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Gravitational field energy density for spheres and black holes

D. Lynden-Bell* and J. Katz *Racah Institute of Physics, Hebrew University of Jerusalem, Mount Scopus, Jerusalem, Israel*

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Summary. We show by physical arguments that static spherical systems have a coordinate-independent field energy density. For Schwarzschild's spacetime it is $+(g^2/8\pi G)$ where $g=\{GM/[(1+m/2\bar{r})^3\bar{r}^2]\}$ and $\bar{r}$ is the isotropic coordinate. The total field energy outside $\bar{r}$ is $GM^2/2\bar{r}$. Schwarzschild's $r=2m$ corresponds to $\bar{r}=½GM/c^2$ so the field energy outside a Schwarzschild black hole totals Mc^2. In this sense all the energy remains outside the hole.

1 Introduction

The principle of equivalence shows that acceleration due to gravity g cannot be given a wholely local definition. Thus field energy in general relativity is controversial and Misner, Thorne & Wheeler (1973) insist that no physical tensor exists which does for gravity what Maxwell's stress tensor does for electricity. However, these authors (p. 604, loc. cit.) admit that for spherical systems the concept of gravitational potential energy is both correct and useful.

Expressions for the conservation laws are commonly given in terms of the Einstein pseudo-tensor (Schrodinger 1954) t_μ^ν or the modifications of it due to Landau & Lifshitz (1966), Bergmann (Bergmann & Thomson 1953) or Møller (1952). However, these quantities are coordinate-dependent and do not represent local physical quantities. It is useful to have at least one test-bed in which field energy density can be defined unequivocally from the physics without any coordinate conditions. The success or failure of more general expressions for conserved quantities, such as those suggested by Penrose (1982) and his school, can be tested on such special cases. It is also interesting to see whether the expressions of Witten (1981) and of Horowitz & Strominger (1983) can be given a local interpretation.

In special relativity there is no field energy and the matter energy is well known to be

$$E_{\mathrm{M}}=\int_\Sigma T_\alpha^\beta \xi^\alpha \sqrt{-g}\, d\Sigma_\beta \tag{1}$$

where ξ^α is the time-like Killing vector of the Minkowski space. Σ is any space-like surface over which the energy is to be evaluated. E_{M} has been written in a form suitable for curvilinear

*Now returned to the Institute of Astronomy, The Observatories, Madingley Road, Cambridge CB3 0HA.

coordinates (even though the space is flat) to expedite the transition to curved space. In special relativity E_M is conserved.

In General Relativity a test particle of rest mass $\Delta\mu$ moving with 4-velocity $u^\alpha=dx^\alpha/d\tau$ in a stationary metric with time-like Killing vector ξ conserves its energy $E_M=\Delta\mu c\, u_\alpha\, \xi^\alpha$. If this particle consists of a localized distribution of matter with stress energy tensor ΔT_α^β this energy is re-expressed as

$$\Delta E_M=\int \xi^\alpha \Delta T_\alpha^\beta \sqrt{-g}\, d\Sigma_\beta=\int \Delta T_0^\beta \sqrt{-g}\, d\Sigma_\beta=\int \Delta T_0^0 \sqrt{-g}\, d^3x$$

where the integral is taken over a spatial slice. Just as in special relativity, the 'matter energy' is the sum of the individual particle energies and so we define E_M by expression (1) above. This definition holds in spaces where ξ is defined, i.e. stationary spacetimes. In more general spacetimes one would need a generalization of ξ but here we need only the simplest *static* case. We take Σ to be a surface t=constant. E_M is not the energy Mc^2 one finds from the Schwarzschild metric at great distances and so we write

$$Mc^2=E=E_M+E_F.$$

We attribute E_F to the energy of the gravitational field. Einstein wrote E_F as an integral over his coordinate-dependent pseudo-tensor. For stationary metrics with the correct asymptotic properties his E_M reduces to ours (equation 1). The classical limit of E_M is interesting:

$$E_M=\int \varrho(c^2+\tfrac{1}{2}v^2-\psi)\, d^3r$$

where $\psi=\Sigma GM/r$ is the (positive) gravitational potential of the Newtonian case. Evidently what is called matter energy in relativity actually contains twice the classical gravitational potential energy. The two occurs because the sum of the individual particle energies counts mutual potential energies twice. Since $V=-\tfrac{1}{2}\int \varrho\psi\, d^3r<0$ the total energy exceeds the sum of the individual particle energies by $E_F=|V|$. In the classical limit we have $E_F=1/8\pi G\int|\nabla\psi|^2 d^3x$. *Relativity avoids the negative field energy of Newtonian theory because its 'matter energy' E_M already contains a $-\int\varrho\psi$ term leaving a positive correction from the field.* The source of gravity is matter energy (which is smaller in a potential well) rather than the rest mass of Newtonian theory. For a discussion of the relation of the Newtonian limit to what is done in electrostatics see the Appendix. We now show that *provided equation (1) is agreed* as the matter energy, the field energy density for spheres follows.

2 Physical argument evaluating field energy density for static spheres

Consider any static spherical spacetime. Cut the metric on a sphere labelled a so as to leave the exterior spacetime unchanged. Replace the interior spacetime by flat space. This can be done by introducing a suitable surface distribution of matter on the sphere. The surface distribution is given by the discontinuities in the gradient of the metric tensor introduced by our cut.

Since the exterior spacetime is unchanged, the total energy as perceived by the Schwarzschild metric at infinity is Mc^2, the same as it was before we cut the system. We can evaluate the field energy of the cut system by taking the difference between its total energy and its matter energy. The latter will have contributions both from the matter exterior to the cut and from the surface distribution we introduced on the cut. We call their sum $E_M(a)$. Thus the field energy of the cut system is

$$E_F(a)=Mc^2-E_M(a).$$

Since a flat spacetime has no field energy, we shall assume that all of it arises from the curved spacetime outside the cut. We assume that outside the cut the field energy of the cut system is the same as the field energy of the original system, since the spacetimes are the same in all that region.

Now change the position of the cut from the sphere with coordinate label a to that with label $a+da$. The change dE_F/da is the field energy in the volume contained between the spheres a and $a+da$ in the original system. From the spherical symmetry this field energy must be uniformly distributed over spheres. Hence, if we divide by the volume between the two spheres in the original system we have found its field energy density. Notice that for static spheres this energy density is *physically* defined and is therefore independent of the coordinates chosen. This is in marked contrast to the field energy density defined by Einstein's pseudo-tensor.

As a simple example we evaluate the field energy density in Schwarzschild's spacetime. Cutting Schwarzschild's spacetime at the sphere whose isotropic radius is a and replacing the interior by a flat spacetime, we have the metric

$$ds^2=A^2\,dt^2-B^2[d\bar{r}^2+\bar{r}^2(d\theta^2+\sin^2\theta\,d\phi^2)]$$

where

$$A=c\begin{cases}\left(1-\dfrac{m}{2r}\right)\bigg/\left(1+\dfrac{m}{2\bar{r}}\right) & \bar{r}\geqslant a\\[2ex] \left(1-\dfrac{m}{2a}\right)\bigg/\left(1+\dfrac{m}{2a}\right) & \bar{r}\leqslant a\end{cases}$$

$$B=\begin{cases}\left(1+\dfrac{m}{2\bar{r}}\right)^2 & \bar{r}\geqslant a\\[2ex] \left(1+\dfrac{m}{2a}\right)^2 & \bar{r}\leqslant a\end{cases}$$

$$\sqrt{-g}=AB^3\,\bar{r}^2\sin\theta$$
$$m=GM/c^2.$$

In the spherical shell,

$$T_0^0=Mc\bigg/\left[4\pi a^2\left(1+\frac{m}{2a}\right)^5\right]\quad\delta(\bar{r}-a)$$

so

$$E_{\rm M}=\int T_0^0\sqrt{-g}\,d^3x=Mc^2-\tfrac{1}{2}GM^2/a.$$

Hence the total gravitational field energy is $\tfrac{1}{2}GM^2/a$ and the amount between a and $a+da$ is $\tfrac{1}{2}GM^2/a^2$. The area of the sphere of isotropic radius a is $4\pi[1+(m/2a)]^4a^2$ and the radial distance corresponding to da is $[1+(m/2a)]^2\,da$ so the gravitational field energy density is

$$\frac{1}{8\pi G}\left[\frac{GM}{a^2(1+m/2a)^3}\right]^2.$$

The field energy density at a general point of Schwarzschild spacetime is given by writing $\bar{r}$ for a. The relationship between Schwarzschild r and isotropic $\bar{r}$ is

$$r=\bar{r}\left(1+\frac{m}{2\bar{r}}\right)^2\ :\ 2\bar{r}=r-m+\sqrt{r(r-2m)}$$

so the total field energy outside r is

$$\frac{GM^2}{2\bar{r}}=\frac{GM^2}{r-m+\sqrt{r(r-2m)}}$$

when $r=2m$, $\bar{r}=m/2$ and the above expressions reduce to Mc^2. Thus the total field energy outside the hole gives the total mass.

Similarly one may demonstrate that the field energy density for a general spherical distribution in the metric

$$ds^2=\exp(2\nu)\,dt^2-\exp(2\mu)[d\bar{r}^2+\bar{r}^2(d\theta^2+\sin^2\theta\,d\phi^2)]$$

is

$$\frac{-c^4}{2\pi G}\exp(-3\mu)\,[\exp(\mu/2)]'\,[\exp(\nu+\mu/2)]' \tag{2}$$

where a dash denotes a derivative with respect to $\bar{r}$. The derivation is straightforward once one has the matter energy on the cut. To obtain this, one uses Einstein's equations to give

$$8\pi T^0_0=-\exp(-2\mu)\left[2\mu''+\mu'^2+\frac{4}{\bar{r}}\mu'\right].$$

Integrating $T^0_0\sqrt{-g}\,d^3x$ across the cut, we find the cut's contribution to E_{M} to be $-\mu'_+a^2\exp[\nu(a)+\mu(a)]$ where μ'_+ is μ' evaluated as $r\rightarrow a$ from above.

At least for all metrics that correspond to static fluid spheres we have proved that expression (2) is positive everywhere.

3 Relationship to other expressions

We have shown that energy density is well-defined for one very simple class of systems. Do the expressions given by Penrose and his school measure the same quantity? The answer is no. The Penrose mass of a Schwarzschild hole is all within the hole, whereas ours is all outside it, see Tod (1983). Witten (1981) has given a positive definite expression for energy in relativity which has been generalized by Horowitz & Strominger (1983). Both give expressions as integrals of positive quantities which one might be led to interpret as energy densities. Witten's expression does not have our matter term in its classical limit; this can be obtained by taking $n=2$ in Horowitz & Strominger's expression but evaluating their integrand in Schwarzschild spacetime leads to a different energy density with part of the 'mass' left inside the hole. Adler, Bazin & Schiffer (1965) evaluated Einstein's pseudo-tensor expression in isotropic coordinates and from it derived the expression $\tfrac{1}{2}GM^2/\bar{r}$ for the field energy outside $\bar{r}$ in Schwarzschild space. They pointed out that all the energy lay outside the hole. Both these results are in agreement with ours. However, theirs is a coordinate-dependent result. Our physical argument demonstrates that anyone who agrees that the 'matter energy' is given by expression (1) must agree to expression (2) for the field energy density whatever coordinates he uses. The relationship of Einstein's expression to the Komar integrals (1959) is elucidated by Katz (1985). Landau & Lifshitz's expression in isotropic coordinates does not give the correct result. Even in a stationary metric their expression denies the possibility of evaluating their *'matter energy'* before coordinates are chosen. This is because their expression has the wrong weight in $\sqrt{-g}$. Hawking's expression (1968) is designed to give the same mass within any sphere around a black hole. Finally, although we gave strong physical arguments leading to formula (1), there may be mathematicians who would have preferred formula (1) to have been weighted by extra redshift factor $|\xi|^{n-2}$. Any such formula replacing (1) is unphysical. Consider a large heavy hollow sphere. Inside it space is flat, special relativity holds. Now consider physical bodies inside with negligible gravity. Their matter energy density is T_0^0 measured locally. However, this energy is worth less on the 'international' energy scale at ∞. The correct redshift factor is the length of the time-like Killing vector ξ. Thus expression (1) gives the correct contributions to the matter energy and any extra $|\xi|^{n-2}$ factor is *wrong*. For any misguided mathematicians who persist, our earlier argument can be used with their formula. Provided $n>1$ all these formulae lead to all the black hole's mass being accounted for by field energy outside the hole.

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Appendix

Dr Schutz has asked us to eludicate the background to our choice of (1) as the matter energy formula. He points out that in electrostatics the energy of a particle is $mc^2+e\phi$ so that summing these and associating the result with the matter as we have done for gravity would yield a matter energy $\Sigma(mc^2+e\phi)$. As the total energy is in reality $\Sigma(mc^2+\frac{1}{2}e\phi)$ we would be left with a field energy of $-\Sigma\frac{1}{2}e\phi=-\int E^2/8\pi\, dV$ which is negative and not correct.

The correct procedure, as determined by the coupling of the stress tensor in general relativity is to associate only Σmc^2 with the matter energy and rewrite the $e\phi$ energy as a field energy. Considering now Newtonian *gravity* and following the procedure correct for electrostatics, we would find a negative field energy $-\int g^2/8\pi G\, dV$ and a matter energy Σmc^2. However, this procedure regards *rest-mass* as the *source* of gravity. If instead of regarding rest-mass as the source we regard the energy $E=m(c^2-\psi)$ as the source, then we get a total matter energy of ΣE leaving the positive field energy $\int g^2/8\pi G$ as derived in the text from the simplest relativistic expression for matter energy – the one used by Einstein.

This *difference* between gravity and electricity stems from the fact that rest mass is not the true source of gravity (a truth that is only fully seen in general relativity).