$. Equations (17) and (23) give

$$\rho(r, z) = \rho_0 \exp \left[(J - \Phi_g + r^2 G^2/2) / c_s^2 \right]. \quad (24a)$$

Expanding the functions of Ψ in a Taylor series, as, for example, $J(\Psi) = J(\Psi_0) + J'(\Psi_0)\Psi_1 + \dots$, and expanding $\Phi_g(r, z) = \Phi_g(r, 0) + (\partial \Phi_g / \partial z^2)z^2 + \dots$, one has

$$\rho(r, z) = \rho(r, 0) \exp \left[-k\Psi_1 - (z/c_s)^2 (\partial \Phi_g / \partial z^2) \right], \quad (24b)$$

where $k(r) \equiv -[J' + r^2(G^2/2)'] / c_s^2$. Owing to the fact that the Taylor expansions enter in the argument of an exponential, terms beyond the first derivative may be neglected (as in the method of steepest descent).

Within the disk, equation (22) can now be reduced to a dimensionless *vertical structure equation for the disk*,

$$\frac{d^2 \tilde{\Psi}_1}{d\zeta^2} = \beta^2 \exp(-\tilde{\Psi}_1 - \zeta^2), \quad (25a)$$

where

$$\tilde{\Psi}_1 \equiv k\Psi_1, \quad \zeta \equiv \frac{z}{h}, \quad h \equiv c_s \left(\frac{\partial \Phi_g}{\partial z^2} \right)^{-1/2}, \quad (25b)$$

and

$$\beta^2 \equiv 4\pi r^2 \rho(r, 0) h^2 [J' + r^2(G^2/2)']^2 / c_s^2. \quad (25c)$$

Of course, h and β depend in general on r .

In the absence of a magnetic field, the vertical (z -direction) half-width (to the $1/e$ point) of the density would be h . For large distances from the central mass ($r \gg 2GM/c^2$), we have $\Phi_g \approx -GM(r^2 + z^2)^{-1/2}$, so that

$$h \approx rc_s [GM/(2r)]^{-1/2} \approx 2^{1/2} r (c_s/V_K), \quad (26)$$

where $V_K \equiv (GM/r)^{1/2}$ is the single-particle Keplerian velocity at r . In considering equation (25) it is helpful to regard h as a fixed length independent of β .

The quantity $[J' + r^2(G^2/2)']$, which enters in the definition of β , can be rewritten as

$$[(\nabla p)/\rho + \nabla \Phi_g - G^2 \nabla(r^2/2)] \cdot (\nabla \Psi) / |\nabla \Psi|^2.$$

This represents the radial magnetic force per unit mass, $F_{mr}(r)$, divided by $|rB_z(r, 0)|$. It is useful to introduce a “velocity deficit or excess,” $\Delta v_m(r)$, associated with the magnetic force via the relation $(\Delta v_m)^2/r = F_{mr}$. Thus we can write

$$\beta^2 = \frac{h^2 (\Delta v_m)^4}{c_s^2 V_A^2} \approx \frac{2 (\Delta v_m)^4}{c_s^2 V_K^2}. \quad (27)$$

where $V_A(r) \equiv B_z(r, 0)[4\pi\rho(r, 0)]^{-1/2}$ is the Alfvén speed at r .

The solution of equation (25) is facilitated by introducing $\psi = \tilde{\Psi}_1 + \zeta^2$, so that

$$\frac{d^2\psi}{d\zeta^2} = 2 + \beta^2 \exp(-\psi). \quad (28a)$$

The relevant solution is

$$\int_0^\psi d\psi' [2\psi' + \beta^2 [1 - \exp(-\psi')]]^{-1/2} = 2^{1/2}\zeta. \quad (28b)$$

The mass density of the disk as a function of ζ is proportional to $\exp(-\psi)$, and this is shown for representative values of β in Figure 1.

The β -dependence of the vertical ($1/e$) half-thickness of the disk density is given by

$$\zeta_{1/2} = 2^{-1/2} \int_0^1 d\psi [2\psi + \beta^2 [1 - \exp(-\psi)]]^{-1/2}. \quad (28c)$$

This dependence is shown in Figure 2. In the weak magnetic field limit, $\beta^2 \ll 1$, the disk thickness $\Delta z_{1/2}$ approaches the value h , which is the thickness in the absence of a magnetic field. Equivalently, $\zeta_{1/2} \rightarrow 1$ as $\beta \rightarrow 0$. *With increasing β the disk thickness decreases monotonically. This is due to the pinch effect of the magnetic field: The magnetic field compresses the disk in the z direction (while exerting a predominantly outward radial force).* For $\beta^2 \gg 1$, we find

$$\zeta_{1/2} \approx 2^{-1/2} \beta^{-1} \operatorname{arccosh}(e^{1/2}) \approx 1.534/\beta.$$

The evaluation of $\beta(r)$ for a specific disk equilibrium is given in § III f.

b) Global Magnetic Field Structure

We now turn to the analysis of the global magnetic field structure of the disk. Owing to the fact that the disk is thin, equation (20) can be replaced by

$$\Delta^* \Psi = -\frac{4\pi r}{c} j_\phi(r) \delta(z), \quad (29a)$$

where $j_\phi(r)$ is the disk current per unit length in the radial direction,

$$j_\phi(r) \equiv \int_{-\infty}^{\infty} dz J_{e\phi}(r, z). \quad (29b)$$

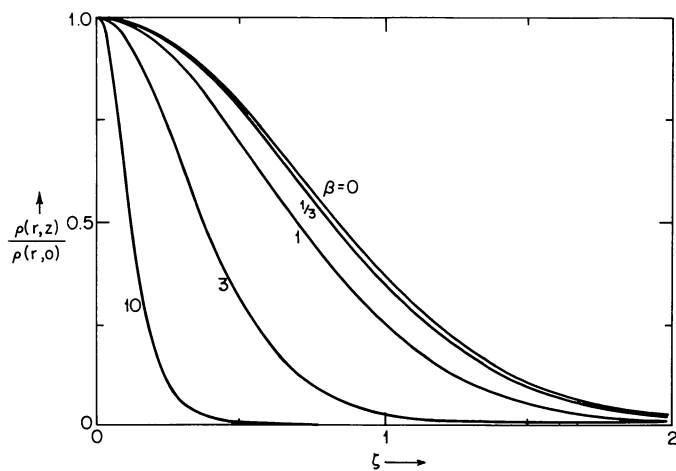


FIG. 1

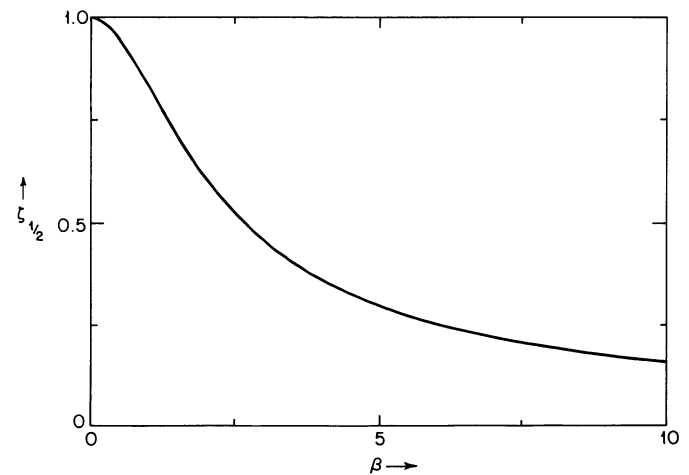


FIG. 2

FIG. 1.—Profiles of the disk density as a function of $\zeta = z/h$ for different values of the magnetic field strength β . In the absence of a magnetic field, the disk half-thickness (to e^{-1}) would be h .

FIG. 2.—Dependence of the dimensionless disk half-thickness, $\zeta_{1/2} = \Delta z_{1/2}/h$, on the magnetic field strength β

Note that equation (29a) implies a discontinuity of the radial magnetic field across the disk.

$$B_r(r, z = 0^+) - B_r(r, z = 0^-) = \frac{4\pi}{c} j_\phi(r). \quad (30)$$

The vertical component of the magnetic field is continuous across $z = 0$. The occurrence of this “cusp” in the field can of course be directly related to the magnetic field line tension acting radially on the disk matter. In the actual disk, this cusp would, of course, be smoothed out over a length scale of the order of the disk thickness.

The solution to equation (29) can be represented as a superposition of the magnetic fields of circular wire loops (Jackson 1975) of different radii:

$$\Psi(r, z) = \frac{2\pi}{c} \int_0^\infty dr' g(r'; r, z) j_\phi(r'), \quad (31a)$$

where

$$\begin{aligned} g(r'; r, z) &\equiv \frac{rr'}{2\pi} \int_0^{2\pi} d\theta \cos \theta (r^2 + r'^2 - 2rr' \cos \theta + z^2)^{-1/2} \\ &= \frac{[(r + r')^2 + z^2]^{1/2}}{2\pi} [(2 - k^2) K(k) - 2E(k)], \end{aligned} \quad (31b)$$

with $k^2 \equiv 4rr'/[(r + r')^2 + z^2]$, and with K and E the complete elliptic integrals of the first and second kinds, respectively.

For subsequent use, notice that we have an inverse relation to equation (31a),

$$j_\phi(r) = -\frac{c}{2\pi r} \int_0^\infty dr' g(r', r) \frac{\partial B_z(r', 0)}{\partial r'}, \quad (32)$$

where $g(r', r) = g(r'; r, z = 0) = g(r, r')$ and $B_z(r, 0) = r^{-1} [\partial \Psi(r, 0) / \partial r]$ (Lovelace 1980). Notice also that $g(r, r') \geq 0$ and

$$g(r, r') = \begin{cases} (2r/\pi) [K(r'/r) - E(r'/r)], & r' < r, \\ (2r'/\pi) [K(r/r') - E(r/r')], & r < r', \end{cases}$$

or

$$g(r, r') = rr' \int_0^\infty d\kappa J_1(\kappa r) J_1(\kappa r'),$$

with J_1 the usual Bessel function of order 1. For $(r - r')^2 \ll r^2$, we have

$$g(r, r') \approx (r/\pi) \ln |r^2(r'^2 - r^2)^{-1}|.$$

For $r' \ll r$,

$$g(r, r') = (r')^2/(2r) + O[(r')^4/r^3].$$

For $r \ll r'$,

$$g(r, r') = r^2/(2r') + O[r^4/(r')^3].$$

Consequently, we have

$$\Psi(r \rightarrow 0, z = 0) = 0 \quad \text{if } \int dr r^{-1} j_\phi(r) < \infty;$$

and

$$\Psi(r \rightarrow \infty, z = 0) = 0 \quad \text{if } \int dr r^2 j_\phi(r) < \infty.$$

The disk surface current density can be written as

$$j_\phi(r) = rc \int_{-\infty}^{\infty} dz \rho(r, z) [J' + r^2 (G^2/2)']. \quad (33)$$

Substituting for J from § II, we obtain the *radial force balance equation*

$$j_\phi(r) = \frac{c\sigma(r)}{B_z(r, 0)} \left(\frac{1}{\sigma} \frac{dP}{dr} - rG^2 + \frac{\partial \Phi_g}{\partial r} \right), \quad (34a)$$

where

$$\sigma(r) \equiv \int_{-\infty}^{\infty} dz \rho(r, z)$$

is the surface mass density of the disk;

$$\begin{aligned} P(r) &\equiv \int_{-\infty}^{\infty} dz p(r, z) \\ &= c_s^2(\Psi_0) \sigma(r) \end{aligned}$$

is the height-integrated pressure;

$$G = G(\Psi_0) = v_\phi/r;$$

and

$$\Psi_0 = \Psi(r, 0).$$

The radial force balance equation (34a) can be written in a more familiar form as

$$\frac{\sigma v_\phi^2}{r} = \frac{dP}{dr} + \sigma \frac{\partial \Phi_g(r, 0)}{\partial r} - \frac{1}{c} j_\phi(r) B_z(r, 0). \quad (34b)$$

In the order given, the different terms represent the “centrifugal force,” the pressure force, the gravitational force, and the magnetic force.

Equations (29) and (34) describe the general equilibrium of a thin disk, including the influence of a poloidal magnetic field. A rather large degree of arbitrariness remains: we are free to specify the functions $\sigma(r)$, $G(\Psi_0)$, and $c_s^2(\Psi_0)$. Before discussing the general equilibria further, we consider some of the basic limitations on magnetized disks. Also, we defer the discussion of the radial dependence of the vertical disk structure to § III f.

c) Virial Equations

Multiplying equation (34b) by $2\pi r^2$ and integrating over r gives the “radial virial equation,”

$$2W_K + \frac{4}{3}W_p + W_g + \int d^3x \frac{B_z^2}{4\pi} = 0. \quad (35)$$

Here,

$$W_K \equiv 2\pi \int_0^\infty r dr \sigma v_\phi^2/2$$

is the bulk kinetic energy of the disk matter;

$$W_p \equiv 2\pi \int_0^\infty r dr \frac{3P}{2}$$

is the “pressure energy” of the disk matter; and

$$W_g \equiv -2\pi \int_0^\infty r dr \sigma r \frac{\partial \Phi_g}{\partial r} < 0$$

is the gravitational potential energy of the disk matter. The magnetic energy contribution in equation (35) is obtained by replacing $rj_\phi(r)B_z(r,0)$ by $\int dz rj_\phi(r)\delta(z)B_z(r,z)$, and by performing a number of integrations by parts.

Similarly, the z -momentum equation can be multiplied by z and integrated over z and r to give the “vertical virial equation,”

$$\frac{2}{3} W_p + 2\pi \int r dr \sigma \frac{\langle z^2 \rangle}{r} \frac{\partial \Phi_g}{\partial r} + \int d^3x \frac{B_r^2 - B_z^2}{8\pi} = 0, \quad (36)$$

where

$$\langle z^2 \rangle \equiv \frac{1}{\sigma(r)} \int_{-\infty}^{\infty} dz z^2 \rho(r, z)$$

is the mean square thickness of the disk. For low-temperature disks, $c_s^2 \ll v_\phi^2$, we have $\langle z^2 \rangle \ll r^2$, so that equation (36) implies

$$\int d^3x B_r^2 = \int d^3x B_z^2,$$

which will be useful later.

The addition of equations (35) and (36) gives the familiar form of the virial theorem (Chandrasekhar 1961, p. 579):

$$2W_K + 2W_p + W_g + W_m = 0, \quad (37)$$

where

$$W_m \equiv \int d^3x \frac{B^2}{8\pi} = \frac{\pi}{c} \int_0^\infty dr j_\phi(r) \Psi_0(r)$$

is the total magnetic energy. In equation (37) we have neglected one term of order $(h/r)^2 W_g$. For low-temperature disks we have, from equation (36),

$$W_m = \frac{2\pi}{c} \int_0^\infty r dr r j_\phi(r) B_z(r, 0).$$

An immediate consequence of the virial equation (37) is that

$$\xi \equiv \frac{W_m}{|W_g|} \leq 1. \quad (38)$$

The ratio ξ is important in a general treatment of the global field configurations. In particular, we are interested in determining how close ξ can be to the limiting value of unity.

d) Axisymmetric Stability

Only one basic stability consideration of magnetized disks is given here, which is analogous to that used in deriving the Rayleigh criterion for the axisymmetric stability of Couette flows (Chandrasekhar 1981, p. 273). Specifically, we examine the “superparticle stability.” We neglect the pressure force ($c_s^2 \ll v_\phi^2$), but return to include its influence in Appendix B. The “superparticle” is considered to be an arbitrary infinitesimal annular ring segment of the disk with radial extent dr , fixed mass $\delta M = 2\pi r dr \sigma(r)$, and current $\delta I = j_\phi(r) dr$. We consider the axisymmetric displacement of this “ringlet,” $r \rightarrow r + dr$. Because the ringlet is infinitesimal, there is no back-reaction due to its displacement. Since the disk remains axisymmetric during the displacement, the angular momentum of the ringlet remains a constant equal to its initial value, $\delta M r v_\phi = \delta M l = \text{constant}$, with l the angular momentum per unit mass.

The equilibrium of the ringlet can be expressed as

$$\frac{\partial}{\partial r} \left[\delta M \left(\frac{l^2}{2r^2} + \Phi_g \right) - L I_d \delta I \right]_{l, I_d, \delta I = \text{const}} = \frac{\partial V_{\text{eff}}}{\partial r} = 0, \quad (39)$$

where $V_{\text{eff}}(r)$ is the effective potential. Here, L denotes the mutual inductance between the ringlet and the disk, so that the magnetic flux through the ringlet is

$$\begin{aligned} cLI_d &= 2\pi \int_0^r r' dr' B_z(r, 0) \\ &= 2\pi \Psi(r, 0). \end{aligned} \quad (40)$$

We have chosen to express the magnetic force on the ringlet, $(\partial L / \partial r) I_d \delta I$, at constant currents as opposed to constant fluxes (see Landau and Lifshitz 1960). Equation (39) is equivalent to equation (34) for $P = 0$.

The condition for the stability of the ringlet displacement is simply $\partial^2 V_{\text{eff}} / \partial r^2 > 0$. This is equivalent to

$$\delta M \left(\frac{3l^2}{r^4} + \frac{\partial^2 \Phi_g}{\partial r^2} \right) - \frac{\partial^2 L}{\partial r^2} I_d \delta I > 0. \quad (41)$$

Using the equilibrium condition (39), the stability condition becomes

$$\frac{\sigma}{r^2} \frac{\partial}{\partial r} \left\{ r^3 \left[\frac{\partial}{\partial r} \Phi_g(r, 0) \right] \right\} > \frac{j_\phi}{cr^3} \left\{ \frac{\partial}{\partial r} [r^4 B_z(r, 0)] \right\}. \quad (42)$$

An explicit formula for the left-hand side of inequality (42) is of interest for the case where $\Phi_g(r, 0) = -GM(r - r_s)^{-1}$, with $r_s \equiv 2GM/c^2$, the Schwarzschild radius. This Φ_g provides a rough non-general-relativistic representation of the gravitational potential outside a Schwarzschild black hole (Paczynski and Witta 1980; Abramowicz, Calvani, and Nobili 1980). From equation (42) one finds

$$\frac{\sigma GM(r - 3r_s)}{(r - r_s)^3} > \frac{j_\phi}{cr^3} \left\{ \frac{\partial}{\partial r} [r^4 B_z(r, 0)] \right\}. \quad (43)$$

For $j_\phi = 0$, this inequality includes the well-known nonmagnetic stability requirement $r > 3r_s = 6GM/c^2$. For specificity, consider now $j_\phi(r) > 0$, so that $B_z(r, 0)$ is positive near the inner edge of the disk. Then, for broadly distributed $j_\phi(r)$ (see § III e), $\partial[r^4 B_z(r, 0)]/\partial r$ is also positive near the inner edge of the disk. Even though the magnetic force per unit mass is outward near the disk inner edge, the radial gradient of this force is positive. Thus, the *magnetic field acts to destabilize the single particle motion near the disk inner edge*. For stability, the disk inner radius must be larger than $3r_s$.

e) Specific Disk Equilibria

In this subsection we investigate disk equilibria described by equations (29) and (34) for conditions where the pressure force is negligible in equation (34). That is, $c_s^2 \ll v_\phi^2$, so that the radial force balance equation is

$$j_\phi(r) = \frac{c\sigma(r)}{B_z(r, 0)} \left[-r\Omega^2 + \frac{\partial \Phi_g(r, 0)}{\partial r} \right]. \quad (44)$$

Here, we use the more obvious notation $\Omega(\Psi) = G(\Psi) = v_\phi/r$. Using equation (31) with $z = 0$, and taking $\Phi_g = -GM(r - r_s)^{-1}$ as in § III d, we obtain the following nonlinear integral equations for $\Psi_0(r) = \Psi(r, z = 0)$:

$$\Psi(r) = \frac{2\pi}{c} \int_{r_0}^{\infty} dr' g(r', r) j_\phi(r') \quad (45a)$$

$$= 2\pi \int_{r_0}^{\infty} dr' g(r', r) \frac{(r')^2 \sigma(r')}{d\Psi(r')/dr'} \left\{ \frac{GM}{r'(r' - r_s)^2} - \Omega^2[\Psi(r')] \right\}, \quad (45b)$$

where the zero subscript on Ψ has been dropped, and where r_0 is the disk inner radius with $r_0 > 3r_s$ from § III d. The dependences $\Omega(\Psi)$ and $\sigma(r)$ must be specified.

The physical content of equation (45) arises from the condition of perfect electrical conductivity, which implies $\Omega = \Omega(\Psi)$. In this regard consider two nearby field lines, AA' , and BB' , as shown in Figure 3, which arch from one radius of the disk $r_1 \geq r_0$, to another, $r_2 > r_1$. Owing to the perfect conductivity of the disk plasma, the radial electric field in the plane of the disk at r_1 is $E_r(1) = -v_\phi(1)B_z(1)/c$, so that the potential difference between A and B is $-\Omega(1)\delta\Psi/c$, where $\delta\Psi \equiv \Psi_B - \Psi_A$. Similarly, at r_2 , $E_r(2) = -v_\phi(2)B_z(2)/c$, and the potential difference between B' and A' is $\Omega(2)\delta\Psi/c$. Thus, even a slight difference in the rotation rate between r_1 and r_2 would give rise to a potential drop along the field-line arches and of course a parallel electric field.

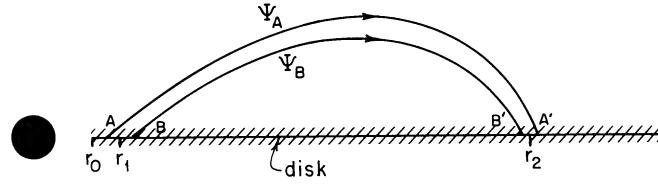


FIG. 3.—Sketch of two field-line arches, illustrating the physical basis for the corotation of the disk matter at r_1 and r_2

However, the field arches also behave as perfect conductors, even though the equilibrium mass density in this region may be negligible. The charge carriers required to enforce perfect conductivity, and therefore zero potential drop along the arches, are readily supplied by even a very tenuous coronal plasma above and below the disk. Thus, we have $\Omega(1) = \Omega(2)$, or, equivalently, $\Omega = \Omega(\Psi)$. Notice that because there is no poloidal flow, the Bernoulli constant $J(\Psi) = \Phi_g - (rG)^2/2$ is allowed to have different values at r_1 and r_2 .

We first consider the qualitative behavior of $\Psi(r)$ for the simplest possible field-line topology. We have $\Psi(0) = 0$ from equation (31b). With increasing r , $\Psi(r)$ increases to a maximum, $\Psi_{\max}$, at $r = r_{oh} > r_0$, which is the O -point of the magnetic field. [Clearly, an equivalent solution is obtained by changing the sign of $\Psi(r)$.] At the O -point we have $\Omega^2 r = \partial \Phi_g / \partial r$. The *reduced magnetic flux of the disk* is

$$\begin{aligned} \Phi_d &= \int_0^{r_{oh}} r dr B_z(r, 0) \\ &= \Psi(r_{oh}) = \Psi_{\max}, \end{aligned} \quad (46)$$

so that the actual magnetic flux is $2\pi\Phi_d$. For r increasing beyond r_{oh} , $\Psi(r)$ decreases and $\Psi(r \rightarrow \infty) = 0$. The more complicated field-line topologies with two or more extrema of $\Psi(r)$ are considered briefly at the end of this subsection.

Notice that $\Psi_c \equiv \Psi(r_0) \leq \Psi_{\max}$ labels the field line which passes through the disk inner radius. The outer radius of this field line, denoted $r_e (> r_{oh})$, is such that $\Psi(r_e) = \Psi_c$. The field lines labeled by Ψ values less than Ψ_c have inner radii less than r_0 and outer radii larger than r_e . For these field lines the perfect conductivity condition $\Omega = \Omega(\Psi)$ is superfluous because there is no plasma for $r < r_0$.

To facilitate the numerical determination of $\Psi(r)$, we introduce dimensionless variables: $\hat{r} \equiv r/r_0$; $\hat{\sigma} \equiv r_0^2 \sigma / M_d$; $\hat{\Psi} \equiv \Psi / \Psi_{\max}$; $\hat{j}_\phi \equiv r_0^2 j_\phi / (c \Psi_{\max})$; $\hat{g} \equiv g/r_0$; $\hat{\Omega}^2 \equiv r_0^3 \Omega^2 / (GM)$; and $\hat{r}_s \equiv r_s / r_0$. Here,

$$M_d = 2\pi \int_{r_0}^{\infty} r dr \sigma(r) \quad (47)$$

is the total disk mass.

One possible approach to the determination of $\Psi(r)$, with $\Omega(\Psi)$ and $\sigma(r)$ specified, is a Picard iteration of equation (45b) based on an initial guess of $\Psi(r)$, that is, $\Psi^{(n+1)} = \int g j(\Psi^{(n)})$. So far we have been unsuccessful in getting this iteration to converge. An alternative approach is to specify $\Omega(\Psi)$ and $j_\phi(r)$ and derive $\sigma(r)$ from equation (44). In terms of dimensionless variables, with the overcarets dropped, equation (44) is

$$\sigma(r) = \frac{\xi_0 j_\phi(r) B_z(r, 0)}{1/(r - r_s)^2 - r \Omega^2(\Psi)}. \quad (48a)$$

The main dimensionless parameter of equation (48a) is

$$\xi_0 \equiv \frac{\Psi_{\max}^2}{G M M_d}, \quad (48b)$$

which is essentially the ratio of the magnetic field energy to the gravitational binding energy of the disk.

A fairly general functional form for $j_\phi(r)$ is considered:

$$j_\phi(r) = K \left(1 - \frac{1}{r}\right)^\delta r^{-m} \exp\left(\frac{1-r}{\lambda}\right), \quad r > 1, \quad (49)$$

and $j_\phi = 0$ otherwise, where $m \geq 0$, $\delta > 0$, $\lambda > 0$, and K are dimensionless constants. The constant K is fixed by the requirement

that $\max(\Psi) = 1$. The total disk current is

$$I_d = \left[\frac{c\Psi_{\max}}{r_0} \right] \int_1^\infty dr j_\phi, \quad (50a)$$

and the disk magnetic moment is

$$m_d = [\pi\Psi_{\max}r_0] \int_1^\infty dr r^2 j_\phi, \quad (50b)$$

where the quantities in square brackets have the dimensions of current and magnetic moment, respectively.

The functional form for $\Omega(\Psi)$ is assumed to be

$$[\Omega(\Psi)]^2 = \omega^2 |\Psi|^n, \quad (51)$$

with ω and n dimensionless constants. Clearly, at the O -point of the magnetic field, $B_z(r_{oh}, 0) = 0$ and $\Psi(r_{oh}) = 1$, so that we must have

$$\omega^2 = [r_{oh}(r_{oh} - r_s)^2]^{-1}. \quad (52)$$

For given values of δ , m , and λ , characterizing j_ϕ , $B_z(r, 0)$ can be derived from equation (45a). The location of the O -point can then be found and the value of Ω^2 determined from equation (52). In turn, $\sigma(r)$ can be obtained from equation (48a) for a given value of n . The value of ξ_0 in equation (48a) is fixed by the requirement that the area integral of σ over the disk is unity. One obvious physical constraint on the parameters (δ, m, λ, n) is that $\sigma(r)$ be nonnegative for $r > 1$.

A further physical constraint is obtained from the requirement of axisymmetric stability of § III d. That is, we require inequality (43) to be satisfied for $r \geq 1$. Because $r_s \leq \frac{1}{3}$, the left-hand side of inequality (43) is always nonnegative. Using equations (48), the stability condition can be expressed in dimensionless variables as

$$S = \frac{r - r_s}{r - 3r_s} \left[1 - \Omega^2 r(r - r_s)^2 \right] \left(4 + \frac{r}{B_z} \frac{dB_z}{dr} \right) < 1. \quad (53)$$

The stability condition is most critical at the inner edge of the disk. Thus, it suffices to look at the stability ratio as $r \rightarrow 1 + 0$. We can then adjust the value of r_s , the only remaining free parameter, to ensure stability right up to the disk inner edge. Note that adjusting r_s amounts to changing the physical length scale r_0 .

The numerical solution of the radial structure of the disk is done using a one-dimensional grid of points along the r -axis. The points are distributed smoothly, but nonuniformly, so that the greatest density of points is in the region near the peak of $\Psi(r)$, and points very far away are also represented, but at larger intervals. The Green's function in equation (45a) is calculated using polynomial approximations for the elliptic functions K and E (Abramowitz and Stegun 1964). The logarithmic singularity of K is integrable, and can be handled straightforwardly. Thus, $\Psi(r)$ is obtained from equation (45a) by application of the trapezoidal rule. The magnetic field, $B_z(r, z = 0) = r^{-1}[d\Psi(r)/dr]$, can be directly calculated from

$$B_z(r, 0) = 2\pi \int_1^\infty dr' \frac{1}{r} \frac{\partial g}{\partial r}(r', r) j_\phi(r'). \quad (54)$$

The kernel, $r^{-1}(\partial g/\partial r)$, has a principal-value singularity $P(r' - r)^{-1}$. For $r < 1$ the singularity falls outside the region of integration. For $r > 1$ the principal value ensures the existence of the integral. At $r = 1$ the behavior is determined by the exponent δ in $j_\phi(r)$ of equation (50). Figure 4 shows the behavior of $4B_z$ and $r(\partial B_z/\partial r)$ for three values of δ , all for $\lambda = 0.4$, $m = 0$, and $n = 3$. For $\delta > 1$, B_z and $r(\partial B_z/\partial r)$ are continuous at $r = 1$. Figure 5 shows the radial dependence of the stability ratio S from equation (53) for a barely stable case and a very unstable case, both for $\delta = \frac{1}{2}$.

A check on the accuracy of the numerical solution is provided by the virial equation (37). The energies that occur in the virial theorem are scaled by GMM_d/r_0 , so that equation (37), with $P = 0$, becomes

$$2W_K + W_g + W_m = 0, \quad (55)$$

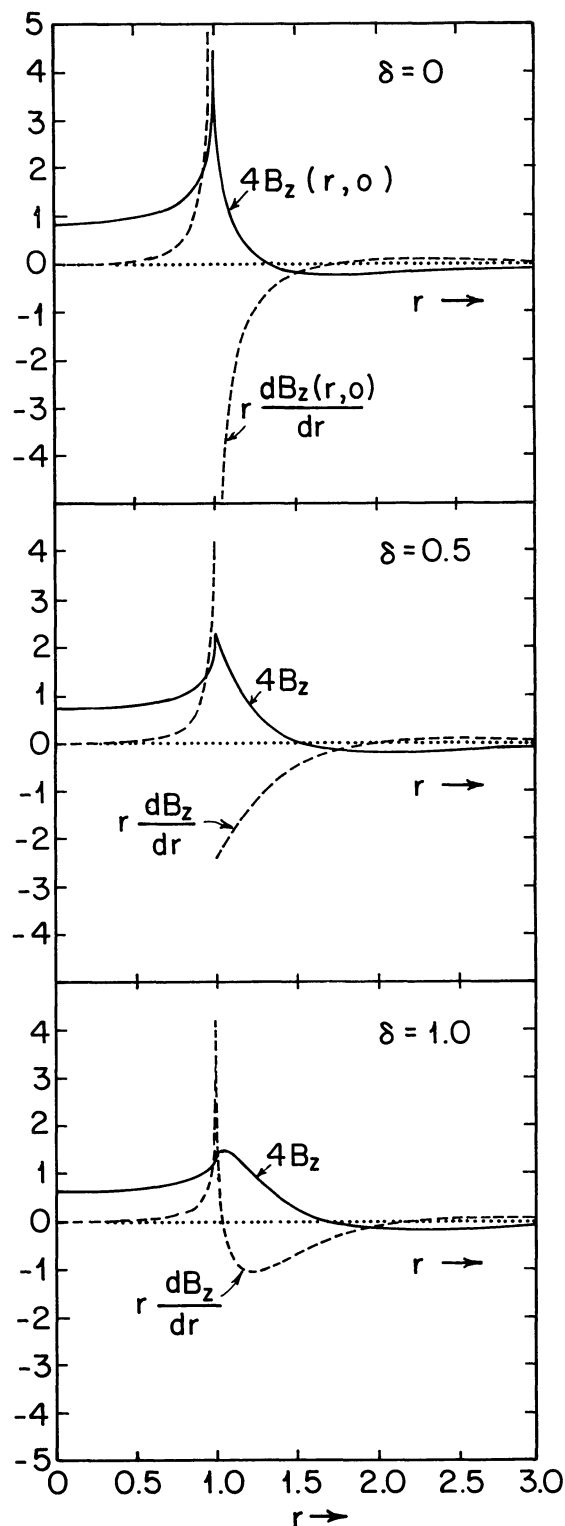


FIG. 4.—Radial profiles of $4B_z$ and $r(dB_z/dr)$, which are important to the determination of the axisymmetric stability of the disk. The vertical and horizontal scales are in the dimensionless units discussed in the text in § III e. The quantity δ determines the variation of the current density near the disk inner edge according to eq. (49). In this figure, $\lambda = 0.4$ and $m = 0$ in eq. (49). For $\delta = 0$, $B_z(r, 0)$ has a logarithmic singularity at $r = 1$. That is, $B_z \propto \ln |1 - r|$ for $|1 - r| \ll 1$. For $\delta = \frac{1}{2}$, B_z is finite and continuous at $r = 1$; dB_z/dr is infinite for r approaching 1 from below but finite for r approaching 1 from above. That is, $B_z \propto \text{constant} - (1 - r)^{1/2}$ for $r \leq 1$, whereas $B_z \propto \text{constant} - (1 - r)$ for $r \geq 1$. For $\delta = 1$, B_z is finite at $r = 1$, but dB_z/dr diverges. That is, $B_z \propto |1 - r| \ln |1 - r|$. For $\delta > 1$, both B_z and dB_z/dr are nonsingular at $r = 1$.

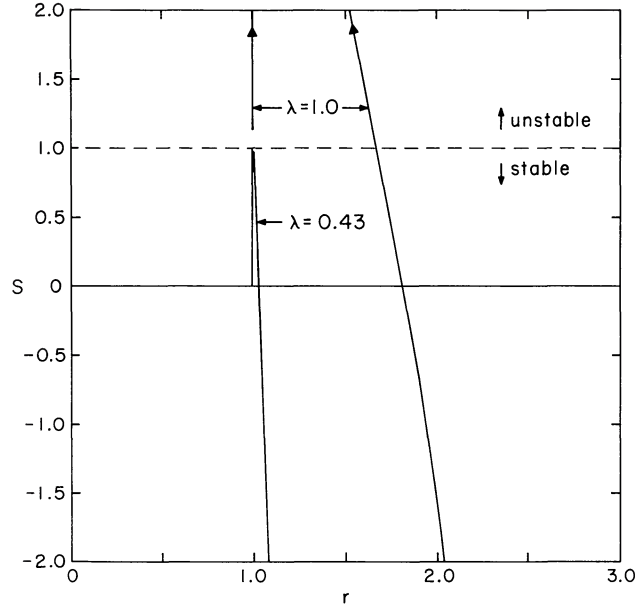


FIG. 5.—Radial dependence of the stability ratio S of eq. (53) for $\delta = \frac{1}{2}$, $r_s = \frac{1}{3}$, and for two values of λ . For $\lambda = 0.43$, we have stability, $S < 1$, for the entire range of r . For $\lambda = 1.0$, there is a substantial range of r where the disk is unstable, $S > 1$.

where

$$W_K \equiv \pi \int_0^\infty r dr \sigma(r) r^2 \Omega^2 [\Psi(r)],$$

$$W_g \equiv 2\pi \int_0^\infty r dr \sigma(r) r^2 \frac{1}{r(r-r_s)^2},$$

$$W_m \equiv 2\pi \xi_0 \int_0^\infty r dr j_\phi(r) r B_z(r, 0)$$

are the dimensionless kinetic, gravitational, and magnetic energies, respectively. For our disk equilibria we find typical errors of less than one part in 10^5 in equation (55).

We now fix $\delta = \frac{1}{2}$, $m = 0$, and $n = 3$ and survey the stable equilibria obtained by varying λ . Of course, r_s (and to a small extent n) must be adjusted to satisfy the criteria discussed above. In Figure 6 we plot various physical quantities of interest as functions of λ . Note that all quantities have been converted back to the physical system of units where r_s is fixed and r_0 , the disk inner radius, varies.

A number of aspects of Figure 6 should be noted: The lower limit of λ for the curves in the figure is set by the constraint that σ be nonnegative. With increasing λ , one would expect a correspondingly broadened current-density distribution. That such broadening does not occur is due to the constraint requiring axisymmetric stability. As λ is increased, the stability condition is satisfied only by decreasing r_s . In terms of the physical variables, with r_s fixed, this corresponds to r_0 increasing with increasing λ . Finally, with regard to Figure 6, notice that the maximum of $\xi \equiv |W_m/W_g|$ for this family of disk equilibria is $\max(\xi) \approx 0.36$. The maximum value allowed by the virial equation (38) is unity.

Figure 7 shows the radial, equatorial profiles of some of the disk parameters for a specific stable equilibrium. Notice that the angular rotation rate of the disk material is reduced (or increased) relative to the value it would have in the absence of a magnetic field for distances interior (or exterior) to the radius of the O -point of the magnetic field. This is a consequence of the tension of the magnetic field lines which pass through the disk.

Figure 8 shows the radial profiles of the centrifugal and magnetic forces normalized to the gravitational force, for the same equilibrium as in Figure 7. Also, for this equilibrium, Figure 9 shows the flux surfaces in the poloidal plane.

Figures 10 and 11 illustrate the nature of some more complicated field-line topologies, having two and three extrema of $\Psi(r)$, respectively. In the case of Figure 10, there are two O -points in the equatorial plane, whereas for Figure 11 there is, for increasing r , an O -point, an X -point, and an O -point. Thus, the first case may be denoted OO and the second OXO . Figures 4–9 correspond simply to O . Note that the Ψ 's in Figures 10 and 11 do not necessarily correspond to a nonnegative surface mass density, unlike those of Figures 5–9. The required calculation of the self-consistent equilibria is deferred to a future work.

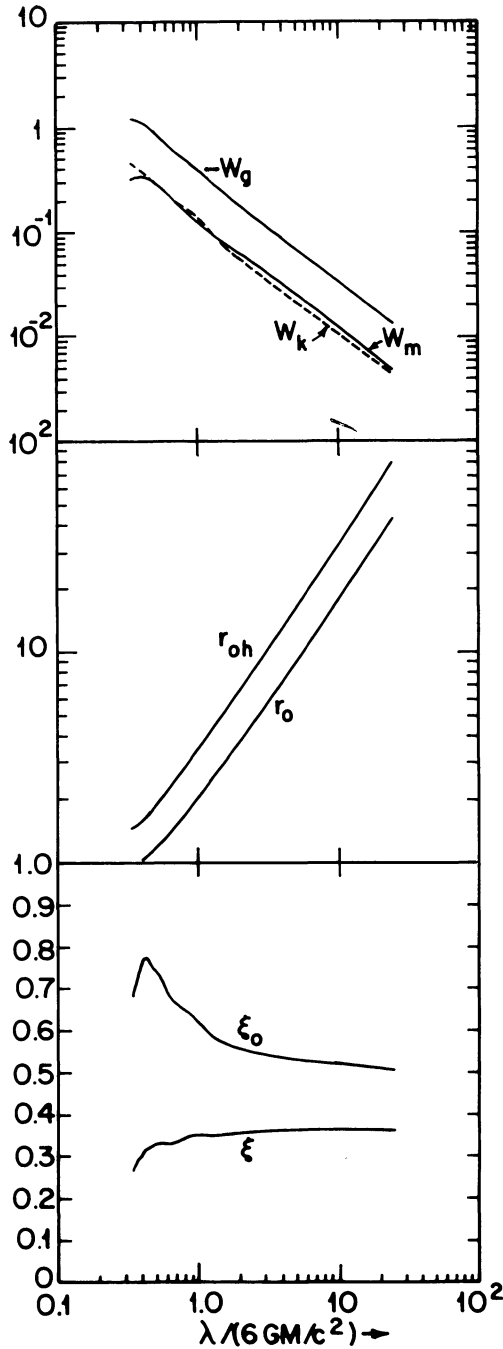


FIG. 6

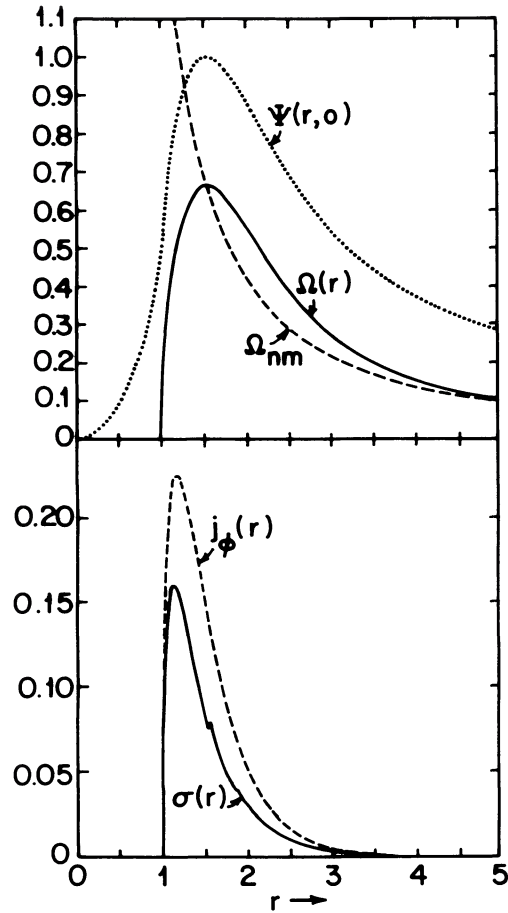


FIG. 7

FIG. 6.—Dependences of the physical parameters of a family of stable disk equilibria on the length λ which characterizes the current density distribution according to eq. (49). In eq. (49) we have set $\delta = \frac{1}{2}$, $m = 0$, and, in eq. (51), $n = 3$. The top panel of the figure shows the total gravitational ($-W_g$), magnetic (W_m), and kinetic (W_k) energies measured in units of $M_d c^2 / 6$. For large values of λ , $\lambda(6GM/c^2)^{-1} > 3$, the energies vary approximately as $\lambda^{-1.05}$. The middle panel of the figure shows the disk inner radius (r_0) and the radius of the O -point of the magnetic field (r_{oh}), measured in units of $3r_s = 6GM/c^2$. For large λ , r_0 and r_{oh} vary approximately as $\lambda^{0.88}$. The bottom panel of the figure shows the dimensionless quantities $\xi = |W_m/W_g|$ and $\xi_0 = (\Psi_{\max})^2 / (GM M_d)$. For large values of λ , ξ approaches a value ~ 0.36 , and ξ_0 approaches a value ~ 0.5 . Notice that the virial theorem of § IIIc requires $\xi < 1$.

FIG. 7.—Radial, equatorial profiles of the flux function (Ψ), the angular rotation rate of the disk matter (Ω), the angular rotation rate the disk matter would have in the absence of a magnetic field (Ω_{nm}), the disk current density (j_ϕ), and the surface mass density (σ) for a stable disk equilibrium characterized by $\lambda = 0.4$, $\delta = \frac{1}{2}$, and $m = 0$ in eq. (49), and $n = 3$ in eq. (51). The “low” rotation rate of the disk matter inside r_{oh} , that is, $\Omega/\Omega_{nm} < 1$, arises from the radially outward force due to the tension of the magnetic field lines. On the other hand, for $r > r_{oh}$, this force is inward and $\Omega/\Omega_{nm} > 1$. Note that at the disk inner radius $\Omega/\Omega_{nm} \approx 0.167$.

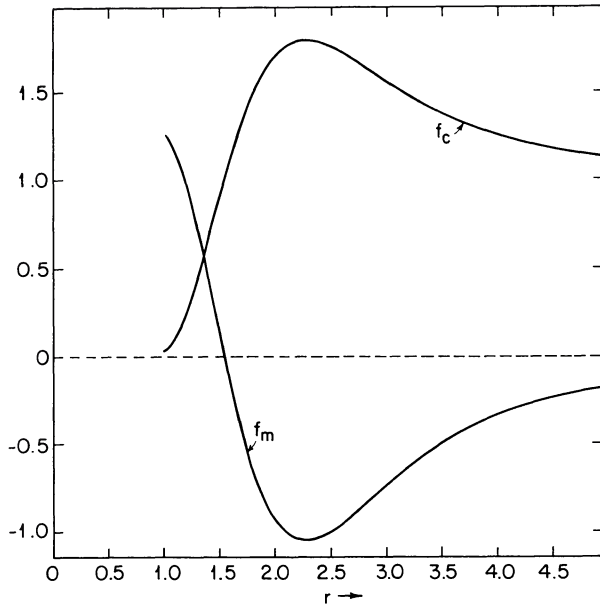


FIG. 8

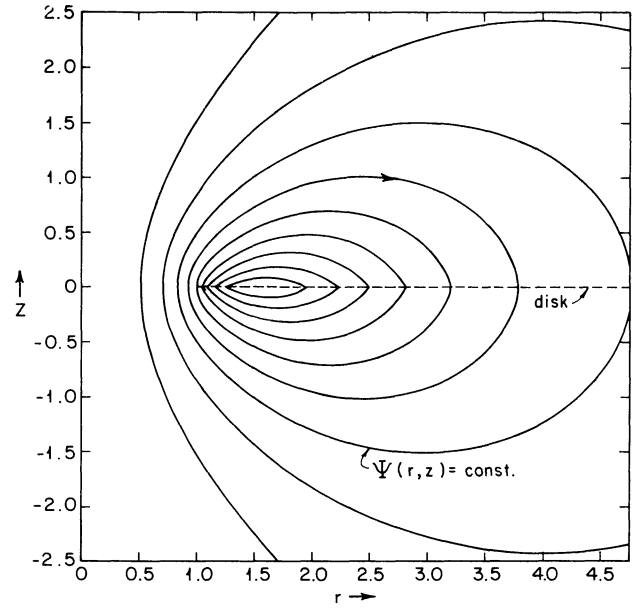


FIG. 9

FIG. 8.—Radial profiles of the centrifugal force (f_c) and the magnetic force (f_m) measured in units of the gravitational force $\sigma(r)GM(r-r_s)^{-2}$ for the same disk equilibrium as in Fig. 6. Note that a positive f -value corresponds to an outward force, and that the radial force balance implies $f_c + f_m = 1$.

FIG. 9.—Flux surfaces for the same disk equilibrium as in Figs. 6 and 7. The difference in the values between successive contours is $0.1\Psi_{\max}$. The $\Psi(r, z) = 0$ contour coincides with the z -axis. Note that B_r is discontinuous at $z = 0$ in the range of r where the current density is nonzero.

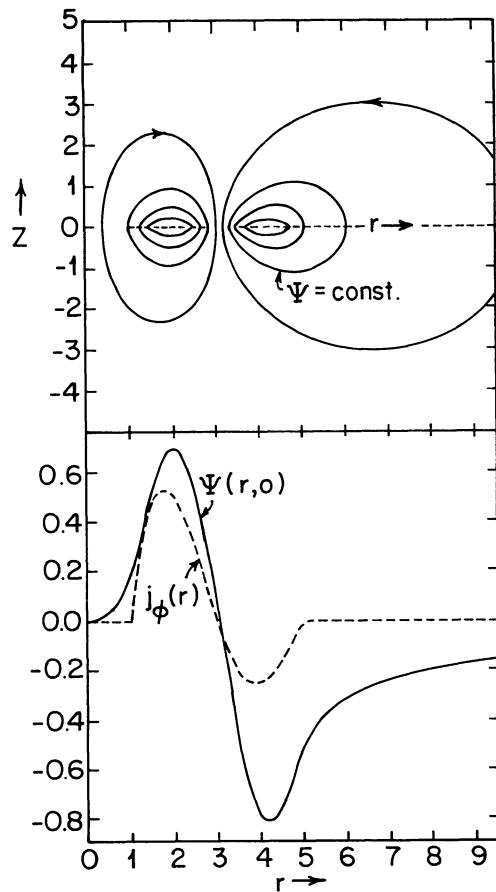


FIG. 10

FIG. 10.—Illustration of a field configuration with two O -points in the equatorial plane

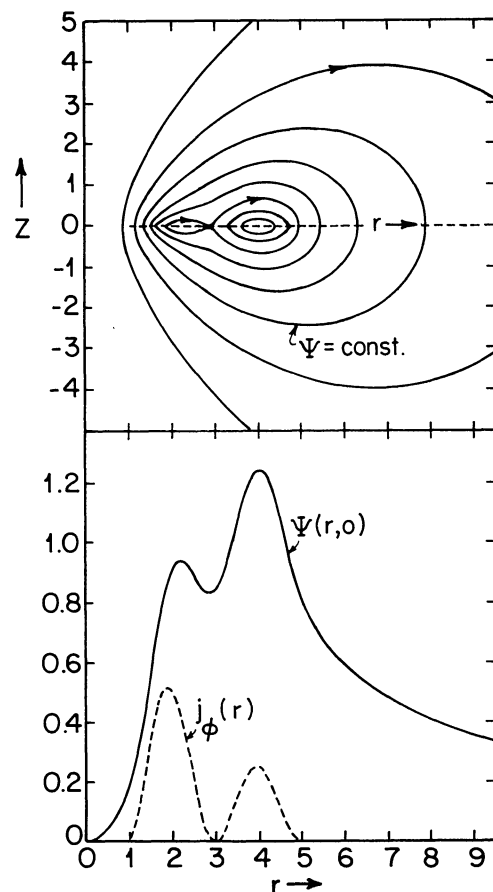


FIG. 11

FIG. 11.—Illustration of a field configuration with an O -point, an X -point, and an O -point in the equatorial plane

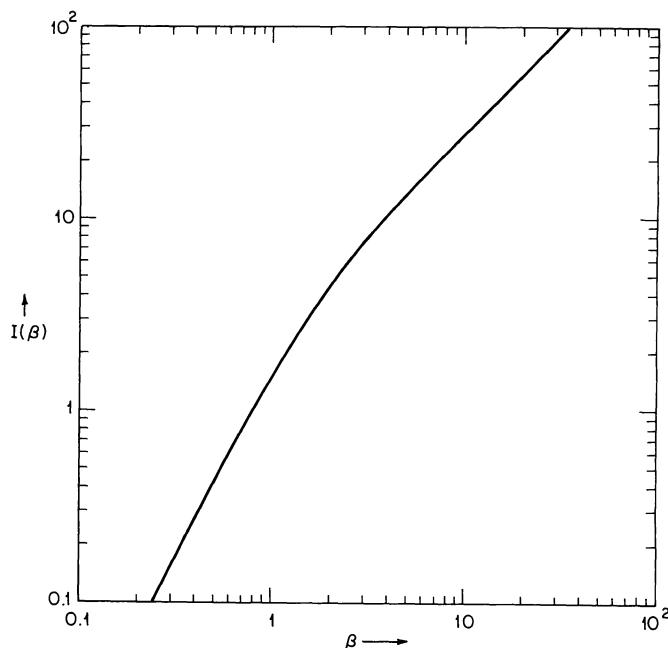


FIG. 12.—Dependence of dimensionless function of the Bennett pinch equation (eq. [56]) on magnetic field strength β

f) Bennett Pinch Relation

Here we return to consider the radial dependence of the disk half-thickness $\Delta z_{1/2} = \xi_{1/2} h$, which in turn is determined by $\beta(r)$ from equation (28c).

From § IIIa we have

$$\begin{aligned} j_\phi(r) &= rc \int_{-\infty}^{\infty} dz \rho(r, z) [J' + r^2 (G^2/2)'] \\ &= [c^2 c_s^2 \sigma(r) I(\beta) / (4\pi h)]^{1/2}, \end{aligned} \quad (56)$$

where $I(\beta)$ is a dimensionless function,

$$\begin{aligned} I(\beta) &\equiv 2\beta^2 \int_0^\infty d\xi \exp(-\psi) \\ &= 2^{1/2} \beta^2 \int_0^\infty d\psi \{2\psi + \beta^2 [1 - \exp(-\psi)]\}^{-1/2} \exp(-\psi). \end{aligned} \quad (57)$$

Figure 12 shows $I(\beta)$ obtained by numerical integration of equation (57). For $\beta^2 \ll 1$, we have $I(\beta) \approx \pi^{1/2} \beta^2$, whereas, for $\beta^2 \gg 1$, $I(\beta) \approx 2^{3/2} \beta$.

With $j_\phi(r)$ evaluated as illustrated in § IIIe, and with c_s , h , and σ assumed known as functions of r , equation (56) can be used to determine $\beta(r)$. Equation (28c) can then be used to obtain $\xi_{1/2}(r)$. In particular, for $\beta \gg 1$, we have

$$\beta(r) \approx 4.44 [j_\phi(r)]^2 [\sigma(r) c_s^2 c^2 / h]^{-1}. \quad (58a)$$

In terms of the dimensionless variables of § IIIe,

$$\beta \approx 8.88 \xi_0 \frac{j_\phi^2}{\sigma} \left(\frac{r}{h} \right) (r - r_s)^2. \quad (58b)$$

For the disk equilibrium of Figures 5–8, with the quantities evaluated at the maximum of j_ϕ , r_m , we find $\beta \approx 2.3(r_m/h)$.

Also, for $\beta \gg 1$,

$$z_{1/2} \approx 0.345 [\sigma(r) c_s^2 c^2] [j_\phi(r)]^{-2} \ll h. \quad (59)$$

Equation (59) clearly exhibits the pinch effect of the magnetic field: the disk thickness varies as j_ϕ^{-2} . Equation (59) is evidently a form of the well-known Bennett pinch relation (see, for example, Krall and Trivelpiece 1973).

IV. RELATIVISTIC MHD EQUILIBRIA

This section generalizes the results of § II to axisymmetric MHD equilibria with relativistic flow and/or pressure and a weak gravitational field ($|\Phi_g|/c^2 \ll 1$). Here, we derive the generalized relativistic Grad-Shafranov equation from the basic equations of general relativity. The work of this section has been extended to strong gravitational fields by Mobarry and Lovelace (1986). Our notation follows that of Landau and Lifshitz (1962). An application of the results of this section is given in § V.

The basic equations are given below.

(1) Particle conservation:

$$(nu^\alpha)_{;\alpha} = 0, \quad (60)$$

where n is the “proper” number density, that is, the density in the reference frame comoving with the fluid; u^α is the fluid 4-velocity with $g_{\alpha\beta}u^\alpha u^\beta = -1$ and $g_{\alpha\beta}$ the metric tensor; and the semicolon denotes covariant differentiation.

(2) Energy-momentum conservation:

$$[T_f^{\alpha\beta} + T_{em}^{\alpha\beta}]_{;\alpha} = 0, \quad (61)$$

where T_f and T_{em} are the energy-momentum tensors for the fluid and the electromagnetic field, respectively. For the fluid,

$$T_f^{\alpha\beta} = nwu^\alpha u^\beta + pg^{\alpha\beta}, \quad (62)$$

where $w = (e + p)/n$ is the enthalpy per particle, e is the proper internal energy density, and p is the proper pressure (Landau and Lifshitz 1959, chap. 15).

(3) Maxwell equations:

$$F^{\alpha\beta}_{;\alpha} = \frac{4\pi}{c} J_e^\beta,$$

$$F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} + F_{\alpha\beta,\gamma} = 0,$$

and

$$T_{em}^{\alpha\beta}_{;\alpha} = -F^\beta_\gamma J_e^\gamma, \quad (63)$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor, and $J_e^\alpha = (\rho_e c, \mathbf{J})$ is the 4-vector electric current density with $J_e^\alpha_{;\alpha} = 0$. A comma denotes ordinary differentiation.

(4) Equation for perfect electric conductivity:

$$u_\alpha F^{\alpha\beta} = 0. \quad (64)$$

(5) Equation of state for an ideal gas:

$$S = k_B (\Gamma - 1)^{-1} \ln(pn^{-\Gamma}), \quad (65)$$

where S is the proper entropy per particle; $\Gamma = 5/3$ for nonrelativistic motion in the comoving frame, or $\Gamma = 4/3$ for ultrarelativistic motion. Also, notice that we have $e = nmc^2 + (\Gamma - 1)^{-1}p$ and $w = mc^2 + \Gamma p[n(\Gamma - 1)]^{-1}$, where m is the particle rest mass.

By contracting u_β with equations (61), (63), and (64), and using the thermodynamic identity $dw = dp/n + TdS$, one obtains

$$(nSu^\alpha)_{;\alpha} = 0, \quad (66)$$

which is a generalization of equation (7) for the adiabatic motion of the fluid. As in the nonrelativistic case of § II, an alternative to equation (66) is the limit where the plasma is isothermal on any given flux surface. This limit is considered further in § V. By

contracting $g_{\delta\alpha} + u_\delta u_\alpha$ with equation (61), we obtain the Euler equation

$$nwu^\beta(u_\alpha)_{;\beta} = -p_{,\alpha} - u_\alpha u^\beta(p)_{,\beta} + F_{\alpha\beta}J_e^\beta/c, \quad (67)$$

which is a generalization of equation (2).

We now simplify equations (60), (66), and (67) to the limit of large distances from a spherically symmetric mass M , $r \gg 2GM/c^2$. Also recall that we are considering equilibria, so that $\partial/\partial t = 0$. In a nonrotating (laboratory) reference frame the metric tensor can be expressed as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $\eta_{\alpha\beta} = (-1, 1, 1, 1)$ along the main diagonal, $h_{\alpha\beta} = -2\Phi_g c^{-2}(1, 1, 1, 1)$ with $|h_{\alpha\beta}| \ll 1$, and $\Phi_g = -GM/r$ (Misner, Thorne, and Wheeler 1970). This corresponds to the weak field limit of an isotropic coordinate system in that the spatial components $g_{\alpha\beta}$ are equal. We let $u^\alpha = \tilde{u}^\alpha + \delta u^\alpha$, where $\eta_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta = -1$ and $\tilde{u}_\alpha = \eta_{\alpha\beta}\tilde{u}^\beta$, so that, to first order in h , $\delta u^\alpha = -\frac{1}{2}h^{\alpha\beta}u_\beta$. Also, we let

$$g \equiv 1 - \Phi_g/c^2, \quad (68)$$

where $g > 1$. Thus, $u^0 = \tilde{u}^0 g$ and $u^i = \tilde{u}^i/g$ for $i = 1, 2, 3$.

The weak field approximation to equation (60) is

$$(ng\tilde{u}^\alpha)_{,\alpha} = 0. \quad (69)$$

The approximation to equation (66) is

$$(ng\tilde{u}^\alpha S)_{,\alpha} = 0. \quad (70)$$

The approximation to the Euler equation (67) is

$$nw\tilde{u}^\beta(\tilde{u}_\alpha)_{,\beta} = -[p_{,\alpha} + \tilde{u}_\alpha\tilde{u}^\beta(p)_{,\beta}] + f_\alpha^g + f_\alpha^{\text{em}}, \quad (71)$$

where f_α^g is the gravitational force,

$$f_\alpha^g = \frac{1}{2}nw\tilde{u}^\beta\tilde{u}^\gamma[h_{\beta\gamma,\alpha} - h_{\beta\alpha,\gamma}], \quad (72a)$$

with $\tilde{u}^\alpha f_\alpha^g = 0$, and where f_α^{em} is the electromagnetic force,

$$f_\alpha^{\text{em}} = \frac{a(\alpha)}{c}F_{\alpha\beta}J_e^\beta, \quad (72b)$$

with $\tilde{u}^\alpha f_\alpha^{\text{em}} = 0$ and $a(\alpha) = (g^2, 1, 1, 1)$ for $\alpha = 0, 1, 2, 3$. Equation (71) includes all terms to first order in Φ_g but neglects terms of higher order such as Φ_g^2 and $\Phi_g|\nabla\Phi_g|$.

In 3-space notation, with the tildes dropped,

$$\begin{aligned} f_0^g &= nw\gamma^2(\mathbf{v} \cdot \nabla)(\Phi_g/c^2), \\ f_i^g &= -nw\gamma^2\nabla(\Phi_g/c^2) - nw(\gamma^2 - 1)(1 - \mathbf{v}\mathbf{v}/v^2) \cdot \nabla(\Phi_g/c^2), \end{aligned} \quad (72c)$$

where $u^0 = \gamma$ and $u^i = \gamma\mathbf{v}/c$, with $\mathbf{v}$ the flow velocity and $\gamma = (1 - v^2/c^2)^{-1/2}$ the Lorentz factor of the flow. Also, in 3-space notation, and with the tildes dropped,

$$\begin{aligned} f_0^{\text{em}} &= [a(0)/c]F_{0\beta}J_e^\beta = -g^2\mathbf{E} \cdot \mathbf{J}_e/c \\ &= +(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{J}_e/c^2, \\ f_i^{\text{em}} &= [a(1)/c]F_{i\beta}J_e^\beta = \rho_e\mathbf{E} + (\mathbf{J}_e \times \mathbf{B})/c \\ &= -\rho_e(\mathbf{v} \times \mathbf{B})/(cg^2) + (\mathbf{J}_e \times \mathbf{B})/c. \end{aligned} \quad (72d)$$

Here, $\rho_e = J_e^0/c$ is the electric charge density and $\mathbf{J}_e = J_e^i$ is the current density (in the laboratory reference frame); the perfect conductivity condition is

$$\mathbf{E} + (\mathbf{v} \times \mathbf{B})/(cg^2) = 0; \quad (72e)$$

the laboratory electromagnetic fields are

$$\mathbf{E} = (E_x, E_y, E_z) = (F^{01}, F^{02}, F^{03})$$

and

$$\mathbf{B} = (B_x, B_y, B_z) = g^4 (F^{23}, F^{31}, F^{12});$$

and the Maxwell equations are

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= 0, \\ [\nabla \times (\mathbf{B}/g^2)]/g^2 &= 4\pi \mathbf{J}_e/c, \\ [\nabla \cdot (g^2 \mathbf{E})]/g^2 &= 4\pi \rho_e.\end{aligned}\tag{73}$$

We now examine the modifications of the results of § II required for relativistic flows. We still have $E_\phi = 0$ owing to the axisymmetry, and $v_p = \kappa(r, z)\mathbf{B}_p(r, z)$ owing to the perfect conductivity. Thus, the continuity equation (69) and $\nabla \cdot \mathbf{B} = 0$ imply

$$4\pi\rho\kappa = F(\Psi),\tag{74}$$

where

$$\rho \equiv nmg\gamma$$

is an effective mass density in the laboratory reference frame, and m is the mean plasma particle rest mass, and, as in § II,

$$B_r = -\frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad B_z = \frac{1}{r} \frac{\partial \Psi}{\partial r}.\tag{75}$$

Equation (74) is formally the same as equation (11).

In place of equation (12) we have

$$(v_\phi - \kappa B_\phi)/(rg^2) = G(\Psi),\tag{76}$$

and equation (13) remains unchanged. Hence, the flow velocity can be expressed as

$$\mathbf{v} = \frac{F(\Psi)}{4\pi\rho(r, z)} \mathbf{B}_p(r, z) + \left[\frac{F(\Psi)}{4\pi\rho(r, z)} B_\phi(r, z) + rg^2 G(\Psi) \right] \hat{\phi}.\tag{77}$$

The toroidal component of the Euler equation (71), the thermodynamic identity $dw = dp/n + TdS$, and equations (69) and (70), imply angular momentum conservation on each flux surface,

$$rB_\phi/g^2 - FWrv_\phi = H(\Psi),\tag{78}$$

where

$$W \equiv wg\gamma(mc^2)^{-1}$$

is dimensionless and $W > 0$. Equation (78) evidently reduces to equation (15) in the nonrelativistic limit where $\gamma \rightarrow 1$, $w \rightarrow mc^2$, and $g \rightarrow 1$.

The “zero” component of the Euler equation (71), and equations (69), (72), and (73), give Bernoulli’s equation. First notice that

$$\nabla \cdot [nw\gamma^2 \mathbf{v} + c(\mathbf{E} \times \mathbf{B})(4\pi g^2)^{-1}] = 0,$$

which implies

$$[FW - GrB_\phi c^{-2}]g^{-2} = I(\Psi),\tag{79}$$

where $I(\Psi)$ is a function only of Ψ . This equation represents the constancy of the energy flux density on each flux surface. The term FWg^{-2} represents the transport of energy by the matter, whereas the term $-GrB_\phi(cg)^{-2}$ represents the transport by the Poynting flux. As an example, in the dynamo model (Lovelace 1976; Blandford 1976) the energy flux of the matter and that of the electromagnetic field are in the same direction ($F > 0$ and $GB_\phi > 0$) with the Poynting flux dominant near the central object.

Using equation (78), we rewrite equation (79) as

$$\begin{aligned} \frac{W}{g^2} \left(1 - \frac{g^2 Grv_\phi}{c^2} \right) &= \frac{I}{F} + \frac{GH}{c^2 F} \\ &= 1 + \frac{J(\Psi)}{c^2}, \end{aligned} \quad (80)$$

where $J(\Psi) \equiv c^2 I/F + GH/F - c^2$. Recalling in the nonrelativistic limit that $\gamma - 1 \approx v^2/(2c^2) \ll 1$, that $W/g^2 \approx (w/mc^2)\{1 + [v^2/(2c^2)] + \Phi_g/c^2\}$, and that $w - mc^2 = m[f(dp/\rho)]_S \ll mc^2$, we verify that this $J(\Psi)$ coincides with that of equation (17).

Also, as in § II, we have $S = S(\Psi)$, owing to equations (69) and (70). Thus, equation (65) implies

$$w = mc^2 \left\{ 1 + \frac{\Gamma(n/n_0)^{\Gamma-1}}{\Gamma-1} \exp \left[S(\Psi) \frac{(\Gamma-1)}{k_B} \right] \right\}, \quad (81)$$

in place of equation (18), where n_0 is a reference density.

In place of equations (16a) and (16b), we have

$$rB_\phi = g^2 \left[H + (rg)^2 GFW \right] \left(1 - \frac{g^2 F^2 W}{4\pi\rho} \right)^{-1} \quad (82a)$$

and

$$rv_\phi = \left[(rg)^2 G + \frac{g^2 FH}{4\pi\rho} \right] \left(1 - \frac{g^2 F^2 W}{4\pi\rho} \right)^{-1}. \quad (82b)$$

As in § II, the singularity of equations (82) at $g^2 F^2 W/(4\pi\rho) \equiv M_a^2 = 1$ is associated with the shear Alfvén waves. In order to have a field line or streamline which passes through the Alfvén surface (where $M_a = 1$), the numerators of equations (82a) and (82b) must vanish on this surface. Thus $H = -(rg)^2 FGW$ and $r^2 G = -FH/(4\pi\rho)$ at the Alfvén surface. If s denotes the arc length along the poloidal projection of a field line which penetrates the Alfvén surface, then $d\Psi/ds = 0$, and from l'Hospital's rule we have

$$(rB_\phi)_A = +H \frac{d[\ln(r^2 g^2 W)]_A/ds}{d[\ln(g^2 W/\rho)]_A/ds}$$

and

$$(rv_\phi)_A = -r_A^2 G \frac{d[\ln(\rho r^2)]_A/ds}{d[\ln(g^2 W/\rho)]_A/ds}$$

at the Alfvén surface.

We have, from equation (80),

$$W = g^2 \left(1 + \frac{J}{c^2} \right) \left(1 - \frac{g^2 Grv_\phi}{c^2} \right)^{-1}. \quad (83)$$

In general, $W > 0$. We define $x \equiv Grv_\phi/(g/c)^2$. Thus, for the situation considered in § V where $1 + J/c^2 > 0$, we have $x < 1$. In general, we have $x^2 \leq (g^2 rG/c)^2 < 1 + (B_\phi/B_p)^2$ from equation (76). Equations (82) and (83) evidently lead to a quadratic formula for rv_ϕ or for rB_ϕ . Also, defining $A \equiv (rG/c)^2 g^4 + FGHg^4(4\pi\rho c^2)^{-1}$ and $B \equiv (1 + J/c^2)F^2 g^4(4\pi\rho)^{-1}$, we find

$$1 = \frac{A}{x} + \frac{B}{1-x}. \quad (84)$$

For, say, $A > 0$ and $B > 0$, and for a given point in space, we may have (1) no solution to equation (84) for x (an excluded region for the plasma); (2) a single solution for x ; or (3) two solutions for x , $0 < x_1 < x_2 < 1$, and a corresponding pair of solutions for rB_ϕ . The fact that two solutions may exist is a consequence of special relativity.

The relativistic Grad-Shafranov equation is obtained as in § II from the poloidal component of the Euler equation (71) parallel to $\nabla\Psi$. A fairly direct calculation from equations (68) and (71) gives

$$\begin{aligned} & \frac{\rho c^2}{(\gamma g)^2} \nabla\Psi \cdot \nabla W - \left(\frac{\rho W}{g^2} \right) \nabla\Psi \cdot [\mathbf{v} \times (\nabla \times \mathbf{v})] \\ & = p \nabla\Psi \cdot \left(\frac{\nabla S}{k_B} \right) + \nabla\Psi \cdot \left(\rho_e \mathbf{E} + \frac{1}{c} \mathbf{J}_e \times \mathbf{B} \right) - 2\rho W (\nabla\Psi \cdot \nabla\Phi_g). \end{aligned} \quad (85)$$

A further manipulation of equation (85) gives the *generalized Grad-Shafranov equation*:

$$\begin{aligned} & \left[1 - \left(\frac{g^2 r G}{c} \right)^2 - \frac{g^2 W F^2}{4\pi\rho} \right] \Delta^* \Psi - \frac{1}{2r^2} \nabla \left[\frac{(rg)^4 G^2}{c^2} \right] \cdot \nabla\Psi - g^2 F \nabla \left(\frac{FW}{4\pi\rho} \right) \cdot \nabla\Psi + 2 \left(\frac{\nabla\Phi_g}{c^2} \right) \cdot \nabla\Psi \\ & = -g^4 (H + Wrv_\phi F) [H' + (Wrv_\phi) F'] - 4\pi\rho r^2 g^4 [J' + (Wrv_\phi) G'] + 4\pi r^2 g^4 p \left(\frac{S'}{k_B} \right). \end{aligned} \quad (86)$$

As in § II, the functions dependent only on Ψ , that is, F , G , H , J , and S , must be supplied. The primes denote, for example, $F' = dF/d\Psi$. In the nonrelativistic limit, $W \rightarrow 1$ and $g \rightarrow 1$, and equation (86) coincides with equation (19) as it should.

The mathematical nature of the operator on the left-hand side of equation (86) can be extracted as in § II by straightforward but lengthy manipulations in which equations (74), (76), (78), (82b), and (83) are used to express $\nabla(W/\rho)$ in terms of $\nabla(\nabla\Psi)^2$. Retaining only the highest (second) order derivatives of Ψ , the left-hand side of equation (86) is

$$P \left(A_{rr} \frac{\partial^2 \Psi}{\partial r^2} + A_{rz} \frac{\partial^2 \Psi}{\partial r \partial z} + A_{zz} \frac{\partial^2 \Psi}{\partial z^2} \right),$$

where

$$P \equiv 1 - \left(\frac{g^2 r G}{c} \right)^2 - \frac{g^2 W F^2}{4\pi\rho},$$

$$A_{rr} \equiv 1 - \frac{v_p^2 v_z^2}{R},$$

$$A_{rz} \equiv \frac{2v_p^2 v_r v_z}{R},$$

$$A_{zz} \equiv 1 - \frac{v_p^2 v_r^2}{R},$$

and

$$R \equiv v_a^2 C_s^2 f - v_p^2 \left[C_s^2 + v_A^2 - v_a^2 \left(\frac{g^2 r G}{c} \right)^2 \right] + v_p^4.$$

Here $v_A \equiv [(B_p^2 + B_\phi^2)/(4\pi\rho g^2 W)]^{1/2}$ is introduced as the effective Alfvén speed; $v_a = [B_p^2/(4\pi\rho g^2 W)]^{1/2}$ is the effective Alfvén speed projected into the poloidal plane; $C_s \equiv c_s \gamma_s / \gamma$ is the effective sound speed in the laboratory reference frame; $c_s = c[(\Gamma - 1)^{-1} + mc^2/(\partial p/\partial n|_s)]^{-1/2} \leq 3^{-1/2}c$ is the speed of sound in the comoving reference frame of the fluid; $f \equiv 1 - (g^2 r G/c)^2$; $\gamma_s \equiv [1 - (c_s/c)^2]^{-1/2}$; and $v_p \equiv (v_r^2 + v_z^2)^{1/2}$. Note that v_A and v_a depend implicitly on the flow speed in that $W \propto \gamma$. Also, from equation (76) we have $g^2 r G/c = (v_\phi/c) \mp (v_p/c)(B_\phi/B_p)$, where the minus (or plus) sign corresponds to $F > 0$ (or $F < 0$). In general, $(v_a)^2 (g^2 r G/c)^2 \leq (v_A)^2$. In the nonrelativistic limit, R coincides with the formula of § II as it should.

As in § II, equation (86) is elliptic (or hyperbolic) depending on whether the discriminant, $D \equiv (A_{rz})^2 - 4A_{rr}A_{zz}$, is negative (or positive). We readily find $D = -4[1 - (v_p^4/R)]$. For simplicity, consider the limit where $C_s \ll (v_p, v_a)$, so that

$$D = \frac{v_a^2 - v_a^2 (g^2 rG/c)^2}{v_p^2 + v_a^2 (g^2 rG/c)^2 - v_a^2}.$$

As mentioned, the numerator of D is nonnegative. For a given sign of F , the denominator of D vanishes at one nonnegative value of v_p given by

$$v_p^2 = v_a^2 \left\{ 1 + b^2 - \left[(v_\phi/c) \mp b(v_p/c) \right]^2 \right\}.$$

where $b \equiv B_\phi/B_p$, and the minus (plus) sign corresponds to $F > 0$ (or $F < 0$). This root corresponds to a flow speed equal to the fast magnetosonic wave speed. (The slow magnetosonic wave speed is zero.) Because $v_a^2 \propto 1/\gamma$, we have $(v_p/c)^2 < 1 - (v_\phi/c)^2$.

A “force-free” limit of equation (86) may be valid in limited spatial regions of a global solution. In this limit, the inertial forces—proportional to the particle rest mass—are negligible compared with the electromagnetic forces. The dimensionless parameter which must be small compared with unity in the force-free limit is

$$\epsilon \equiv \left(\frac{v_p}{v_a} \right)^2 \approx \frac{4\pi\rho\gamma v_p^2}{B^2} \ll 1.$$

Neglecting all terms of order s and the term $2\nabla\Phi_g \cdot \nabla\Psi$, and setting $g=1$ for simplicity, a force-free limit of equation (86) is obtained:

$$\left[1 - \left(\frac{rG}{c} \right)^2 \right] \Delta^* \Psi - \frac{1}{2r^2} \left\{ \nabla \left[\frac{(r^2 G)^2}{c^2} \right] \right\} \cdot \nabla \Psi = -HH', \quad (87)$$

where $rB_\phi = H(\Psi)$, $v_\phi = rG(\Psi) + c\chi$, $v_p = \chi rB_p/H$, and where $\chi \equiv FH/(4\pi\rho rc)$. With the simplification $W = \gamma$, the energy equation (83) can be written as

$$\left(1 - \frac{Grv_\phi}{c^2} \right)^2 = Q(\Psi) \left(1 - \frac{v_p^2 + v_\phi^2}{c^2} \right),$$

where $Q(\Psi) \equiv [1 + J(\Psi)/c^2]^2$. This equation determines χ in terms of $rG(\Psi)$, $H(\Psi)$, rB_p , and $Q(\Psi)$.

Equation (87) is equivalent to the “pulsar equation” (Michel 1973; Scharlemann and Wagoner 1973; a limit of the pulsar equation was given earlier by Cohen, Coppi, and Treves 1973). However, the “pulsar equation” neglects without discussion the term $-[r^2/(2c^2)]\nabla(G^2) \cdot \nabla\Psi$ on the left-hand side of equation (87). This contribution arises from the space charge density at the boundary surface between the private magnetic flux of the pulsar and the external world, and from the volume distributed space charge density on the external field lines due to the nonconstancy of G in this region.

A subcase of equation (86) of interest to us in the next section is that corresponding to *no* poloidal flow [$v_p = 0$ or $F(\Psi) = 0$] and *no* toroidal magnetic field [$B_\phi = 0$ or $H(\Psi) = 0$], and isothermal conditions on any given flux surface:

$$\begin{aligned} & \left[1 - \left(\frac{g^2 rG}{c} \right)^2 \right] \Delta^* \Psi - \frac{1}{2r^2} \left\{ \nabla \left[\frac{(rg)^4 G^2}{c^2} \right] \right\} \cdot \nabla \Psi + 2 \left(\frac{\nabla\Phi_g}{c^2} \right) \cdot \nabla \Psi \\ & = -4\pi\rho r^2 g^4 \left[J' + W(rg)^2 \left(\frac{G^2}{2} \right)' \right] - 4\pi r^2 g^4 \left(\frac{\rho}{m} \right) ST'. \end{aligned} \quad (88)$$

In this case, only three functions of Ψ , that is, G , J , and T , need be supplied. Notice that with $F=0$, the flow velocity is $\mathbf{v} = g^2 rG(\Psi) \hat{\phi}$, and that $(W/g^2)[1 - (g^2 rG/c)^2] = 1 + J(\Psi)/c^2$. The magnitude of the flow velocity in a physical solution of (88) must be less than the speed of light; equivalently, $[g^2 rG(\Psi)/c]^2 < 1$.

V. THIN RELATIVISTIC DISKS

Here we apply the results of § IV to magnetized disks, including the possibility of relativistic azimuthal motion, but excluding both poloidal flow and a toroidal field [$F(\Psi) = 0$ and $H(\Psi) = 0$ in eq. (86)]. In this section the central object is assumed to be an aligned, rotating magnetized star. The star is characterized by its mass, M_* ; magnetic moment, $m_* > 0$; angular rotation rate, Ω_* ; and radius, r_* . We consider the disk vertical structure in § Va, the radial force balance in § Vb, and specific disk equilibria in § Vc, and further aspects of the field behavior at the speed-of-light cylinder in § Vd.

The disk thickness Δz is assumed small compared with r ; the conditions for this are then established.

A number of further assumptions are made:

1. The disk has reflection symmetry about $z = 0$.
2. The gravitational potential $\Phi_g(r, z)$ is due to the central object ($M_* \gg 4\pi\sigma r^3/\Delta z$).
3. The pressure p in the comoving reference frame is small compared with nmc^2 .
4. The equation of state of the plasma on a given flux surface is isothermal, with $p = nk_B T(\Psi)$, so that the enthalpy per particle is

$$w = mc^2 + m [c_s(\Psi)]^2 \ln \left(\frac{\rho}{\gamma \rho_0} \right), \quad (89)$$

where $c_s = [k_B T(\Psi)/m]^{1/2}$ is the Newtonian sound speed in the comoving reference frame, and ρ_0 is a constant reference density. Assumption 3 corresponds to $c_s^2 \ll c^2$.

5. The variation of c_s with Ψ is gradual in comparison with that of $J(\Psi)$ and $G(\Psi)$, so that the term $-4\pi r^2 g^4 (\rho/m) ST'$ in equation (88) is negligible.

6. The gravitational field is weak, $g - 1 = -\Phi_g/c^2 \ll 1$. Consequently, the term $2(\nabla \Phi_g/c^2) \cdot \nabla \Psi$ in equation (88) is negligible. In all of the other terms of equation (88), the g -factors are for the present retained.

a) Vertical Structure of Disk

From § IV and the above condition 4, we find

$$J(\Psi) = \gamma^{-1} \left[c^2 + c_s^2 \ln \left(\frac{\rho}{\gamma \rho_0} \right) + \Phi_g \right] - c^2, \quad (90)$$

$$W = (g\gamma)^2 \left(1 + \frac{J}{c^2} \right) = \gamma \left[1 + \left(\frac{c_s}{c} \right)^2 \ln \left(\frac{\rho}{\gamma \rho_0} \right) - \frac{\Phi_g}{c^2} \right], \quad (91)$$

and

$$\gamma = \left[1 - \left(\frac{g^2 r G}{c} \right)^2 \right]^{-1/2}, \quad (92)$$

where $G(\Psi) = v_\phi (g^2 r)^{-1}$. As in § IIIa, we invert equation (90) to obtain

$$\rho(r, z) = \rho_0 \gamma \exp \left\{ \left[(\gamma - 1) c^2 + \gamma J - \Phi_g \right] / c_s^2 \right\} \quad (93)$$

in place of equation (24a).

Writing $\Psi(r, z) = \Psi_0(r) + \Psi_1(r, z)$, as in § IIIa, we obtain the *vertical structure equation*,

$$\frac{d^2 \tilde{\Psi}_1}{d\zeta^2} + q \left(\frac{d\tilde{\Psi}_1}{d\zeta} \right)^2 + s \zeta \frac{d\tilde{\Psi}_1}{d\zeta} = \beta^2 \exp(-\tilde{\Psi}_1 - \zeta^2), \quad (94)$$

in place of equation (25a). Here,

$$\tilde{\Psi}_1 \equiv k_1 \Psi_1,$$

$$k_1 \equiv -\gamma \left[J' + W(rg)^2 (G^2/2)' \right] / c_s^2,$$

$$\zeta \equiv z/h,$$

$$h \equiv c_s (2\gamma^2 - 1)^{-1/2} (\partial \Phi_g / \partial z^2)^{-1/2},$$

$$q \equiv \left(\frac{c_s}{c} \right)^2 \frac{\gamma r^2 g^4 (G^2/2)'}{J' + W(rg)^2 (G^2/2)'},$$

$$s \equiv 2 \left(\frac{c_s}{c} \right)^2 \left(1 - \frac{1}{\gamma^2} \right) \left(1 - \frac{1}{2\gamma^2} \right)^{-1},$$

$$\beta^2 \equiv 4\pi\rho(r, 0) r^2 h^2 g^4 \gamma^3 \left[J' + W(rg)^2 (G^2/2)' \right]^2 / c_s^2.$$

The density of the disk is $\rho(r, z) = \rho(r, 0) \exp(-\tilde{\Psi}_1 - \zeta^2)$. The new terms in equation (94) proportional to q and s arise from the space charge density contribution $(2r^2)^{-1}[\nabla(r^4 G^2/c^2)] \cdot \nabla \Psi$ in equation (88). Because $q, s = O[(c_s/c)^2] \ll 1$, we neglect these terms in the subsequent analysis. Equation (94) then has the same form as equation (25a). As in § III, the dimensionless parameter β provides a local measure of the strength of the magnetic field. Of course, in the nonrelativistic limit we recover the expression for β of § IIIa.

The solution of equation (94) is given by equation (28). Thus, the vertical $(1/e)$ half-thickness of the disk density has the dependence on β given by equation (28c) and shown in Figure 2. For $\beta^2 \ll 1$, the disk thickness takes on its nonmagnetic value $\Delta z_{1/2} = h = c_s(2\gamma^2 - 1)^{-1/2}(\partial \Phi_g / \partial z^2)^{-1/2}$. For $\gamma \gg 1$, we have $h = c_s 2^{-1/2} \gamma^{-1}(\partial \Phi_g / \partial z^2)^{-1/2}$; the Lorentz factor dependence is explained by the fact that the vertical velocity spread in the “laboratory” reference frame is a factor of γ smaller than the spread, c_s , in the comoving reference frame. For $\beta \gg 1$, the disk thickness is reduced, $\Delta z_{1/2} \approx 1.534 h / \beta$, owing to the pinch effect of the magnetic field.

b) Radial Force Balance of Disk

With a rearrangement of terms and the above condition 5, the Grad-Shafranov equation (eq. [88]) is

$$\Delta^* \Psi = -\frac{4\pi r}{c} j_\phi(r, z) = \frac{\gamma^2}{2r^2} \left\{ \nabla \left[\frac{(rg)^4 G^2}{c^2} \right] \right\} \cdot \nabla \Psi - \frac{4\pi r}{c} j_{\phi d}, \quad (95a)$$

where

$$j_{\phi d}(r, z) \equiv \rho c r g^4 \gamma^2 [J' + W(rg)^2 (G^2/2)'] \quad (95b)$$

represents the part of the total azimuthal current-density, $j_\phi(r, z)$, which is proportional to the disk mass density.

Following the steps of § IIIb, we integrate equation (95a) over the range of z encompassing the disk matter to obtain the *radial force balance equation for the disk*:

$$\sigma W g^2 \frac{v_\phi^2}{r} = \frac{g^4}{\gamma} \frac{dP}{dr} + \sigma W \left(2 - \frac{1}{\gamma^2} \right) \frac{\partial \Phi_g(r, 0)}{\partial r} - \frac{1}{c\gamma^2} j_{\phi d}(r) B_z(r, 0), \quad (96a)$$

in place of equation (34b).

In the approximation where $g=1$, $v_\phi = rG(\Psi)$, and $W = \gamma$, we have

$$\sigma \gamma \frac{v_\phi^2}{r} = \frac{1}{\gamma} \frac{dP}{dr} + \sigma \left(2\gamma - \frac{1}{\gamma} \right) \frac{\partial \Phi_g}{\partial r} - \frac{1}{c\gamma^2} j_{\phi d}(r) B_z(r, 0). \quad (96b)$$

Here

$$\sigma(r) \equiv \int_{-\infty}^{\infty} dz \rho(r, z), \quad j_{\phi d}(r) \equiv \int_{-\infty}^{\infty} dz j_{\phi d}(r, z), \quad P \equiv [c_s(\Psi_0)]^2 \sigma(r),$$

with $\Psi_0 \equiv \Psi(r, 0)$. In the order given, the different terms in equation (96) represent the “centrifugal force,” the pressure force, the gravitational force, and the electromagnetic force. Equation (96) neglects the contribution from the first term on the right-hand side of equation (95a), which arises from the volume and surface distributed space charge density. The prefactor of γ^{-2} of the electromagnetic force arises from the fact that this term combines the magnetic and electric forces, which nearly cancel.

In place of the Bennett pinch relation (56), we now have

$$j_{\phi d}(r) = [\gamma c^2 c_s^2 \sigma(r) I(\beta) / (4\pi h)]^{1/2}, \quad (97)$$

where $I(\beta)$ is defined by equation (57). Thus, for $\beta \gg 1$, we have

$$\Delta z_{1/2} \approx 0.345 [\gamma \sigma(r) c_s^2 c^2] [j_{\phi d}(r)]^{-2} \ll h, \quad (98)$$

in place of equation (59).

c) Specific Disk Equilibria

This subsection applies the results of § Vb to disk equilibria around an aligned, rotating magnetized star. For simplicity, we assume that the pressure force in the disk is negligible and that the gravity is weak. Thus, the basic equations are (95a) with $g=1$, and equation (96b) with $P=0$.

The determination of $\Psi(r, z)$ from equations (95a) and (96b) is inherently more complicated than in the nonrelativistic problem treated in § III. This is because of the first term on the right-hand side of equation (95a) which arises from the space charge density. Here we consider an iteration method for solving equation (95a) wherein we first solve for $\Psi^{(0)}$, where

$$\Delta^* \Psi^{(0)} = - \frac{4\pi r}{c} j_{\phi d}^{(0)}(r) \delta(z), \quad (99)$$

and where $j_{\phi d}^{(0)}(r)$ satisfies equation (96b) with $B_z^{(0)} = r^{-1} \partial \Psi^{(0)} / \partial r$. In subsequent iterations, $n=1, 2, \dots$,

$$\Delta^* \Psi^{(n+1)} = \left[\frac{\gamma^2}{2r^2} \nabla \left(\frac{r^4 G^2}{c^2} \right) \right]^{(n)} \cdot \nabla \Psi^{(n)} - \frac{4\pi r}{c} j_{\phi d}^{(n)}(r) \delta(z), \quad (100)$$

where $G^{(n)} = G[\Psi^{(n)}]$ and $\gamma^{(n)} = \gamma[G^{(n)}]$. The dimensionless quantity, which must be small compared with unity in order for this iteration to converge rapidly, is the ratio of the magnetic field energy to the magnitude of the gravitational binding energy of the disk. In the present work, we assume that this ratio is small. Therefore, only equation (99) is used to obtain an approximation to the flux function. Hence, the zero superscripts are dropped. Subsequently, we return to discuss the qualitative influence of the space charge term in § Vd.

By analogy with equation (31), the Green's function solution of equation (99) for $\Psi(r) = \Psi(r, z=0)$ is

$$\Psi(r) = \Psi_*(r) + \Psi_b(r) + \frac{2\pi}{c} \int_{r_*}^{\infty} dr' g(r, r') j_{\phi d}(r'). \quad (101)$$

Here g is given by equation (31b) with $z=0$; $\Psi_*(r) = m_*/r$ is the flux function of the star (due to its internal currents), m_* its magnetic moment, and r_* its radius; and $\Psi_b(r) = m_b/r$ is introduced to account for the boundary condition (§ II) that the magnetic flux due to the currents in the disk does not penetrate the star. The constant m_b is evidently given by

$$m_b = - \frac{2\pi r_*}{c} \int_{r_*}^{\infty} dr' g(r_*, r') j_{\phi d}(r'). \quad (102)$$

The radial force balance equation (96b) is rewritten as

$$\sigma(r) = \frac{(cr\gamma^3)^{-1} j_{\phi d}(r) B_z(r, 0)}{(2 - 1/\gamma^2)(GM_*/r^3) - \Omega^2(\Psi)}. \quad (103)$$

Here, the gravitational potential of the star is $\Phi_g = -GM_*/r$; the more familiar notation $\Omega(\Psi) = G(\Psi)$ is adopted for the angular rotation rate of the disk matter; and $\gamma = [1 - (r\Omega/c)^2]^{-1/2}$.

The field lines which pass through the star define what we refer to as the “private” flux surfaces of the star. For these Ψ -surfaces, $\Omega(\Psi) = \Omega_*$, which is the angular rotation rate of the star. For the other, nonprivate flux surfaces, $\Omega = \Omega(\Psi)$ as in § III.

There is evidently a large degree of arbitrariness in the solutions to equations (101)–(103). However, some *basic physical constraints* on the solutions follow by inspection of equations (101)–(103): First, note that for $\Omega = \Omega_*$, the vanishing of the denominator of equation (103) corresponds to an exact balance of the centrifugal and gravitational forces acting on a mass element corotating with the star. The balance occurs at the “centrifugal radius,”

$$r_c \equiv \left\{ \frac{GM_*}{\Omega_*^2} \left[1 + \left(\frac{r_c \Omega_*}{c} \right)^2 \right] \right\}^{1/3}. \quad (104)$$

At the centrifugal radius, we have either (1) $B_z(r_c, 0) = 0$ (an O or an X -point of the magnetic field at r_c) or (2) $j_{\phi d}(r_c) = 0$. Second, note that the private flux of the star cannot penetrate the disk beyond the “light-cylinder radius,”

$$r_l \equiv \frac{c}{\Omega_*}. \quad (105)$$

This is simply due to the fact that the disk matter on the private flux corotates with the star with azimuthal velocity $\Omega_* r$. Close to the star, the stellar magnetic field may be assumed dominant, so that the private field lines of the star loop through the disk. With increasing r , the looping of the private field lines through the disk must cease at some distance inside the light cylinder. Therefore, we conclude that there must be at least one X -point in the equatorial plane at a distance $r = r_x < r_l$.

For typical conditions we have

$$r_* \ll r_c \ll r_l. \quad (106)$$

Thus, to a good approximation the gravitational force is $-GM_*/r^2$ (or it is negligible) and $r_c = [GM_*/\Omega_*^2]^{1/3}$ (Michel and Dessler 1981). We are interested in disk equilibria which extend from r_* to distances larger than r_l . Therefore, we can neglect Ψ_b in equation (101) under the weak limitation $B_{zd}(r_*, 0) \ll m_*(r_*)^{-3}$, where B_{zd} is the magnetic field due to the disk current.

In order to facilitate the numerical determination of $\Psi(r)$, we introduce dimensionless variables analogous to those of § IIIe. The basic length scale is taken to be r_c , and the basic flux scale is taken to be $\Psi_0 \equiv m_*/r_c$. The dimensionless variables are $\hat{r} = r/r_c$; $\hat{\Psi} = \Psi/\Psi_0$; $\hat{j}_{\phi d} = (r_c)^2 j_{\phi d}/(c\Psi_0)$; $\hat{g} = g/r_c$; $\hat{\Omega}^2 = (r_c)^3 \Omega^2/(GM_*)$; and $\hat{\sigma} = (r_c)^2 \sigma/M_d$, where M_d is the total disk mass.

In dimensionless variables, with overcarets implicit, equation (101) becomes

$$\Psi(r) = r^{-1} + 2\pi \int_{r_*}^{\infty} dr' g(r, r') j_{\phi d}(r'). \quad (107)$$

Equation (103) becomes

$$\sigma(r) = \frac{\xi_0 (r\gamma^3)^{-1} j_{\phi d}(r) B_z(r, 0)}{[r^{-3} - \Omega^2(\Psi)]}, \quad (108)$$

where

$$\xi_0 = \frac{\Psi_0^2}{GM_* M_d} = \frac{m_*^2 (\Omega_*)^{4/3}}{(GM_*)^{5/3} M_d}. \quad (109)$$

Following the approach of § IIIc, a radial virial equation can be obtained directly from equations (107) and (108). In dimensionless variables, it is

$$2W_K + W_g + W_{dd} + W_{d*} = 0. \quad (110)$$

Here

$$W_K \equiv \pi \int_{r_*}^{\infty} r dr \sigma \gamma (r\Omega)^2 > 0$$

is the kinetic energy of the disk matter;

$$W_g \equiv -2\pi \int_{r_*}^{\infty} r dr \sigma r \frac{\partial \Phi_g}{\partial r} < 0$$

is the gravitational potential energy of the disk matter;

$$W_{dd} \equiv 2\pi \xi_0 \int_{r_*}^{\infty} r dr \gamma^{-2} r j_{\phi d}(r) B_{zd}(r, 0) > 0$$

is the self-magnetic energy of the disk; and

$$W_{d*} \equiv -2\pi \xi_0 \int_{r_*}^{\infty} r dr \gamma^{-2} r j_{\phi d}(r) r^{-3} \geq 0,$$

is the magnetic interaction energy between the disk and the star. Also, $B_{zd} \equiv r^{-1}(\partial \Psi_d / \partial r)$, where Ψ_d is the flux function arising from the disk current. For $z = 0$, Ψ_d is given by the second term on the right-hand side of equation (107).

The total magnetic energy of the disk is

$$W_m = W_{dd} + W_{d*}. \quad (111)$$

From equation (110), we conclude that

$$\xi \equiv \frac{W_m}{|W_g|} \leq 1, \quad (112)$$

which is the analog of equation (38). Notice that the self-magnetic energy of the star, which exceeds W_m by a factor $\sim \xi_0 (r_c/r_*)^3 \gg 1$, does not enter in equation (110).

In order to generate illustrative solutions to equations (107) and (108) numerically, we consider the simplest possible field configuration which is consistent with the above-mentioned physical constraints, and which extends from r_* to distances well beyond r_l . Specifically, we consider the current density profile:

$$j_{\phi d}(r) = K(r-1) \exp(-r/\lambda), \quad (113)$$

where K and λ are dimensionless constants. The choice of $j_{\phi d}(r=1) = 0$ means that an O or an X -point need not be located at $r=1$. Instead, we have, under the conditions discussed below, an X -point at $r_x (\neq 1)$ and an O -point at $r_{oh} > r_x$. Both points are in the equatorial plane. The field configuration can be denoted XO following the convention of § IIIe.

Rather than use K and λ directly to characterize the disks, we use Φ_d and λ , where Φ_d is the (reduced) total flux of the disk,

$$\Phi_d \equiv \max_r [\Psi_d(r, 0)]. \quad (114)$$

Thus, the total disk flux is $2\pi\Phi_d$. Owing to the simple functional form of $j_{\phi d}(r)$ of equation (113), we have $\Phi_d \sim K\lambda^3$ for $\lambda \gg 1$. Thus, equation (107) can be used to evaluate $\Psi(r)$ for given values of Φ_d and λ .

The range of values of the parameters (Φ_d, λ) is restricted to the hatched region of Figure 13, owing to a number of considerations: Note that, for $r \ll 1$, we have a dipole field, so that $\Psi(r) \sim 1/r$. With increasing r , Ψ decreases to a minimum at r_x , which is the X -point. For larger r , Ψ increases to a maximum at r_{oh} , which is the O -point. For even larger r , $\Psi(r)$ decreases to zero. The line denoted A in Figure 13 corresponds to the condition $\Psi(r_{oh}) \geq \Psi(r_x)$, which is simply a requirement for the existence of the XO field configuration. For large λ , this inequality is roughly $\Phi_d \geq 1/\lambda$, which is equivalent to the requirement that the magnetic moment of the disk, $m_d \sim \Phi_d \lambda$, exceed the magnetic moment of the star.

The line denoted B in Figure 13 corresponds to the above-mentioned physical constraint $r_x < r_l$. For large λ , this inequality is roughly $\Phi_d > \lambda^2/r_l^3$. Clearly, the combination of lines A and B implies a minimum Φ_d value,

$$\min(\Phi_d) \sim r_l^{-1}, \quad (115)$$

which occurs at $\lambda \sim r_l$. In terms of the physical variables, equation (115) is $\min(\Phi_d) \sim m_*/r_l$.

The line denoted C in Figure 13 corresponds to the condition $r_x > 1$. For large λ , this corresponds roughly to $\Phi_d < \lambda^2$. We introduce this inequality as a natural limitation on the family of disk equilibria described by equation (113).

For given (Φ_d, λ) , as limited by Figure 13, a disk equilibrium can be calculated following the method used in § IIIe. First, equation (107) is used to calculate $\Psi(r)$, and thus to determine r_x and r_{oh} . Then, in order to obtain $\sigma(r)$ from equation (108), we need $\Omega(\Psi)$. For $r < r_x$, the disk corotates with the star, so that $\Omega = 1$. For $r > r_x$, we consider the same dependence of Ω on Ψ as

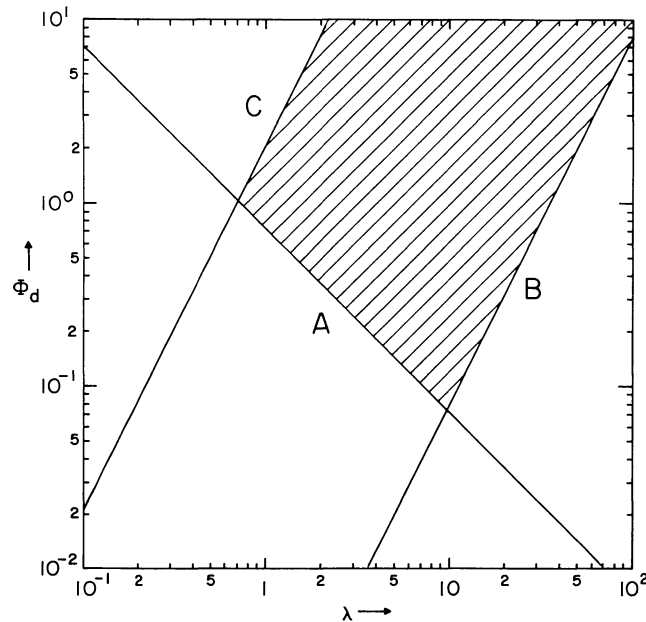


FIG. 13.—The hatched region represents the range of allowed values of the length scale λ of the current density of eq. (113), and of the reduced magnetic flux of the disk, Φ_d , defined in eq. (114). The lines A , B , and C represent restrictions imposed by different considerations discussed in the text.

in § IIIe. That is,

$$\Omega^2 = \begin{cases} 1, & r < r_x, \\ \omega^2 |\Psi|^n, & r > r_x, \end{cases} \quad (116)$$

where ω and n are dimensionless constants. Notice that in equation (108) both the numerator and the denominator vanish at $r=1$, giving a finite $\sigma(1)$ value. At $r=r_{oh}$, the numerator of equation (108) vanishes, and, therefore, the denominator must also vanish as in § IIIe. The vanishing of the denominator implies

$$\omega^2 = |\Psi(r_{oh})|^{-n} (r_{oh})^{-3}. \quad (117)$$

For given Φ_d , λ , and n , we can now derive $\sigma(r)$.

The normalization of σ requires $2\pi r dr \sigma = 1$. In turn, this implies the value of ξ_0 in equation (109). For $\lambda \gg 1$, it can be readily shown that $\xi_0 \sim \Phi_d^{-2}$. Condition (115) is thus equivalent to $\xi_0 \leq r_l^2$. Combining equations (109) and (115) yields

$$\frac{\Phi_d}{\min(\Phi_d)} \sim \frac{(GM_* M_d)^{1/2}}{|m_* \Omega_* / c|}. \quad (118a)$$

Using values pertinent to a neutron star, we find

$$\frac{\Phi_d}{\min(\Phi_d)} \sim 2.5 \times 10^6 \left(\frac{M_*}{M_\odot} \right)^{1/2} \left(\frac{M_d}{10^{-6} M_\odot} \right)^{1/2} \left(\frac{P_*}{1s} \right) \left(\frac{10^{30} \text{ G cm}^3}{m_*} \right), \quad (118b)$$

where $P_* = 2\pi/\Omega_*$ is the period of the star.

For the numerical calculation of a disk equilibrium, we evidently need to specify Φ_d , λ , r_l , r_* , and n . To simplify the discussion, we fix $r_l = 10$, $r_* = 0.1$, and $n = 3$. This relatively “small” value of r_l is chosen in order to reduce the time required in the numerical calculations. The small but finite value of r_* facilitates the evaluation of the energies in the virial equation (110), but it has no noticeable influence on the disk equilibria. [Notice that for $r \ll 1$, we have $\sigma(r) \sim \xi_0 K/r$, so that W_g has a contribution $\sim -2\pi\xi_0 K \ln(1/r_*)$ from the lower limit of the r integration. This is canceled in the virial equation by a lower r limit contribution to W_{d*} which is of the order of $+2\pi\xi_0 K \ln(1/r_*)$.] Finally, we have chosen $n = 3$, which is the same as the value of the analogous parameter used in § IIIe.

We have numerically generated disk equilibria for a sequence of values of λ for $\Phi_d = 1, 10, 10^2$. With the radial grids used (500 grid points), the virial equation (110) is typically satisfied to one part in 10^4 or better. Values of $\Phi_d \sim 10^6$ and $\lambda \sim 10^4$, suggested by equation (118b), appear to require an alternative numerical approach.

Figure 14 shows the dependence of some of the main physical quantities on $(\lambda^2/\Phi_d)^{1/3}$. For large λ , the variable $(\lambda^2/\Phi_d)^{1/3}$ is useful in that it is restricted to the range from 1 to r_l independent of Φ_d .

Figure 15 shows the radial, equatorial profiles of the basic parameters for a specific disk equilibrium. Notice that there is a narrow “gap” in the surface mass density at $r = r_x$, and that the angular rotation rate of the disk jumps from its corotation value of unity for $r = r_x - 0$ to a much lower value at $r = r_x + 0$. [Disk equilibria without such a gap in $\sigma(r)$ can be generated; however, they require a singular behavior of the current density, $j_{\phi d} \propto (r_x - r)^{-1/2}$, for r close to but less than r_x .]

The analysis of the stability of these disks, and the incorporation of the influence of stability requirements, is deferred to a future work. However, some qualitative points are noted here: For radial distances between r_x and r_{oh} , the disks may suffer from the axisymmetric instability discussed in § III d. In the region of the X -point we expect to have magnetic field line reconnection with the associated instability and energy conversion processes (see, for example, Priest 1981). Also, for radial distances in the vicinity of r_x , the Kelvin-Helmholtz instability may be important (see, in this regard, Ghosh and Lamb 1979a). The Kelvin-Helmholtz instability will be in the hypersonic limit (Roy Choudhury and Lovelace 1984); however, the influence of the instability may be small because of the small mass density in the region of the gap.

d) Further Aspects of the Speed-of-Light Cylinder

Figure 16 shows the field lines in the poloidal plane for the same disk equilibrium as in Figure 15. Notice that this field-line configuration involves an inconsistency with the full Grad-Shafranov equation (95a) because of the first term on the right-hand side of the equation, which is due to the space charge density. As mentioned below equation (88), the requirement that the plasma flow speed be less than the speed of light is $(g^2 r G/c)^2 < 1$. This space charge contribution acts to prevent the private field lines (or flux surfaces) of the star from crossing the speed-of-light cylinder. Thus, in a complete solution $\Psi(r, z)$, obtained from, say, equation (100), the private field lines of the star which cross the speed-of-light cylinder in Figure 16 would be “forced” to enter the X -point. In the absence of field-aligned currents ($H = 0$), no asymptotic ($z \rightarrow \infty$) solutions are possible with a finite amount of the private flux remaining within the light cylinder. With field-aligned currents included, and for, say, $H(\Psi) = K\Psi$, there are for

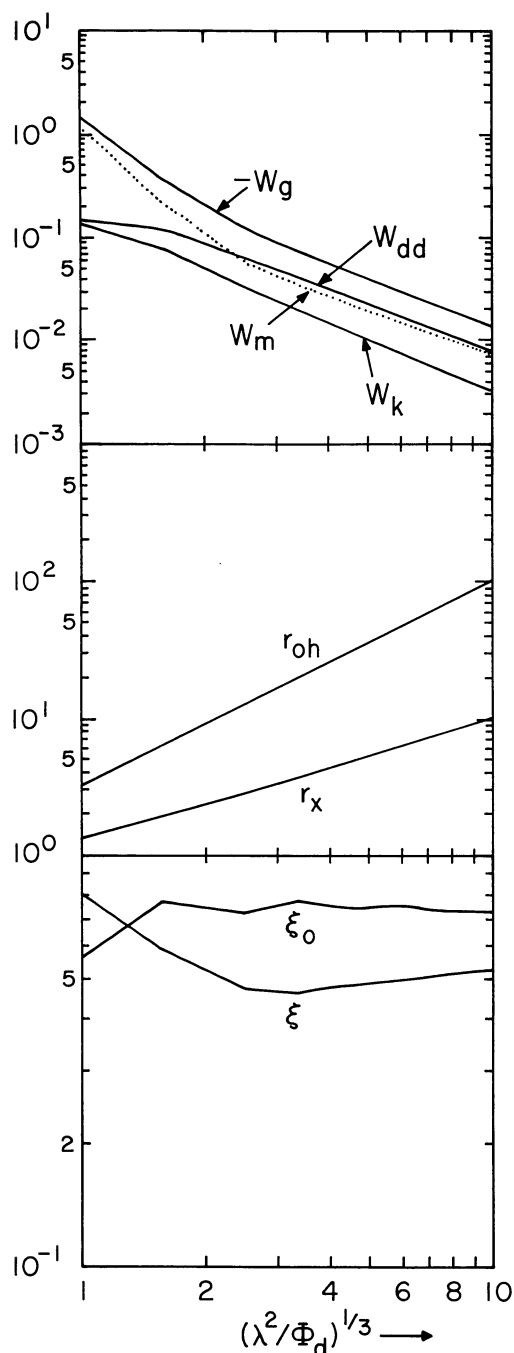


FIG. 14

FIG. 14.—Dependence of various physical quantities of the disk equilibria characterized by eq. (113) on the scaled variable $(\lambda^2/\Phi_d)^{1/3}$ for $\Phi_d = 1$. The top panel of the figure shows the total gravitational ($-W_g$), kinetic (W_k), magnetic (W_m), and self-magnetic (W_{dd}) energies measured in units of $GM_\star M_d/r_c = (GM_\star \Omega_\star)^{2/3} M_d$. The middle panel of the figure shows the radius of the X-point, r_x , and the radius of the O-point, r_{oh} . The dependences are approximately fitted by $r_x \propto \lambda^{0.56}$ and $r_{oh} \propto \lambda$. The bottom panel of the figure shows $\xi_0 \equiv \Psi_0^2/(GM_\star M_d)$ and $\xi \equiv |W_m/W_g|$. The virial equation implies that $\xi < 1$.

FIG. 15.—Radial, equatorial profiles of the total flux function (Ψ), the disk current density ($j_{\phi d}$), the disk surface mass density multiplied by r ($r\sigma$), and the angular rotation rate of the disk matter (Ω) for a disk equilibrium characterized by $\Phi_d = 1$ and $\lambda = 2$. All of the scales correspond to the dimensionless variables of § Vc except for $j_{\phi d}$, which has been multiplied by 92.73. The disk matter corotates with the star out to the radius of the X-point, $r_x = 1.97$. Just interior to the X-point, the azimuthal velocity of the disk matter is $0.197c$. Just exterior to the X-point, the azimuthal velocity is $6.83 \times 10^{-3}c$. The O-point of the magnetic field is at $r_{oh} = 6.32$, where the azimuthal velocity has its Keplerian value, $3.98 \times 10^{-2}c$. The fraction of the total disk mass inside r_x is 6.59×10^{-2} .

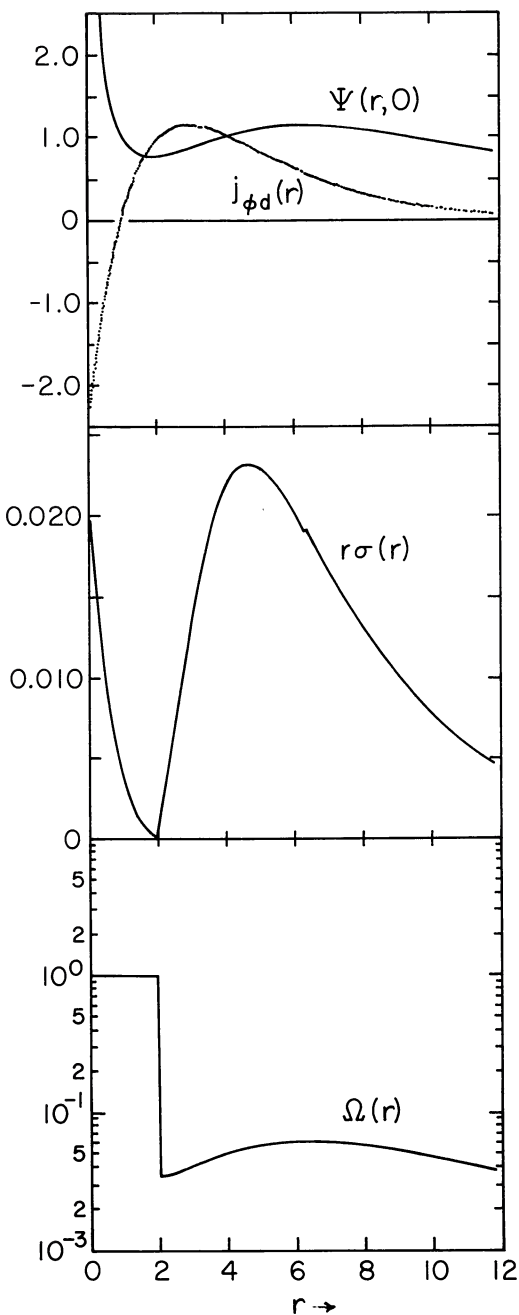


FIG. 15

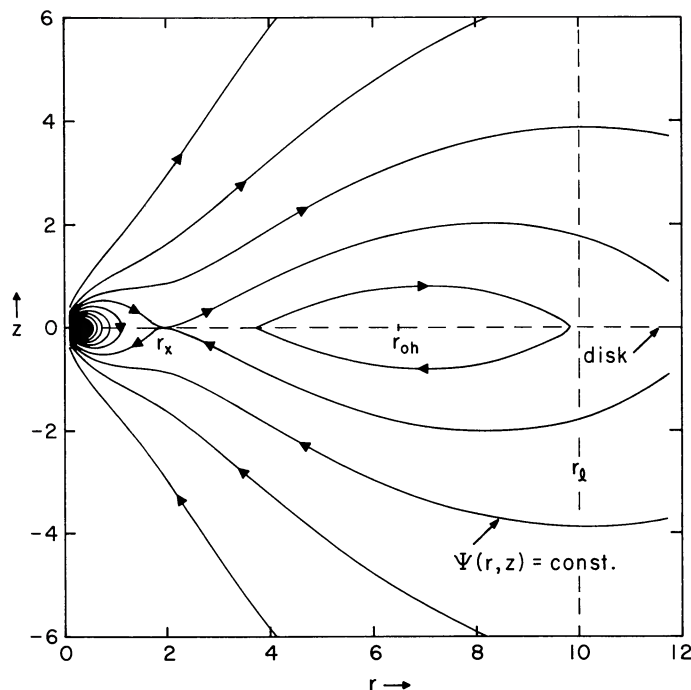


FIG. 16.—Flux surfaces for the same disk equilibrium as in Fig. 15. The difference in the values of Ψ between successive contours is $0.2\Psi_0$, where $\Psi_0 \equiv m_*/r_c$.

$K > 2/r_l$ asymptotic and global solutions to the full Grad-Shafranov equation (86) with a finite amount of the private flux remaining within the light cylinder as $z \rightarrow \infty$ (Sulkanen and Lovelace 1986; Lovelace, Wang, and Sulkanen 1986).

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APPENDIX A

EQUILIBRIA WITH ANISOTROPIC PRESSURE

Here, we obtain the Grad-Shafranov equation including nonrelativistic flow, gravity, and an anisotropic pressure. This case is evidently of interest for low-density, collisionless plasmas. The only change in the basic equations (1)–(6) of § II is in the Euler equation, which we now write as

$$\rho(\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \cdot \mathbf{p} - \rho \nabla \Phi_g + \frac{1}{c} \mathbf{J}_e \times \mathbf{B}, \quad (\text{A1})$$

where

$$\mathbf{p} = p_\perp \mathbf{1} + (p_\parallel - p_\perp) \mathbf{b}\mathbf{b} \quad (\text{A2})$$

is the plasma pressure tensor, with $p_\parallel$ the pressure parallel to the magnetic field, $p_\perp$ the pressure perpendicular to the field, and $\mathbf{b} \equiv \mathbf{B}/|\mathbf{B}|$. The form of the pressure (A2) corresponds to a distribution of particle velocities—as seen by an observer moving with the flow velocity $\mathbf{v}$ —which is isotropic in the two dimensions perpendicular to $\mathbf{B}$. This restricted isotropy arises from the gyration of the particles about the magnetic field. In equation (A1) we have

$$\nabla \cdot \mathbf{p} = \nabla p_\perp + \mathbf{B}(\mathbf{B} \cdot \nabla) \left(\frac{p_\parallel - p_\perp}{B^2} \right) + \left(\frac{p_\parallel - p_\perp}{B^2} \right) (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (\text{A3})$$

where $B \equiv |\mathbf{B}|$ (see, for example, Miyamoto 1976, p. 160). Because of the axisymmetry, we have, for example, $(\mathbf{B} \cdot \nabla) B^2 = (\mathbf{B}_\phi \cdot \nabla) B^2$.

The results of § II for the form of the flow velocity and the electrostatic potential, equations (8)–(14), remain unchanged. In place of equation (15), the toroidal component of equation (A1) yields

$$rB_\phi\sigma - Frv_\phi = H(\Psi), \quad (\text{A4})$$

where

$$\sigma \equiv 1 - 4\pi(p_\parallel - p_\perp)B^{-2}. \quad (\text{A5})$$

In place of equations (16),

$$rB_\phi\sigma = \left(H + r^2FG\right)\left(1 - \frac{F^2}{4\pi\rho\sigma}\right)^{-1}, \quad rv_\phi = \left(r^2G + \frac{HF}{4\pi\rho\sigma}\right)\left(1 - \frac{F^2}{4\pi\rho\sigma}\right)^{-1}.$$

In place of equation (17), the component of equation (A1) parallel to $\mathbf{B}$ gives

$$\rho\mathbf{B}\cdot\nabla J = -(\mathbf{B}_p\cdot\nabla)p_\parallel + (p_\parallel - p_\perp)(\mathbf{B}_p\cdot\nabla)\ln B, \quad (\text{A6})$$

where

$$J \equiv v^2/2 + \Phi_g - rv_\phi G$$

appears in place of the $J(\Psi)$ of § II. In contrast with § II, equation (A6) indicates that J is *not* a function dependent only on Ψ .

The poloidal component of equation (A1) parallel to $\nabla\Psi$ can be simplified to give a generalization of equation (19),

$$\begin{aligned} & \left(1 - \frac{F^2}{4\pi\rho\sigma}\right)\Delta^*\Psi - \frac{F}{\sigma}\nabla\left(\frac{F}{4\pi\rho}\right)\cdot\nabla\Psi \\ &= -\frac{4\pi\rho r^2}{\sigma}\left(\frac{\partial J}{\partial\Psi} + rv_\phi\frac{\partial G}{\partial\Psi}\right) - \sigma^{-2}(H + rv_\phi F)\left(\frac{\partial H}{\partial\Psi} + rv_\phi\frac{\partial F}{\partial\Psi}\right) + \sigma^{-3}(H + rv_\phi F)^2\frac{\partial\sigma}{\partial\Psi} \\ & \quad - \frac{4\pi r^2}{\sigma}\frac{\partial p_\perp}{\partial\Psi} - \frac{1}{2}(1 - \sigma)r^2\frac{\partial B^2}{\partial\Psi}, \end{aligned} \quad (\text{A7})$$

where $\partial/\partial\Psi \equiv |\nabla\Psi|^{-2}(\nabla\Psi\cdot\nabla)$.

The solution of equation (A7) requires the specification of the functions F , G , and H , which depend only on Ψ , and, in addition, the functions $p_\parallel$ and $p_\perp$, which are not in general functions of Ψ alone. The determination of $p_\parallel$ and $p_\perp$ is discussed below in a simplified case.

For the limit of no poloidal flow, $F = 0$, and no toroidal magnetic field, $H = 0$, equation (A7) simplifies to

$$\Delta^*\Psi = -\frac{4\pi\rho r^2}{\sigma}\left(\frac{\partial J}{\partial\Psi} + rv_\phi\frac{\partial G}{\partial\Psi}\right) - \frac{4\pi r^2}{\sigma}\frac{\partial p_\perp}{\partial\Psi} - \frac{1}{2}(1 - \sigma)r^2\frac{\partial B_p^2}{\partial\Psi}. \quad (\text{A8})$$

A single particle orbit in the equilibrium (A8) can be written as

$$\mathbf{v}^\alpha = rG(\Psi)\hat{\boldsymbol{\phi}} + v_\parallel^\alpha\mathbf{b} + \mathbf{v}_\perp^\alpha,$$

where α labels the particle, and $\mathbf{v}_\perp^\alpha$ describes the gyration of the particle around the magnetic field with $\mathbf{v}_\perp^\alpha\cdot\mathbf{b} = 0$. The gyration radius is considered to be a “microscopic” length compared with the gross length scales of the equilibrium (see, for example, Longmire 1967). Therefore, in a reference frame rotating with angular rate $G(\Psi)$, the motion of a particle is characterized by two constants: its energy,

$$\begin{aligned} w_\alpha &= \frac{1}{2}m_\alpha(v_\parallel^\alpha)^2 + \mu_\alpha B + q_\alpha\Phi_e(\Psi) + m_\alpha\left\{\Phi_g(r, z) - \frac{1}{2}r^2[G(\Psi)]^2\right\} \\ &= \frac{1}{2}m_\alpha(v_\parallel^\alpha)^2 + V_\alpha(s, \mu) \end{aligned} \quad (\text{A9})$$

(with s the distance along a magnetic field line and V_α an effective potential), and its magnetic moment,

$$\mu_\alpha = \frac{1}{2} m_\alpha (v_\perp^\alpha)^2 / |B_p|. \quad (\text{A10})$$

A particle's gyrocenter remains on a given flux surface, so that the electrostatic potential term in (A9) is a constant. The term $-m_\alpha r^2 G^2/2$ in equation (A9) is evidently the centrifugal potential.

At the point of the minimum of $V_\alpha(s, \mu)$ as a function of s for a given μ , denoted $s_m = s_m(\mu)$, we let $f_{\alpha m}(w, \mu; \Psi) dw d\mu$ represent the number of particle guiding centers per unit volume moving in the $+\mathbf{b}$ direction in the phase-space element $dw d\mu$. An equal number of guiding centers is assumed to be moving in the $-\mathbf{b}$ direction. At any other point along the field line, the number density of guiding centers is proportional to $B(s)[w - V_\alpha(s, \mu)]^{-1/2}$ times $f_m dw d\mu$ (Longmire 1967). Thus,

$$p_\parallel = 4 \sum_\alpha \int_0^\infty d\mu \int_{w \geq V_\alpha(s, \mu)}^\infty dw f_{\alpha m}(w, \mu; \Psi) \frac{B(s)}{B(s_m)} [w - V_\alpha(s_m, \mu)]^{1/2} [w - V_\alpha(s, \mu)]^{1/2} \quad (\text{A11a})$$

and

$$p_\perp = 2 \sum_\alpha \int_0^\infty d\mu \int_{w \geq V_\alpha(s, \mu)}^\infty dw \mu f_{\alpha m}(w, \mu; \Psi) \frac{[B(s)]^2}{B(s_m)} \left[\frac{w - V_\alpha(s_m, \mu)}{w - V_\alpha(s, \mu)} \right]^{1/2}. \quad (\text{A11b})$$

The functions $f_{\alpha m} \geq 0$ are arbitrary except for the condition of charge neutrality,

$$0 = \sum_\alpha q_\alpha \int_0^\infty d\mu \int_{w \geq V_\alpha(s, \mu)}^\infty dw \frac{f_{\alpha m}}{B(s_m)} \left[\frac{w - V_\alpha(s_m, \mu)}{w - V_\alpha(s, \mu)} \right]^{1/2}. \quad (\text{A12})$$

Equations (A7) and (A11) represent a closed description of an equilibrium with flow, gravity, and an anisotropic pressure. The functions F , G , and H of Ψ must be specified, as well as the functions $f_{\alpha m}(w, \mu; \Psi)$ with $\alpha = \pm$.

APPENDIX B

AXISYMMETRIC STABILITY WITH $P \neq 0$

For a generalization of the stability condition of § III d to include the pressure, the pressure force must first be included in the equilibrium equation (37),

$$\delta M \left[\frac{\partial P}{\partial r} + \sigma g_r - \frac{1}{c} j_\phi(r) B_z(r, 0) \right] = 0, \quad (\text{B1})$$

where $g_r \equiv [\partial \Phi_g(r, 0)/\partial r] - l^2/r^3$ is the effective gravity and

$$P(r) \equiv \int_{-\infty}^\infty dz p(r, z).$$

Second, the buoyancy force acting on the displaced ringlet,

$$\delta M g_r \frac{\partial}{\partial r} \left\{ \ln \left[\frac{P/h}{(\sigma/h)^\gamma} \right] \right\}, \quad (\text{B2})$$

must be added to the left-hand side of inequality (39). The considerations leading to (B2) are analogous to those applied to the convective stability of stars (Schwarzschild 1958). Here, γ is the adiabatic index for the ringlet displacement, and $h = h(r)$ is the disk thickness. The contribution (B2) to inequality (39) is stabilizing, for example, for $g_r < 0$ and $\partial \ln [(P/h)(\sigma/h)^{-\gamma}]/\partial r > 0$.

REFERENCES

- | | |
|---|--|
| <p>Abramowicz, M. A., Calvani, M., and Nobili, L. 1980, <i>Ap. J.</i>, 242, 772.</p> <p>Abramowitz, M., and Stegun, I. A., 1964, <i>Handbook of Mathematical Functions</i> (Washington, DC: NBS), pp. 591–592.</p> <p>Aly, J. J. 1984, in <i>AIP Conf. Proc. 144</i> (New York: AIP), in press.</p> <p>Blandford, R. D. 1976, <i>M.N.R.A.S.</i>, 176, 465.</p> <p>Braginskii, S. I. 1965, in <i>Reviews of Plasma Physics</i>, ed. M. A. Leontovich (New York: Consultants Bureau), 1, 205.</p> <p>Chan, K. L., and Henriksen, R. N. 1980, <i>Ap. J.</i>, 241, 534.</p> <p>Chandrasekhar, S. 1956a, <i>Ap. J.</i>, 124, 232.</p> | <p>Chandrasekhar, S. 1956b, <i>Proc. Nat. Acad. Sci.</i>, 42, 1.</p> <p>———. 1961, <i>Hydrodynamic and Hydromagnetic Stability</i> (London: Oxford).</p> <p>Cohen, R. H., Coppi, B., and Treves, A. 1973, <i>Ap. J.</i>, 179, 269.</p> <p>Coppi, B., and Pegoraro, F. 1979, <i>Ann. Phys.</i>, 119, 1979.</p> <p>Coroniti, F. V. 1981, <i>Ap. J.</i>, 244, 587.</p> <p>Coroniti, F. V. 1984, in <i>IAU Symposium 111, Unstable Current Systems in Astrophysics</i>, ed. M. R. Kundu (Dordrecht: Reidel), p. 453.</p> |
|---|--|

- Dessler, A. J. (ed.) 1983, *Physics of the Jovian Magnetosphere* (Cambridge: Cambridge University Press).
- Eardley, D. M., and Lightman, A. P. 1975, *Ap. J.*, **200**, 187.
- Galeev, A. A., Rosner, R., and Vaiana, G. S. 1979, *Ap. J.*, **229**, 318.
- Ghosh, P., and Lamb, F. K. 1979a, *Ap. J.*, **232**, 259.
- . 1979b, *Ap. J.*, **234**, 296.
- Gisler, G., Norman, M., and Lovelace, R. 1986, in preparation.
- Grad, H., and Rubin, H. 1958, *International Atomic Energy Agency Conf. Proc. 31* (Geneva), p. 190.
- Grimm, R. C., Dewar, R. L., and Manickan, J., 1983, *J. Comput. Phys.*, **49**, 94.
- Hameiri, E. 1983, *Phys. Fluids*, **26**, 230.
- Heinemann, M., and Olbert, S. 1978, *J. Geophys. Res.*, **83**, 2457.
- Jackson, J. D. 1975, *Classical Electrodynamics* (New York: Wiley), p. 177.
- Krall, N. A., and Trivelpiece, A. W. 1973, *Principles of Plasma Physics* (New York: McGraw-Hill), p. 100.
- Kuo-Petravic, L. G., Petravic, M., and Roberts, K. V. 1974, *Phys. Rev. Letters*, **32**, 1019.
- . 1975, *Ap. J.*, **202**, 762.
- Lamb, F. K. 1984, in *AIP Conf. Proc. 144* (New York: AIP), in press.
- Landau, L. D., and Lifshitz, E. M. 1959, *Fluid Mechanics* (London: Pergamon).
- . 1960, *Electrodynamics of Continuous Media* (London: Pergamon), p. 134.
- . 1962, *Classical Theory of Fields* (Oxford: Pergamon), chaps. 10, 11.
- Longmire, C. L. 1967, *Elementary Plasma Physics* (New York: Interscience), chap. 3.
- Lovelace, R. V. E. 1976, *Nature*, **262**, 649.
- . 1980, *Plasma Phys.*, **22**, 977.
- Lovelace, R. V. E., and Ruchti, C. B. 1983, in *AIP Conf. Proc. 101, Positron-Electron Pairs in Astrophysics*, ed. M. L. Burns, A. K. Harding, and R. Ramaty (New York: AIP), p. 314.
- Lovelace, R. V. E., Wang, J. C. L., and Sulkanen, M. E. 1986, *Ap. J.*, submitted.
- Lüst, R., and Schlüter, A. 1954, *Zs. Ap.*, **34**, 263.
- Mestel, L. 1961, *M.N.R.A.S.*, **122**, 473.
- . 1968a, *M.N.R.A.S.*, **138**, 359.
- . 1968b, *M.N.R.A.S.*, **140**, 177.
- Mestel, L., Phillips, P., and Wang, Y. M. 1979, *M.N.R.A.S.*, **188**, 385.
- Mestel, L., and Wang, Y. M. 1979, *M.N.R.A.S.*, **188**, 799.
- Michel, F. C. 1973, *Ap. J. (Letters)*, **180**, L133.
- Michel, F. C., and Dessler, A. J. 1981, *Ap. J.*, **251**, 654.
- Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1970, *Gravitation* (San Francisco: Freeman), chap. 18.
- Miyamoto, K. 1976, *Plasma Physics for Nuclear Fusion* (Cambridge: MIT Press).
- Mobarry, C. M., and Lovelace, R. V. E. 1986, *Ap. J.*, in press.
- Paczyński, B., and Witta, P. S. 1980, *Astr. Ap.*, **88**, 23.
- Priest, E. R. 1981, in *Solar Flare Magnetohydrodynamics* (New York: Gordon & Breach), chap. 3.
- Roy Choudhury, S., and Lovelace, R. V. E. 1984, *Ap. J.*, **283**, 331.
- Sakimoto, P. J., and Coroniti, F. V. 1981, *Ap. J.*, **247**, 19.
- Scharlemann, E. T., and Wagoner, R. V. 1973, *Ap. J.*, **182**, 951.
- Schwarzschild, M. 1958, *Structure and Evolution of Stars* (Princeton: Princeton University Press), p. 44.
- Scott, H. A., and Lovelace, R. V. E. 1982, *Ap. J.*, **252**, 765.
- Schafraanov, V. D. 1966, in *Reviews of Plasma Physics*, ed. M. A. Leontovich (New York: Consultants Bureau), **2**, 103.
- Shapiro, S. L., and Teukolsky, S. A. 1983, *Black Holes, White Dwarfs, and Neutron Stars* (New York: Wiley), p. 428.
- Sloan, J. H., and Smarr, L. L. 1985, in *Numerical Astrophysics: A Festschrift in Honor of James R. Wilson*, ed. J. Centrella et al. (Portola Valley: Jones & Bartlett), p. 52.
- Sulkanen, M. E., and Lovelace, R. V. E. 1986, in preparation.
- Tsinganos, K. C. 1981, *Ap. J.*, **245**, 764.
- Whitham, G. B. 1974, *Linear and Nonlinear Waves* (New York: Wiley), p. 259.
- Woltjer, L. 1959, *Ap. J.*, **130**, 405.
- Zehrfeld, H. P., and Green, B. J. 1972, *Nucl. Fusion*, **12**, 569.

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