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Letter to the Editor

Synchrotron radiation in random magnetic fields

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SUMMARY. An exact analytical expression for the emissivity function of vacuum synchrotron radiation in random magnetic fields is derived. With this expression the calculation of the spontaneously emitted power and the synchrotron absorption coefficient is reduced to one quadrature (instead of three before) for any given energy distribution function of the radiating particles.

Key words: synchrotron radiation – magnetic fields – emission coefficient – relativistic electrons

1. Introduction

In almost all powerful cosmic synchrotron sources, as radio galaxies and active galactic nuclei, the measured degree of linear polarization is rather small (Kellermann and Pauliny-Toth, 1981; Miley, 1980; Angel and Stockman, 1980). This usually (e.g. Moffet, 1975) is interpreted by a small degree of homogeneity of the magnetic field B, so that the field line directions are nearly randomly distributed. A field configuration is proposed which consists of a small unidirectional part and a large random component. The spectral power radiated at position $\underline{r}$ into a particular linear polarization mode σ ($\sigma = 1,2$) at frequency ν then is expressed as the sum of the homogeneous (P_h), with percentage q, and the random (P_r), percentage 1-q, contribution:

$$P^{\sigma}(\nu,\theta) = q\ P_h(\nu,\theta) + \frac{1}{2}\ (1-q)\ P_r(\nu) \tag{1}$$

$q = 0$ and $q = 1$ refer to the cases of emitted power in 0% and 100% ordered magnetic field, respectively. θ denotes the angle between field lines and the direction of emission. The factor 1/2 arises because only half of the power $P_r(\nu)$ is measured in each of the two polarization modes.

In a completely random field polarization is absent and $P_r(\nu)$ is calculated by averaging the total spontaneously emitted power for a homogeneous field, $P_h^{tot}(\nu,\theta)$, over all possible values of the polar (θ) and azimuthal (ϕ) angles:

$$P_r(\nu) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\ \sin\theta\ P_h^{tot}(\nu,\theta)$$

$$= \frac{1}{2}\int_0^{\pi} d\theta \sin\theta\ P_h^{tot}(\nu,\theta) \tag{2}$$

with (Schwinger, 1949; Westfold, 1959; Pacholczyk, 1970; Moffet, 1975)

$$P_h^{tot}(\nu,\theta) = \sum_{\sigma=1}^{2} P_h^{\sigma}(\nu,\theta) = \frac{1}{4\pi}\int_0^{\infty} dE\, N(E)\, p(\nu,\theta,E) \tag{3}$$

Equation (2) implicitely assumes that B has stochastic directions on scales small compared to the size of the cosmic source but large compared to the Larmor radii of the radiating particles so that the synchrotron formulas are still applicable. N(E) in equation (3) denotes the energy distribution of the radiating particles, whereas $p(\nu,\theta,E)$ is the spontaneously emitted spectral power of a single electron in vacuum (m: electron mass, e: electron charge, c: speed of light)

$$p(\nu,\theta,E) = 4\pi\ c_2\ B\ \sin\theta\ F(\nu/\nu_c) \tag{4}$$

with

$$c_2 = 3^{1/2}\ e^3 / (4\pi mc^2)\ ,$$

$$F(\nu/\nu_c) = \frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} dt\ K_{5/3}(t), \tag{5}$$

$$\nu_c = \frac{3eB}{4\pi mc}\left(\frac{E}{mc^2}\right)^2 \sin\theta = c_1\, B\, E^2 \sin\theta \tag{6}$$

$K_{5/3}$ denotes a modified Bessel function of order 5/3.

From equations (2)-(5) one notices that the calculation of $P_r(\nu)$ involves the evaluation of three integrals over t, E, and θ, which normally are done numerically. It is the purpose of this letter to demonstrate that two of these integrations can be done rigorously, so that the calculation of the spectral synchrotron power from a cosmic source with random magnetic field is reduced to one quadrature. Bearing in mind the central role of the emissivity function in determining not only the spontaneously emitted power but also the synchrotron self-absorption coefficient, the importance of this result is evident.

2. Synchrotron emissivity in large-scale random magnetic fields

Defining

$$x \equiv \nu/(c_1\, B\, E^2) \tag{7}$$

we may write

$$P_r(\nu) = c_2\ B\int_0^{\infty} dE\ N(E)\ R(x) \tag{8}$$

where

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$$R(x) \equiv \frac{1}{2}\int_0^{\pi} d\theta \, \sin^2\theta \, F\left(\frac{x}{\sin\theta}\right)$$

$$= \frac{x}{2}\int_0^{\pi} d\theta \sin\theta \int_{x/\sin\theta}^{\infty} dt \, K_{5/3}(t) \qquad (9)$$

In the following we prove that $R(x)$ can be integrated to give

$$R(x) = \frac{1}{2}\pi x \, [W_{0,\frac{4}{3}}(x) \, W_{0,\frac{1}{3}}(x) - W_{\frac{1}{2},\frac{5}{6}}(x) \, W_{-\frac{1}{2},\frac{5}{6}}(x)] \qquad (10)$$

where $W_{\lambda,\mu}(x)$ denotes Whittaker's function (Abramowitz and Stegun, 1970, p. 505).

The proof starts by noting that because $\sin\theta$ is symmetric around $\theta = \pi/2$ we may write (9) as

$$R(x) = x \int_0^{\pi/2} d\theta \, \sin\theta \int_{x/\sin\theta}^{\infty} dt \, K_{5/3}(t) \qquad (11)$$

Integrating (11) by parts with respect to θ gives

$$R(x) = x \, [-\cos\theta \int_{x/\sin\theta}^{\infty} dt \, K_{5/3}(t)]_{\theta=0}^{\theta=\pi/2}$$

$$+ x^2 \int_0^{\pi/2} d\theta \cos^2\theta \sin^{-2}\theta \, K_{5/3}\left(\frac{x}{\sin\theta}\right) \qquad (12)$$

The first term evidently is zero. In the second term we substitute $\theta = (\pi/2) - \beta$, so that

$$R(x) = I_1(x) - I_2(x) \qquad (13)$$

with

$$I_1(x) = x^2 \int_0^{\pi/2} d\beta \, \cos^{-2}\beta \, K_{5/3}(x/\cos\beta) \qquad (14)$$

$$I_2(x) = x^2 \int_0^{\pi/2} d\beta \, K_{5/3}(x/\cos\beta) \qquad (15)$$

Since (Gradshteyn and Ryzhik, 1965, p. 741)

$$\int_0^{\pi/2} d\beta \, \frac{\cos(2\lambda\beta)}{\cos\beta} \, K_{2\mu}\left(\frac{x}{\cos\beta}\right) = \frac{\pi}{2x} \, W_{\lambda,\mu}(x) \, W_{-\lambda,\mu}(x) \qquad (16)$$

integral (15) reduces to

$$I_2(x) = \frac{\pi}{2} \, x \, W_{\frac{1}{2},\frac{5}{6}}(x) \, W_{-\frac{1}{2},\frac{5}{6}}(x) \qquad (17)$$

For the calculation of integral (14) we substitute

$$\cosh t = \cos^{-1}\beta \qquad (18)$$

implying $\sin\beta = \tanh t$, and

$$d\beta/dt = \cos^2\beta \, \cosh t \, .$$

We obtain

$$I_1(x) = x^2 \int_0^{\infty} dt \, \cosh t \, K_{5/3}(x \cosh t) \qquad (19)$$

which solves with (Gradshteyn and Ryzhik, 1965, p. 727)

$$\int_0^{\infty} dt \, \cosh(2\mu t) \, K_{2\nu}(2a \cosh t) = \frac{1}{2} \, K_{\mu+\nu}(a) \, K_{\mu-\nu}(a) \qquad (20)$$

to

$$I_1 = \frac{x^2}{2} \, K_{4/3}\left(\frac{x}{2}\right) \, K_{1/3}\left(\frac{x}{2}\right) \qquad (21)$$

Combining (17) and (21) yields for (13)

$$R(x) = \frac{x^2}{2} \, K_{4/3}\left(\frac{x}{2}\right) K_{1/3}\left(\frac{x}{2}\right) - \frac{\pi x}{2} \, W_{\frac{1}{2},\frac{5}{6}}(x) \, W_{-\frac{1}{2},\frac{5}{6}}(x) \qquad (22)$$

Modified Bessel functions are related to Whittaker functions as (Abramowitz and Stegun, 1970, p. 377)

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\nu}(2z) \qquad (23)$$

which allows us to write (22) in the neater form given in equation (10). This completes our proof of equation (10).

Using the asymptotic expansions of Whittaker functions for small and large arguments we derive

$$R(x) \cong \begin{cases} \frac{2^{1/3}}{5}\Gamma^2\left(\frac{1}{3}\right) x^{1/3} = 1.80842 \, x^{1/3} & \text{for } x \ll 1 \\ \frac{\pi}{2} \, e^{-x} \left(1 - \frac{99}{162} \, x^{-1}\right) & \text{for } x \gg 1 \end{cases} \qquad (24)$$

Figure 1 shows the exact calculation of the emissivity function $R(x)$ in comparison with its zeroth order asymptotic expansions (24). For $x < 10^{-2}$ and $x > 10$ $R(x)$ is well approximated by (24) (error less than 5 percent) whereas at $x = 0.28$ the largest deviation (67 percent) occurs. In Table 1 $R(x)$ is tabulated in the range $0.01 \leqq x \leqq 10$. For smaller and greater values of x the asymptotic expansions should be used.

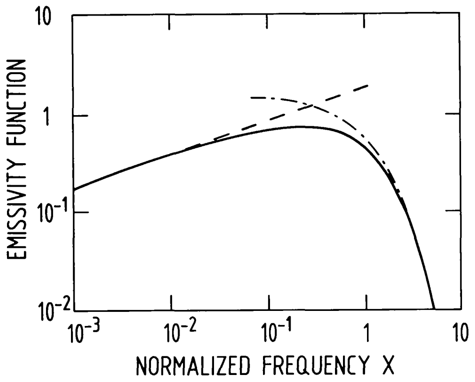

Fig. 1: Emissivity function $R(x)$ from equation (10) as a function of normalized frequency x from equation (7) (full curve) in comparison with its zeroth order asymptotic expansions from equation (24) for small x (dashed curve) and large x (dashed-dotted curve).

It was known for a long time that for power law energy distribution functions of the radiating electrons, $N(E) = N_o \, E^{-s}$, $P_r(\nu)$ in equation (8) can be integrated exactly (Moffet, 1975). So we finally demonstrate that with our emissivity function $R(x)$ we obtain the same result. For $N(E) = N_o \, E^{-s}$ with (7) and (22) equation (8) can be written as

$$P_r(\nu) = \frac{1}{2} \, c_2 \, B \, N_o \left(\frac{\nu}{c_1 B}\right)^{\frac{1-s}{2}} G'(s) \qquad (25)$$

with

$$G'(s) \equiv \frac{1}{2}\int_0^{\infty} dx \, x^{\frac{s+1}{2}} \, K_{4/3}\left(\frac{x}{2}\right) K_{1/3}\left(\frac{x}{2}\right) - \qquad (26)$$

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Table 1

$R(x) = \frac{1}{2}\pi x\,[W_{0,\frac{4}{3}}(x)\,W_{0,\frac{1}{3}}(x) - W_{\frac{1}{2},\frac{5}{6}}(x)\,W_{-\frac{1}{2},\frac{5}{6}}(x)]$

x	R(x)	x	R(x)	x	R(x)
0.01	0.371	0.7	0.559	3.0	0.068
0.02	0.455	0.8	0.517	3.5	0.042
0.03	0.508	0.9	0.477	4.0	0.026
0.04	0.546	1.0	0.439	4.5	0.016
0.05	0.576	1.1	0.403	5.0	0.010
0.06	0.600	1.2	0.370	5.5	0.0059
0.07	0.620	1.3	0.338	6.0	0.0036
0.08	0.636	1.4	0.309	6.5	0.0022
0.09	0.650	1.5	0.283	7.0	0.0013
0.1	0.661	1.6	0.258	7.5	0.00081
0.2	0.711	1.7	0.235	8.0	0.00049
0.3	0.705	1.8	0.214	8.5	0.00030
0.4	0.678	1.9	0.195	9.0	0.00018
0.5	0.641	2.0	0.178	9.5	0.00011
0.6	0.600	2.5	0.111	10.0	0.000068

$$- \frac{\pi}{2}\int_0^\infty dx\, x^{\frac{s-1}{2}}\, W_{\frac{1}{2},\frac{5}{6}}(x)\, W_{-\frac{1}{2},\frac{5}{6}}(x) \qquad (26)$$

Using (Gradshteyn and Ryzhik, 1965, p. 858; Erdélyi et al., 1954, p. 334)

$$\int_0^\infty dx\, x^{\rho-1}\, W_{K,\mu}(x)\, W_{-K,\mu}(x) = \frac{\Gamma(\rho+1)\,\Gamma(\frac{\rho}{2}+\frac{1}{2}+\mu)\,\Gamma(\frac{\rho}{2}+\frac{1}{2}-\mu)}{2\Gamma(1+\frac{\rho}{2}+K)\,\Gamma(1+\frac{\rho}{2}-K)}$$

$$(\mathrm{Re}\,\rho > 2\,|\mathrm{Re}\,\mu| - 1) \qquad (27)$$

$$\int_0^\infty dx\, x^{\rho-1}\, K_\mu(ax)\, K_\nu(ax) = 2^{\rho-3}\, a^{-\rho}\, [\Gamma(\rho)]^{-1}$$

$$\cdot\, \Gamma[\tfrac{1}{2}(\rho+\mu+\nu)]\,\Gamma[\tfrac{1}{2}(\rho-\mu+\nu)]\,\Gamma[\tfrac{1}{2}(\rho+\mu-\nu)]\,\Gamma[\tfrac{1}{2}(\rho-\mu-\nu)]$$

$$(\mathrm{Re}\,\rho > |\mathrm{Re}\,\mu| + |\mathrm{Re}\,\nu|) \qquad (28)$$

and straightforward algebra of gamma functions gives

$$G'(s) = \frac{1}{2}\pi^{1/2}\,\frac{\Gamma(\frac{s+5}{4})}{\Gamma(\frac{s+7}{4})}\,G(s) \qquad (29)$$

where

$$G(s) = 2^{\frac{s-3}{2}}\,\frac{s+7/3}{s+1}\,\Gamma(\frac{3s-1}{12})\,\Gamma(\frac{3s+7}{12}) \qquad (30)$$

which agrees with the well-known result.

3. Applicability of results

The formalism developed in this paper considers cosmic radio sources that are represented by a magnetic field of arbitrary strength with a random distribution of field line directions on scales small compared to the size of system ($\sim 10^{17}$ cm for active galactic nuclei) but large compared to the Larmor radii of the radiating electrons ($r_L \cong 10^6$ cm $(\gamma/10^3)\,(B/1G)^{-1}$; γ: Lorentz factor of relativistic electrons). In particular, the formalism can be applied to inhomogeneous synchrotron models for radio sources (e.g. Marscher, 1977) with any assumed radial variation of the magnetic field strength appearing both, in the argument of the emissivity function R(x) through x (see eq. (7)), and as multiplication factor in front of the integral in equation (8).

However, with respect to opacity effects a more thorough discussion is required. In general, the synchrotron intensity is calculated from solving the radiation transfer equation

$$\frac{dI(\underset{\sim}{r},\nu)}{ds} = P(\underset{\sim}{r},\nu,\theta) - \mu(\underset{\sim}{r},\nu,\theta)\,I(\underset{\sim}{r},\nu) \qquad (31)$$

where the synchrotron absorption coefficient μ is determined by the same emissivity function $F(\nu/\nu_c)$ determining $P(\underset{\sim}{r},\nu,\theta)$:

$$\mu(\underset{\sim}{r},\nu,\theta) = -\frac{c^2 c_2\, B\sin\theta}{2\nu^2}\int_0^\infty dE\,\frac{d}{dE}\left(\frac{N(E,\underset{\sim}{r})}{E^2}\right)E^2\,F\left(\frac{\nu}{\nu_c}\right) \qquad (32)$$

If absorption processes are neglected ($\mu \equiv 0$) the intensity calculation is reduced to the integration of the emissivity $P(\underset{\sim}{r},\nu,\theta)$ along the line of sight, and the angle-averaged emissivity $P_r(r,\nu)$ would directly yield the synchrotron intensity from a source with large-scale random magnetic field. In case of sources with absorption one may average equation (31) over all possible values of the polar (θ) and azimuthal (ϕ) angles to obtain

$$\frac{1}{2}\int_0^\pi d\theta\,\sin\theta\,\frac{dI(\underset{\sim}{r},\nu)}{ds} = P_r(r,\nu) - \frac{1}{2}\int_0^\pi d\theta\,\sin\theta\,\mu(\underset{\sim}{r},\nu,\theta)\,I(\underset{\sim}{r},\nu) \qquad (33)$$

In cases where $I(\underset{\sim}{r},\nu) = I(r,\nu)$ is independent of the angles, equation (33) reduces to

$$\frac{dI_r(r,\nu)}{ds} = P_r(r,\nu) - \mu_r(r,\nu)\,I_r(r,\nu) \qquad (34)$$

and the synchrotron intensity is determined by the averaged emissivity coefficient P_r and absorption coefficient μ_r which both are related to the emission coefficient R(x).

So to summarize: our formalism of using the emissivity function R(x) averaged over field line inclinations along the line of sight in sources with large-scale random magnetic fields for the computation of the emitted synchrotron intensity is without restrictions applicable to optically thin sources, whereas for optically thick sources the formalism is justified when the local intensity is independent of θ and ϕ.

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