Astron. Astrophys. 167, 379–386 (1986)

ASTRONOMY AND ASTROPHYSICS

Critical orbits in the elliptic restricted three-body problem

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Received March 14, accepted June 12, 1986

Summary. The main purpose of this numerical investigation of quasi-periodic orbits in the elliptic restricted three-body problem is to establish the regions of stable and unstable motions for possible planetary orbits in double star systems. There are three types of stable motion for the third, massless, body which are interesting as regards this problem: the P-type (Planet-type) sorrounding both primary bodies, the S-type (Satellite-type) orbiting one of the primaries, and the L-type (Librator-type) librating around the Lagrangian equilibrium points L_4 or L_5 [which are stable only in cases where the mass ratio of the primaries $m_1/(m_1+m_2)$ is less than 0.04]. The problem in question depends on two parameters, namely the mass ratio of the primaries and the eccentricity of their orbit; therefore it is, for the moment, too complex to be studied as a whole. As the first approximation the mass ratio was fixed ($m_1/m_2=1$). In the numerical experiment thousands of orbits were integrated for at least hundreds of periods in the elliptic restricted three-body problem for different eccentricities of the primaries.

The results show a region of stability far away from the primaries; then, as one approaches them, within a certain distance stable and unstable regions are found close together. This limit is called the Upper Critical Orbit (UCO). The grey region of "chaotic" motion, chaotic in the sense of unpredictability, is limited by the Lower Critical Orbit (LCO), within which all the integrated orbits were found to be unstable. What has been established here is the dependence of the distance of the LCO and UCO to the barycentre as a function of eccentricity e of the primaries. A least squares parabolic fit for the LCO and the UCO to the discrete numerical results gives the following expressions:

$$\mathrm{UCO}=2.37+2.76\,e-1.04\,e^2,$$

and

$$\mathrm{LCO}=2.09+2.79\,e-2.07\,e^2.$$

These results can be interpreted in the following way. Outside the UCO all the orbits should be stable, if they initially had only low eccentricity planet-like orbits. We therefore predict that the region outside the UCO is the region of possible planetary motion in double stars in our model of equally massive primaries on eccentric orbits.

Key words: elliptic restricted problem – stability – quasi-periodic orbits

1. Introduction

The possible existence of planetary orbits in double star systems is a question that still has no satisfying answer. Some numerical work has been undertaken by Harrington (1975 and 1977) in the general three-body problem, and by Szebehely (1980) and Szebehely and McKenzie (1981) in the circular problem making use of the Jacobian constant. Moreover efforts have been made by Hadjidemetriou (1975) to determine planetary orbits where only one primary is massive and the two "planets" have comparable, but small masses. But there exists no detailed study of this question in the framework of the more realistic elliptic restricted problem for three bodies. As most double stars have strong eccentricities any results for the circular restricted problem, which in fact neglects these effects, cannot be adapted to the underlying problem.

The appropriate method to establish stable regions of possible planetary motion and unstable regions, as well as the separatix between them, is an analytical study of the stability of the periodic orbits (PO). This has been done in a series of papers by Hénon (1968, 1969) and Hénon and Guyot (1970) for the whole mass ratio range of the primaries for all the Strömgren families of periodic orbits in the circular restricted problem. For the elliptic problem there exist some approaches where the linear stability of selected periodic orbits is studied (Shelus and Kumar, 1970). Also a detailed investigation for different eccentricities and mass ratios has been carried out by Broucke (1969) to find families of POs and their linear stability. The important point for POs in the elliptic restricted problem is the "strong" criterion for symmetric POs as it is cited in this article (Broucke, 1969): "An orbit is periodic if it has two perpendicular crossings with the syzygy axis and if the crossings are at moments when the primaries are at an apse."

Since the time interval is always a multiple of π all the POs have periods of $2k\pi$. Being interested primarily in the boundaries of stability around the primaries for possible planetary orbits, Broucke's results cannot be used for this determination, because the integrated orbits which are periodic for any integer $k=1,2,3\ldots$ are not dense in the physical plane as they are in the circular problem. It would be necessary to go to even higher commensurabilities (corresponding to greater values of k) to establish this kind of limiting orbit separating stable and unstable POs in the physical plane. This study of higher commensurabilities is still under discussion (Erdi, 1985), and it will probably be one of the next steps in our theoretical study of stability zones in the elliptic restricted three-body problem.

Our method of finding stable and unstable regions for the motions of planets in binary systems in the model of the elliptic restricted three-body problem is a purely numerical one: as in general a stable PO in phase space is surrounded by a quasi PO, we look for quasi POs and then conclude that we are in the vicinity of a stable PO. This region will be denoted in what follows as a stable one. By integrating a great variety of such orbits, and separating the non stable orbits (i.e., escape orbits) from the stable ones (i.e., quasi PO) we have found in an earlier paper (Dvorak, 1984) certain stability zones depending on the eccentricity for specially selected initial conditions. In this paper a more general picture will be derived. To include all possible types of orbits in the elliptic restricted problem the POs should be examined following the Strömgren families a–m (Strömgren, 1935). Since the main reason for this work is to establish stability limits for planetary orbits in double stars it is sufficient to concentrate on three possible types of orbits which include only some of the Strömgren families, but which are more appropriate to the problem in question (Dvorak, 1982, 1984):

a) the P-type orbit – the Planet-type orbit, where the third body surrounds both primaries;

b) the S-type orbit – the Satellite-type orbit, where the third body is orbiting one of the primaries;

c) the L-type orbit – the Librator-type orbit, where the third body is librating around one of the triangular Lagrangian points.

In the paper of Szebehely (1980) three different types of planetary orbits in binaries are also distinguished: a satellite orbit, an inner planet orbit (both of them are incorporated into the S-type in this study) and an outer planet orbit (i.e., P-type). A detailed study concerning the S-type orbit is still in progress. The L-type is not of great interest for planets in double stars, because the condition for the mass ratio $m_1/(m_1+m_2)$ to be smaller than 0.04, for stable orbits around the triangular Lagrange points L_4 or L_5, is, in general, not fulfilled for binaries. Moreover there exist several studies concerning librating Trojan orbits in the circular and the elliptic restricted problem (e.g., Rabe, 1967; Erdi, 1978, 1981). The main goal of this study is to find for the P-type orbit the stable and unstable regions in the physical plane, and, correspondingly, the limiting (critical) orbit separating these stable and unstable regions. Due to the numerical method used, which will be described in more detail in the next section, we can just give lower and upper bounds for such limiting orbits. Because to establish such curves which depend on both parameters (mass ratio and eccentricity) seems for the moment too complex for a numerical study we decided to concentrate on one interesting specific case. The main goal for this theoretical study is the following one: To derive the dependence of the upper and lower limiting orbits on the eccentricity of the orbit of the binary, for the elliptic restricted problem, where the primaries have equal masses.

2. Numerical method

As numerical method we used the Lie-series method recently developed for the elliptic restricted three-body problem by Delva (1985). Many tests with other programs and theoretical investigations by Lichtenegger (1984) and Hanslmeier and Dvorak (1984) were carried out to ensure the accuracy and efficiency of our method in comparison with others (e.g., the well known n-body program by Schubart and Stumpff, 1966). To establish stable and unstable regions (with escape orbits) we first have to define what we understand by stability: "A stable orbit is defined as an orbit having elliptic orbital elements with an eccentricity smaller than 0.3 throughout the whole integration time of 500 periods of the primary bodies."

This can be regarded as a kind of numerical Laplace stability: "All solutions stay in bounded regions of the phase space and no collisions and no escapes of the bodies occur." The restriction that the eccentricity should be smaller than 0.3 is just a limit given by numerical experience. All the numerical experiments have shown that every orbit with an eccentricity above this value is finally subject to perturbations which throw the massless body out of the binary system with a hyperbolic velocity after one or more close approaches to one of the primary bodies. Another interesting result is the following: The perturbations act in a similar way on the eccentricities and the semimajor axes of the stable orbits as for the "escape" orbits during their "phases of stability". Thus no difference at all is recognized between a stable or unstable orbit before the escape. In the sense of the definition of stability given above an orbit with an eccentricity greater than 0.3 is not stable even with a large semimajor axis (e.g. 10 times greater than the semimajor axis of the primaries), which is in fact a stable one. The definition of stability used in this work enables us to find low eccentric orbits (with $e<0.3$) around the double star as possible candidates for planetary orbits.

Our model is the planar elliptic restricted three-body problem with equal masses for the primaries. At first representative initial conditions for the third body, a certain subset of the set of all initial conditions, have to be fixed. The first attempt (Dvorak, 1984) was to integrate orbits where the initial position of the third body was on the connecting line of the primaries, and the velocity directed perpendicularly to it. Different velocities correspond to different eccentricities and different semimajor axes of the osculating elements (defined as usual, where the two primary masses are thought to be replaced by one single mass in the barycentre with $m=m_1+m_2$). We fixed circular initial conditions of the planet with the argument that nearly circular orbits are the ones we might expect for planetary orbits around double stars (e.g., the planetary orbits in the solar system are almost circular).

As the primaries are in eccentric orbits two different starting positions were selected for them: the apoapsis and periapsis, being the initial conditions for the primaries to show the two extremes concerning stability or escape of the third body (Benest, 1984). To better establish the different regions of stability around the primaries the initial conditions of the planet were chosen in the following way:

position 1: on the line of apses
position 2: on a line inclined 45 degrees to it
position 3: on a line perpendicular to the line of apses
position 4: on a line inclined 135 degrees to it

as shown in Figs. 1 and 2. The velocities are always perpendicular to the actual position line and correspond to circular initial velocity. For the elliptic problem with nonequal primary masses 4 additional positions (inclined 180, 225, 270, and 315 degrees to the syzygy axis) should be considered. Consequently for the problem under study ($m_1=m_2$) it is necessary to test, 4 different initial positions for the massless body and 2 different positions for the primaries (apoapsis and periapsis). They will be denoted by A1 (position 1, apoapsis) to P4 (position 4, periapsis). We now fix a position and integrate orbits with different distances from the barycentre i.e., different initial semi-major axes), with an interval of 0.05 AU. For instance we integrated as a first approach for $e=0.0$ the sequence of orbits from 1.85 to 2.5 (see Fig. 1), for $e=0.5$ the orbits from 2.95 to 3.5 AU (Fig. 2). In this initial condition diagram the time interval of escape is marked with 1 to

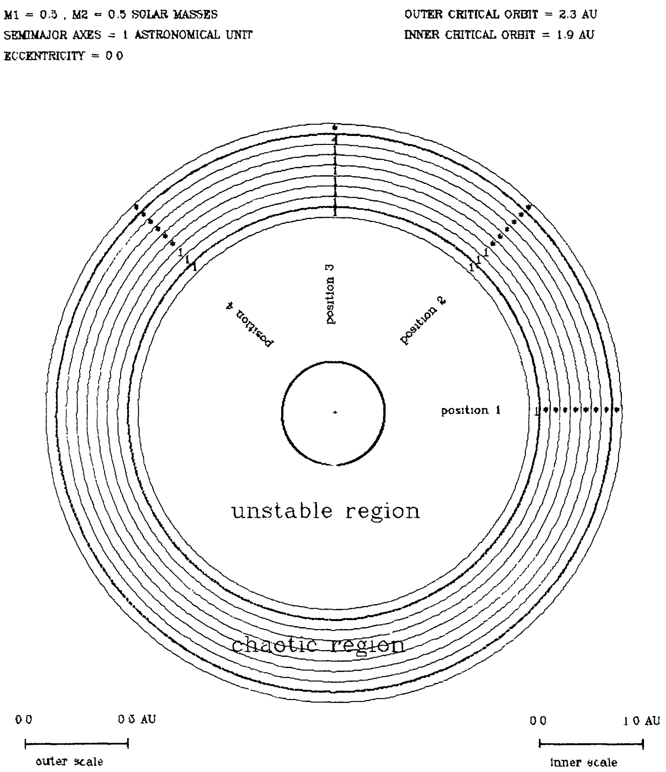

Fig. 1. Diagram of the initial conditions for planetary orbits in double stars. The 4 starting positions indicated show quite different stability behaviour. The inner circle (the two ellipses in Fig. 2 respectively) shows the orbit of the primaries. Between the outer concentric circles stable orbits are "*" and unstable orbits are numbers. These numbers 1 to 9 correspond to instability occurring after multiples of 50 revolutions of the primaries; e.g. position 1 number 7 means that this orbit was found to be unstable after 350 revolutions of the double star system. The two full circles (bold) mark the lower respectively upper critical orbit

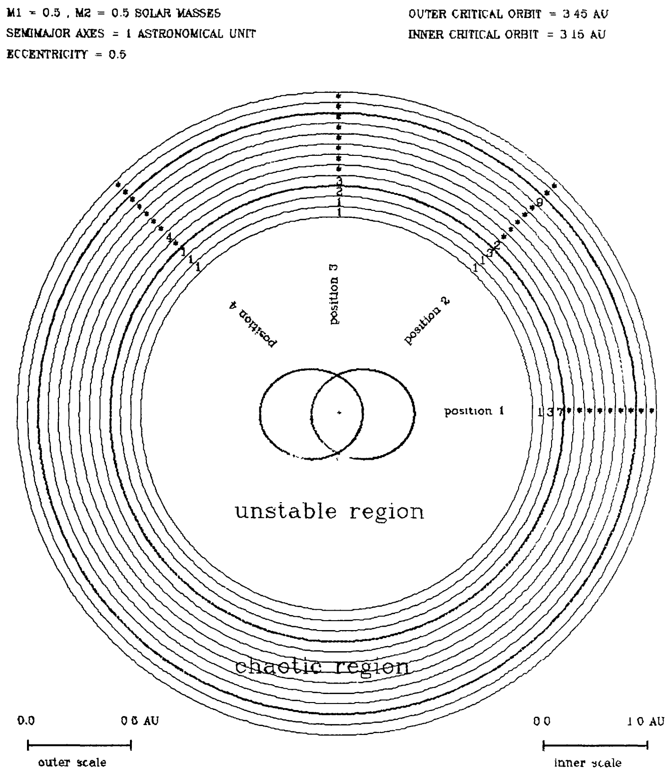

Fig. 2. See explanation Fig. 1 (for apoapsis)

10 corresponding to the number or revolutions of the primaries in units of 50 periods; the stable orbits correspond to "*" (in the figures) or "+" (in the tables). As one can see immediately, different positions give quite different results for the stability of the orbits (see Figs. 1 and 2).

As a next step the analysis should show the efficiency of the method concerning the dependence of stability on the integration time. In other words: After what time of integration can we be more or less sure to have found at least most of the escape orbits (e.g., 90 %)? As an example we have chosen the case $e = 0.5$ where there seemed to be a rather complex structure of the grey region lying in between the stable and unstable regions (Fig. 2), where one finds stable as well as unstable orbits. Our aim is to find their main features in the shortest possible integration time. The results of the test calculations are given in Tables 2 and 3. As one can notice even after 100 periods most of the structure is visible, but on the other hand it can be misleading: in P1 two unstable orbits are present, which have disappeared after 200 more revolutions. Also the LCO (lower critical orbit, defined as the largest orbit unstable in all 8 starting positions) and the UCO (upper critical orbit, defined as the orbit with the smallest semimajor axis stable in all 8 positions) would be wrongly estimated after 100 revolutions. 500 periods integration time for the primaries seem to give a rather good picture concerning stable and unstable regions, the numerical value of the UCO and LCO, and the existence of both islands and lakes.

Table 1. Stability of the initial conditions. The 4 different columns for $e = 0.0$ (and the 4 different ones for the apoapsis and the periapsis in the case $e = 0.5$) correspond to the different positions 1 to 4 in Figs. 1 and 2. Stable orbits are marked by "+", the number marking an unstable orbit corresponds to the total number of revolutions of the primaries, in units of 50, within which the orbit became unstable

distance	e = 0.0	distance	e = 0.5	
			apoapsis	periapsis
			+ + + +	+ + + +
		3.6	+ + + +	+ + + +
			+ + + +	+ + + +
2.5		3.5	+ + + +	+ + + +
	+ + + +		+ + + +	3 + + +
2.4	+ + + +	3.4	+ 9 + +	1 + + 7
	+ + + +		+ + + +	5 + + +
2.3	+ + + +	3.3	+ + + +	2 + + +
	+ + 4 +		+ + + +	3 + 4 +
2.2	+ + 1 +	3.2	+ + + +	1 + + +
	+ + 1 +		+ + + +	1 + + +
2.1	+ + 1 +	3.1	+ 2 3 4	1 7 + +
	+ + 1 +		7 3 2 1	1 2 [illegible] 2
2.0	+ 1 1 1	3.0	3 1 1 1	1 1 + 1
	+ 1 1 1		1 1 1 1	1 1 4 1
1.9	1 1 1 1	2.9	1 1 1 1	1 1 1 1
	1 1 1 1		1 1 1 1	1 1 1 1
1.8	1 1 1 1	2.8	1 1 1 1	1 1 1 1

Table 2. Numerical test of the method (for $e=0.5$ and periapsis). Columns 1 to 10 correspond to the number of revolutions of the primaries in units of 100. Stable orbits are marked by "+" and unstable ones by "o". For the 4 different positions of the third body the time scale of stability is indicated (e. g. orbit 3.4 in position P4 becomes unstable between 300 and 400 revolutions of the primaries)

POSITION P1											POSITION P2									
1	2	3	4	5	6	7	8	9	10	AU	1	2	3	4	5	6	7	8	9	10
+	+	+	+	+	+	+	+	+	+	3.7	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.6	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.5	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
o	o	o	o	o	o	o	o	o	o	3.4	+	+	+	+	+	+	+	+	+	o
+	+	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o	3.3	+	+	+	+	+	+	+	+	+	+
+	o	o	o	o	o	o	o	o	o		+	+	+	+	+	+	+	+	+	+
o	o	o	o	o	o	o	o	o	o	3.2	+	+	+	+	+	+	+	+	+	+
o	o	o	o	o	o	o	o	o	o		+	+	+	+	+	+	+	+	+	+
o	o	o	o	o	o	o	o	o	o	3.1	+	+	+	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o	3.0	o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o

POSITION P3											POSITION P4									
+	+	+	+	+	+	+	+	+	+	3.7	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.6	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.5	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.4	+	+	+	o	o	o	o	o	o	o
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.3	+	+	+	+	+	+	+	+	+	+
+	o	o	o	o	o	o	o	o	o		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.2	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.1	+	+	+	+	+	+	+	+	+	o
+	+	+	+	+	+	+	+	+	+		o	o	o	o	o	o	o	o	o	o
+	+	+	+	+	+	+	+	+	+	3.0	o	o	o	o	o	o	o	o	o	o
+	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o

Table 3. Numerical test of the method (for $e=0.5$ and apoapsis). For explanation see Table 2

POSITION A1											POSITION A2									
1	2	3	4	5	6	7	8	9	10	AU	1	2	3	4	5	6	7	8	9	10
+	+	+	+	+	+	+	+	+	+	3.7	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.6	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.5	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.4	+	+	+	+	o	o	o	o	o	o
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	o	o	o
+	+	+	+	+	+	+	+	+	+	3.3	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.2	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.1	o	o	o	o	o	o	o	o	o	o
+	+	+	o	o	o	o	o	o	o		+	o	o	o	o	o	o	o	o	o
+	o	o	o	o	o	o	o	o	o	3.0	o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o

POSITION A3											POSITION A4									
+	+	+	+	+	+	+	+	+	+	3.7	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.6	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.5	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.4	+	+	+	+	+	+	o	o	o	o
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	o	o	o
+	+	+	+	+	+	+	+	+	+	3.3	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	3.2	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+		+	+	+	+	+	+	+	+	+	+
+	o	o	o	o	o	o	o	o	o	3.1	+	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o	3.0	o	o	o	o	o	o	o	o	o	o
o	o	o	o	o	o	o	o	o	o		o	o	o	o	o	o	o	o	o	o

Figure 3 shows numerically the dependence of the escape orbits discovered with respect to the integrated time span.

An interesting feature is the appearance of two new excapes even after 1000 revolutions in P2 and P4. Therefore we can conclude that even 900 periods would not be sufficient. In fact during a long time integration over 5000 periods one orbit became unstable between 2500 and 3000 periods (the orbit 3.35 AU for the position P2, so that the lake at 3.4 turned out to be larger). Another point is the "certainty" of having found the nearest stable orbits in all 8 positions for defining the LCO respectively the most distant unstable orbits (UCO). For this reason with an interval of 0.05 AU 8 additional orbits outside the UCO and 8 inside the LCO were calculated. As no stable or no unstable orbits were found in these regions we thought it would be a reasonable compromise to limit ourselves in all the other computations to search for 4 stable (unstable) orbits for all 8 positions to define the UCO (LCO), to minimize computer time (only the stable orbits are cumbersome in this respect); but not all of these integrated orbits are shown in Tables 2 and 3. As an important consequence of these experiments the results presented here should be treated with caution because they are established purely numerically.

3. Results

As a result of our numerical experiments for different values of the eccentricity from 0.0 to 0.9, UCOs and LCOs were established, and between these a "grey" region was found. For an orbit within this region no conclusion at all can be drawn. Escape orbits are found as well as stable orbits, lakes of unstable orbits are found in larger regions of stable orbits, and, vice versa, islands of stability are found in the ergodic sea [a well known phenomenon according to many numerical experiments in dynamical systems, e.g., Hénon and Heiles (1964) and Contopoulos (1967)]. The "main land" of stable single periodic orbits (corresponding in our results to the quasiperiodic orbits), with initial conditions corresponding to circular orbits in the two body case, is outside this zone of chaos (see Fig. 1 and 2).

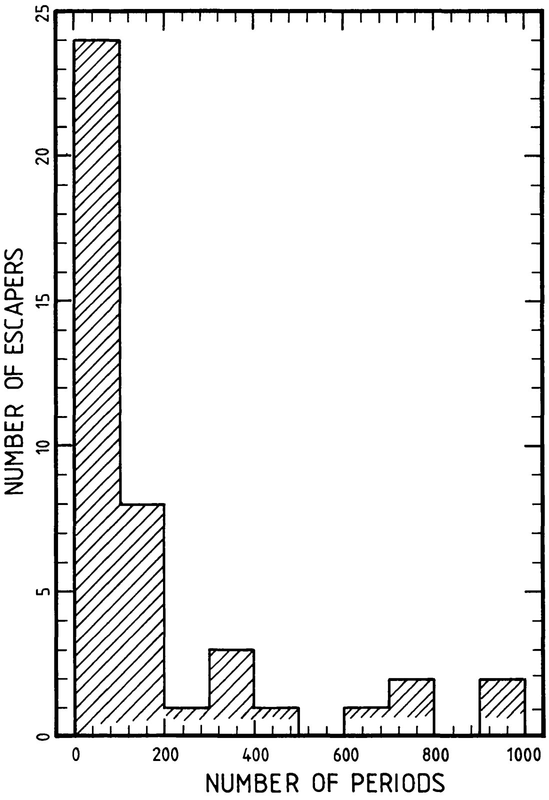

Fig. 3. A Nomogram showing the number of escape orbits (ordinate) in depending on the integrated times scales for the test case $e=0.5$ for all eight positions (Tables 2 and 3)

For $e=0.0$ we can compare our numerical results with the theoretical ones by Hénon and Guyot (1970). We found (Table 1) 1.9 as the LCO and 2.3 as the UCO. We note that our value found numerically for the LCO is very close to the analytically derived value for the Critical Periodic Orbit (CPO) of 1.91 AU. How can we now explain the existence of escape orbits in the region of stable PO? It is in fact not surprising that we have in the vicinity of the CPO [separatrix between stable and unstable POs as defined by Hénon and Guyot (1970)] orbits which do not lie on the torus around the stable PO in phasespace; only in sufficiently large distance from the CPO do we have orbits which lie on a torus around a stable PO. The UCO of 2.3 is close to the value found by Szebehely and McKenzie (1981) in the circular restricted problem using the Hill stability criterion. They found a critical value of 2.165 for the lower limit for the radius; the massless body will never come close to the primaries (for $m_1=m_2$).

As mentioned above for eccentric orbits of the primary bodies we have to distinguish for the starting positions of the planets between two different positions of the primaries: either the two massive bodies are at apoapsis or periapsis. The maximum and the minimum distances of the third body to the primaries, and the corresponding different velocities, lead to different forces acting on the planet. The results of the integration are presented in detail in Tables 1 and 4a–4c. Table 5 summarizes the main features concerning the width of the grey zone, the number of the unstable lakes, the area they cover, and also the location of the LCO and the UCO by the the corresponding initial position.

In general no conclusion can be drawn for the stability dependence either on the positions of the planet (1–4), or on the

Table 4. Stability of the initial conditions. For explanation see Table 1

e = 0 .1 apoapsis	e = 0 .1 periapsis	distance	e = 0.2 apoapsis	e = 0.2 periapsis
			+ + + +	+ + + +
		3.1	+ + + +	+ + + +
			+ + + +	+ + + +
		3.0	+ + + +	+ + + +
			+ 3 + +	+ + + +
		2.9	+ + + +	+ + + +
+ + + +	+ + + +		+ + + +	+ + + +
+ + + +	+ + + +	2.8	+ + + +	+ + + +
+ + + +	+ + + +		+ + 3 +	+ + + +
+ + + +	+ + + +	2.7	+ 2 2 1	1 + + 9
+ + 2 6	+ + + 6		4 1 1 1	1 2 1 1
+ + 1 2	2 1 3 4	2.6	3 1 1 1	1 1 1 1
5 5 1 2	1 6 2 2		1 1 1 1	1 1 1 1
2 1 2 2	2 1 2 1	2.5	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1			
1 1 1 1	1 1 1 1	2.4		

e = 0 .3 apoapsis	e = 0 .3 periapsis	distance	e = 0.4 apoapsis	e = 0.4 periapsis
			+ + + +	+ + + +
		3.3	+ + + +	+ + + +
+ + + +	+ + + +		+ + + +	+ + + +
+ + + +	+ + + +	3.2	+ + + +	+ + + +
+ + + +	+ + + +		2 + + +	+ + + +
+ + + +	+ + + +	3.1	+ + + +	+ + + +
+ + 2 +	2 + + +		+ 3 2 +	2 8 + +
+ 8 3 5	3 3 + 4	3.0	+ 2 1 1	1 2 + 4
8 5 2 1	+ 1 6 2		3 2 1 1	3 1 3 2
3 4 3 1	+ 2 2 2	2.9	1 2 1 1	1 1 3 1
+ 2 3 1	3 2 2 6		1 1 1 1	1 1 2 1
+ 3 2 2	1 3 2 8	2.8	2 1 1 1	1 1 1 1
+ 2 1 1	1 2 2 3			
3 1 1 1	1 1 2 1	2.7		
1 1 1 1	1 1 1 1			
1 1 1 1	1 1 1 1	2.6		
1 1 1 1	1 1 1 1			

position of the primaries at their apoapsis and periapsis. Whereas in one position for a fixed eccentricity the orbits are stable, the picture can be completely different for the same position but with another eccentricity. Nevertheless there seems to be a certain priority for the positions 1 and 3 to determine the CO. The LCO is determined in most of the cases through the position A1 (5 cases) and P3 (5), and the UCO through the positions P1 (4 cases), A3 (3) and P3 (3). All the other positions only define a CO once or twice. It should be mentioned that sometimes a CO can even be defined through 3 positions, e.g., the LCO for $e=0.2$ is defined through the last stable orbit in the positions A1, P2 and P3 at a distance of 2.7 AU from the barycentre. But in most cases, especially for higher eccentricities, the COs are defined through one single stable orbit (i.e., LCO) or one single unstable orbit (i.e., UCO), which is in almost all cases (except $e=0.0$, $e=0.1$ and $e=0.5$) an unstable lake.

Concerning the lakes it is evident that they become more and more numerous with increasing width of the grey zone and consequently their number in general increases with the eccentricity of the primaries. In a stability diagram of the initial

Table 4b. Stability of the initial conditions. For explanation see Table 1

e = 0.6		distance	e = 0.7	
apoapsis	periapsis		apoapsis	periapsis
			+ + + +	+ + + +
		4.0	+ + + +	+ + + +
			+ + + +	+ + + +
		3.9	+ + + +	+ + + +
			9 + + +	+ + + +
		3.8	9 + + +	+ + + +
			+ + + +	+ + + +
+ + + +	+ + + +	3.7	+ + + +	+ + + +
+ + + +	+ + + +		+ + + +	2 + + +
+ + + +	+ + + +	3.6	+ + + +	2 + + +
+ + + +	+ + + +		+ + + +	3 + + +
+ + + +	3 + + +	3.5	+ + + +	2 + + +
+ + + +	+ + + +		4 3 + 2	1 + + +
+ 4 + 2	1 + + +	3.4	+ 4 + 2	1 5 + 6
+ 4 7 9	1 4 + 5		+ + 3 1	1 5 + 5
+ + + +	2 + + +	3.3	+ + + +	1 3 + 7
+ + + +	1 + + +		4 + + 9	1 8 + +
+ 0 + 2	1 + 5 +	3.2	3 6 + 2	1 3 6 +
+ 8 + 8	1 + + +		2 4 4 7	1 2 2 7
2 4 2 1	1 4 + 3	3.1	2 7 3 2	1 3 3 3
1 2 1 1	1 2 + 2		3 1 1 2	1 2 5 3
2 1 1 1	1 1 + 1	3 0	1 1 1 1	2 1 ? 3
1 1 1 1	1 1 + 1		2 1 1 1	1 2 2 1
1 2 2 1	1 2 1 1	2.9	1 1 2 1	1 1 1 1
1 1 1 1	1 1 1 1		1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1	2.8	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1			

Table 4c. Stability of the initial conditions. For explanation see Table 1

e = 0.8		distance	e = 0.9	
apoapsis	periapsis		apoapsis	periapsis
+ + + +	+ + + +			
+ + + +	+ + + +	4.1	+ + + +	+ + + +
+ + + +	+ + + +		+ + + +	+ + + +
+ + + +	+ + + +	4.0	+ + + +	+ + + +
+ + + +	5 + + +		+ + + +	+ + + +
+ + + +	+ + + +	3.9	9 + + +	+ + + +
8 + + +	+ + + +		+ + + +	+ + + +
+ + + +	+ + + +	3.8	+ + + +	+ + + +
+ + + +	+ + + +		+ + + +	2 9 + 7
+ + + +	2 + + +	3.7	+ + + +	1 + + +
+ + + +	1 + + +		+ + + +	1 + + +
+ + + +	1 + + +	3.6	6 + + +	5 + + +
+ + + +	1 + + +		6 + + +	1 + + +
3 + + +	1 + + +	3.5	9 + + +	1 + + +
+ + + 2	1 + + +		+ 2 + 2	1 + + +
+ 3 + 3	1 3 + 3	3.4	+ 3 + 2	1 2 4 4
+ 2 2 3	1 5 + 4		+ 4 + 2	1 2 4 4
+ 8 5 2	1 3 + 3	3.3	1 2 2 3	1 2 2 2
+ 6 7 +	1 5 + 3		1 4 5 1	1 + 2 2
1 3 + 1	1 6 + 3	3.2	1 2 5 1	1 3 2 2
1 2 + 1	1 2 + 3		2 1 + 1	1 1 + 1
2 1 + 1	1 1 4 2	3.1	1 1 + 1	1 1 4 1
1 1 + 1	1 1 3 1		1 1 + 1	1 3 2 1
2 1 2 1	1 1 4 1	3.0	1 1 2 1	1 1 2 1
1 1 1 1	1 1 1 1		1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1	2.9	1 1 2 1	1 1 1 1
1 1 1 1	1 1 1 1		1 1 1 1	1 1 1 1

Table 5. Characteristics of the grey zone. For all the calculated orbits for different eccentricities of the primaries (column 1) the width of the grey "chaotic" region (column 2) and the number of the unstable lakes (column 3) in the "main land" of stable quasi periodic orbits are shown. In column 3 also the area of a lake is given in parentheses; e.g. 1 big lake can consist of 2 unstable orbits, like in Table 4b for A1 and $e = 0.7$ the orbits 3.8 and 3.85 were found to be unstable (area = 2). The last 2 columns show the largest orbit which is unstable in all 8 positions (LCO) and the nearest orbit which is stable in all 8 positions (UCO). In parentheses in these 2 columns we indicate the position which defines the critical orbit. All distances are measured from the barycentre

eccentricity	width of the grey region	number(area) of lakes	LCO	UCO
0.0	0.35	0	1.9 (1)	2.3 (3)
0.1	0.1	0	2.55 (A1,2)	2.7 (A3,4,P4)
0.2	0.3	1 (1)	2.65 (A1,P2,3)	3.0 (A2)
0.3	0.3	2 (4)	2.7 (A1)	3.1 (A3,P1)
0.4	0.4	1 (1)	2.95 (A1,P3)	3.2 (A1)
0.5	0.5	3 (3)	2.95 (P3)	3.5 (P1)
0.6	0.6	7 (9)	2.9 (P3)	3.55 (P1)
0.7	0.9	7 (16)	2.95 (P3)	3.9 (A1)
0.8	0.95	5 (10)	3.0 (A3)	4.0 (P1)
0.9	0.9	7 (17)	3.0 (A3)	3.95 (A1)

conditions with a smaller interval for the eccentricities of the primaries and the initial positions (i.e., semimajor axis) of the third body (e.g. $\Delta e = 0.02$ instead of $\Delta e = 0.1$ and $\Delta a = 0.01$ instead of $\Delta a = 0.05$ AU) it is highly probable that one would recognize the lakes appearing, growing, shrinking and disappearing (and similarly for the stable islands). These details cannot be seen in this determination of the LCO and UCO; for example, for the position P3 at the distance 3.25 from the barycentre for $e = 0.4$ the orbit is stable, for $e = 0.5$ it becomes unstable between 150 and 200 time steps, and for $e = 0.6$ the orbit is again stable. The lakes themselve's seem to appear more or less randomly in the grey zone. It is certain that some of them would disappear with longer integration times, as they would then be connected with the sea of unstability (which is equivalent to the disappearance of a stable island). But, as a detailed study of this phenomenon shows, new lakes and islands appear with longer integration time. There is no doubt that some of them are real, but at the moment the mechanism for protecting their stability is still under study. The whole phenomenon is well known in dynamical systems with more than two degrees of freedom when approaching the stability limit.

4. Conclusion

What can be concluded from about 2000 integrated orbits for at least 500 periods of the primaries? Concerning the P-type orbits in the elliptic restricted three-body problem there exist orbits which are little perturbed, if they are at least 4 units away from the barycentre of the primaries (with the relative semimajor axis taken as unity). Then coming closer to a critical limiting orbit (LCO and UCO, which depend on the eccentricity of the primaries) the perturbations acting on the third body become more and more important and lead to growing variations δa and δe. Then further decreasing the semimajor axis (initial distance from the barycenter) the grey zone between the COs is entered through the UCO and more and more excape orbits are found. From a certain other limiting critical orbit (LCO) all the integrated orbits with the corresponding initial positions smaller than the LCO are found to be escape orbits.

In Fig. 4 the dependence of the initial semimajor axis of the planet on the eccentricity of the primaries is shown. As already mentioned the LCOs can be regarded as close to the CPOs as defined by Hénon and Guyot (1970).

The UCO is a reasonable good stability limit for planetary orbits in double stars. Outside the UCO we expect stable quasi PO of planets with moderate eccentricities. We now recap on the escape process as it happens according to our numerical experiments: the third body becomes unstable after its eccentricity increases above 0.3. In this case a close approach to one of the primaries occurs, the perturbations still increase the eccentricity and after one or more close encounters with one of the primaries the massless body escapes with a hyperbolic velocity.

In Fig. 4 are also marked the initial positions of the primaries with respect to the barycentre and to the planets; further more the change of the corresponding distances (always for the one which is closer to the planet) with increasing eccentricity is evident from the 2 lower lines. The chaotic region is between the UCO and the LCO. The diagram extends up to $e = 0.9$, although some tests were made for $e = 0.95$. In real double stars such eccentricities have not yet been observed, and are certainly very rare (if they exist at all) and are only of theoretical interest. For the limiting case of $e = 1.0$ an analytical approach seems possible for primaries with equal masses and is still under study. The decrease of the fitted curve

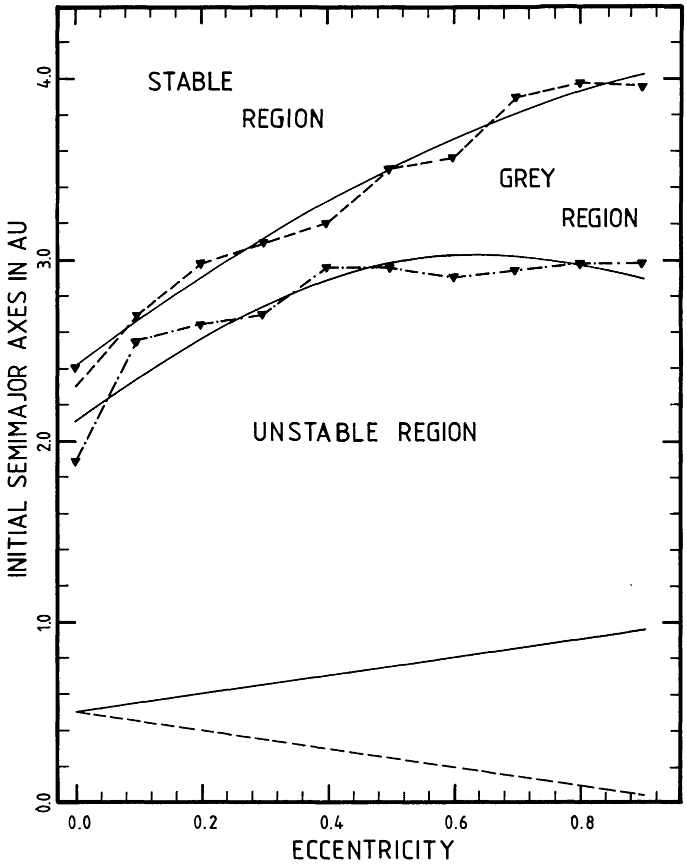

Fig. 4. Critical orbits depending on the eccentricities of the primaries. The triangles mark the UCO and the LCO due to the results of the numerical experiments (Table 5). The two parabolas represent the UCO and LCO given by a least square fit. The two lines at the bottom indicate the initial distance of the nearer primary for starting in the apoapsis (upper line) and in the periapsis (lower dashed line)

Table 6. Parabolic fit. Analytical expression for the upper critical orbit and the lower critical orbit (UCO and LCO) depending on the eccentricity of the binary. The result is a least square parabolic fit of the numerical data: $a_0 + a_1 \cdot e + a_2 \cdot e^2$

CO	a_0	+/-	a_1	+/-	a_2	+/-
LCO	2.09	0.30	2.79	0.53	-2.08	0.56
UCO	2.37	0.23	2.76	0.40	-1.04	0.43

(Fig. 4) of the LCO for higher eccentricities will not reflect the real behaviour of the LCO. It is an indication that close to the case $e = 1.0$ the parabola fit should not be used. Least square solutions of the LCO and the UCO yield the numerical values depending on the eccentricity using the ansatz $a_0 + a_1 * e + a_2 * e^2$ which is shown in Table 6.

It should be mentioned that a more rigorous study using the stability of POs could be undertaken only for selected POs for a fixed value of m_1/m_2 and e. Since the criterion of strong periodicity defined earlier (Broucke, 1969) is valid for the elliptic problem there exist no families of orbits for one specific problem, but only for varying values of e or m_1/m_2.

We finally emphasize again what we mean by stability or, more precisely, what we define as stable motion. Due to the units chosen (AU for the semimajor axis of the primaries and the Solar mass for the sum of the masses of the primaries) the period of revolution of the primaries is 1 year for all different integrated orbits. Therefore stability during the integration time means exact stability over 500 yr. This is in fact a very short time in comparison with the life times of the planetary orbits. But even for longer integration times the test of the numerical method showed no dramatic changes of the stability diagram. It is evident that whatever the length of the integration one never will arrive at a complete picture of stable and unstable regions. The most relevant confirmation of the results presented is the good accord with the circular problem with Hénon and Guyot's work on one hand and with Szebehely's study on the other where both used analytic criteria.

Acknowledgements. First of all I have to thank Drs. Christiane and Claude Froeschlé from Nice Observatory for many fruitful discussions throughout this work and for preparing the manuscript. I further thank Dr. Hans Scholl and Dr. Balint Erdi for critical comments and Drs. Lusting and Lichtenegger for preparing the final manuscript.

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