

ON MODEL-DEPENDENT BOUNDS ON H_0 FROM GRAVITATIONAL IMAGES: APPLICATION TO Q0957 + 561A,B

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ABSTRACT

We consider the images of a background source formed by a foreground mass distribution in a matter-dominated Friedmann cosmology. We present a transformation that leaves observable properties of these gravitational images invariant: superposing a uniform, projected mass distribution on the initial mass distribution, combined with moving the source position and reducing the value of H_0 , both by an appropriate scale factor, leaves unchanged the positions, relative magnifications, apparent luminosities, and difference in propagation times associated with any two images of the source. Using a compound mass distribution to represent a galaxy-cluster lens, we can account for the positions and relative magnifications of the A and B images of Q0957 + 561. Since mass cannot be negative, the transformation applied to this lens model yields bounds on the Hubble constant, given the difference $\Delta\tau_{BA}$ (in years) between the propagation times associated with the two images: $0 < H_0 \leq (98 \pm 3)\Delta\tau_{BA}^{-1} \text{ km s}^{-1} \text{ Mpc}^{-1}$. Temporal variations in the quasar's luminosity always appear first in the A image. Measurements of any such variations, reliably correlated between images, would determine $\Delta\tau_{BA}$. We emphasize that this transformation applied to other plausible lens models may not yield such bounds on H_0 .

Subject headings: cosmology — gravitation — quasars

I. INTRODUCTION

Observations of the images of a distant source formed by a sufficiently simple deflector acting as a gravitational lens can determine the deflector mass and a value for H_0 (Refsdal 1964, 1966 and references therein) for a given cosmology. However, in the one case of a multiply-imaged quasar where the lens has been clearly identified: Q0957 + 561 (Walsh, Carswell, and Weymann 1979), observations have revealed it to be complex—a giant elliptical galaxy, designated G1 (Stockton 1980; Young *et al.* 1980), embedded in a large cluster (Young *et al.* 1980, 1981).

For such a complex lens, the observables may not determine uniquely either the mass distribution of the lens or a value for H_0 . For example, a sufficiently complex model may permit transformations of a subset of its parameters, possibly including H_0 , that leave the observables invariant. In this *Letter*, we present one such transformation that leaves unchanged the image observables—positions, relative magnifications, apparent luminosities, and differences in propagation times associated with image pairs (hereafter “relative time delays”).

In § IIa we relate these image observables to the parameters of the source and of a generalized lens model. In § IIb we apply the transformation to these relations in two parts; the first part scales the value of H_0 , and the second part modifies the mass distribution of the lens and moves the position of the source. The combined transformation leaves unchanged all the image observables. In § III we discuss one lens model for Q0957 + 561 in the context of this transformation. In § IIIa

we model G1 as a singular isothermal sphere and represent the light bending of the cluster as a linear function of impact parameter (Gorenstein *et al.* 1983). We show in § IIIb that this model accounts for the positions and relative magnifications of the A and B images of Q0957 + 561 and yields a bound on H_0 , given the relative time delay between these images and the fact that mass cannot be negative (see also Borgeest and Refsdal 1984). A brief discussion of our results and of possible further applications of them is given in § IIIc. Derivation and discussion of those results only stated here will appear elsewhere (Gorenstein *et al.* 1985, hereafter Paper II).

II. IMAGE OBSERVABLES AND LENS-MODEL TRANSFORMATION

We employ the formulation of ray-optics in a matter-dominated Friedmann cosmology as, for example, presented by Young *et al.* (1980, 1981, and references therein). The quantities D_d , D_s , and D_{ds} will represent, respectively, the observer-deflector, observer-source, and deflector-source angular-size distances appropriate for this cosmology (e.g., Gunn 1967; Young *et al.* 1980). We describe positive displacements to the east and north on the sky by x and y , respectively, and we set $G = c = 1$. We refer to a mass distribution projected on the plane of the sky as a surface mass distribution (mass per steradian). Boldface symbols denote vectors on the plane of the sky.

a) Observables

The ray-trace relation between the angular position Θ_s of a source on the sky in the absence of lensing, a corresponding image position Θ , and the bend angle $\alpha(\Theta)$ due to a gravita-

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tional lens with surface mass distribution $\sigma(\Theta)$ is

$$(\Theta_s - \Theta)(D_s/D_{ds}) = \alpha(\Theta) \quad (1)$$

(see, e.g., Young *et al.* 1980, and Fig. 6 therein). The relationship $\delta\Theta_s = [M]\delta\Theta$ defines the elements of a symmetric 2×2 (magnification) matrix $[M]$ associated with the image, where $\delta\Theta_s$ and $\delta\Theta$ describe, respectively, differential displacements about Θ_s and Θ that are related by equation (1). For a transparent lens and a sufficiently small image, conservation of surface brightness implies that $\mathcal{M}$, the absolute magnification of the image, equals $\det[M]^{-1}$; the apparent luminosity of the image is then $L = |\mathcal{M}|L_s$, where $\mathcal{M}$ is negative for parity-reversed images and where L_s is the apparent luminosity of the source in the absence of lensing. The derivatives of the (Cartesian) components of Θ_s with respect to the components of Θ yield expressions for the elements of $[M]$:

$$[M] = [I] + (D_{ds}/D_s)[A], \quad (2)$$

where $[I]$ is the identity matrix and $[A]$ is the matrix of partial derivatives of the components of $\alpha(\Theta)$ with respect to the components of Θ .

We now consider two images i and j of a source, where the positions, apparent luminosities, and magnification matrices for these images are, respectively, Θ_i and Θ_j , L_i and L_j , and $[M_i]$ and $[M_j]$. Small displacements about Θ_i and about Θ_j are related by $\delta\Theta_i = [M_{ij}]\delta\Theta_j$, where $[M_{ij}] = [M_i]^{-1}[M_j]$ is the *relative* magnification matrix. The elements of $[M_{ij}]$ are observable quantities (Gorenstein *et al.* 1984); the determinant of $[M_{ij}]$ gives the ratio of the absolute magnification of image i to that of image j .

The relative time delay between images i and j is (Cooke and Kantowski 1975; Paper II):

$$\Delta\tau_{ij} = (1 + z_d)D_d \left[-(\Theta_i - \Theta_j) \cdot (\alpha_i + \alpha_j)/2 + P_i - P_j \right], \quad (3)$$

where $\Delta\tau_{ij} \equiv T_i - T_j$; T_k , $k = i, j$, is the time elapsed when light propagates from the source to the observer via image k ; z_d is the redshift of the deflector; $\alpha_k \equiv \alpha(\Theta_k)$; and $P_k \equiv P(\Theta_k)$, where $(1 + z_d)D_d P(\Theta)$ is the delay due to the gravitational potential of the lens and where $\text{grad } P(\Theta) = \alpha(\Theta)$, the gradient being taken with respect to angular coordinates. The term involving the Θ_k 's and α_k 's specifies the geometric contribution to the relative delay. The values for Θ_k , α_k , and D_d determine the spatial position of the source, so the angular position and the angular-size distance of the source do not appear in equation (3). For $\alpha(\Theta)$ described by a sum of terms and for images at fixed positions, equation (3) can be used to compute the contribution that each term makes to the relative time delay.

b) Lens-Model Transformation

We now present a transformation in two parts; neither part affects the redshifts and intrinsic source luminosity, and each part applied separately leaves $\Theta_{i,j}$ and $[M_{ij}]$ unchanged but causes a scaling of $L_{i,j}$ and $\Delta\tau_{ij}$. The first part scales the

Hubble constant, $H_0 \rightarrow hH_0$; each angular size distance D then scales as: $D \rightarrow h^{-1}D$, so the ratio D_s/D_{ds} is invariant. Values of Θ_s and $\alpha(\Theta)$ remain unchanged. Since the luminosity distance scales as h^{-1} , we have $L_s \rightarrow h^2 L_s$. Applying these results to equations (1)–(3), one finds that $\Theta_{i,j} \rightarrow \Theta_{i,j}$, $[M_{i,j}] \rightarrow [M_{i,j}]$, $L_{i,j} \rightarrow h^2 L_{i,j}$, and $\Delta\tau_{ij} \rightarrow h^{-1} \Delta\tau_{ij}$. Dimensional analysis also shows that $\sigma(\Theta) \rightarrow h^{-1} \sigma(\Theta)$.

The second part of the transformation consists of modifying the surface mass distribution $\sigma(\Theta)$ of the lens and moving the position of the source. We scale $\sigma(\Theta)$ and add (or subtract) a circular disk of uniform surface mass density σ_0 to (or from) it:

$$\sigma(\Theta) \rightarrow \varepsilon \sigma(\Theta) + \sigma_0, \quad (4)$$

where ε is a positive scaling variable, the center of the disk is taken at $\Theta = 0$, and the area of the disk is taken to span at least the image formation region. Provided the image formation region remains covered, translation of such a disk only adds to the bending law a constant bend angle that can be ignored (Paper II).

The circular disk acts like a lens with focal length $D_d^2/4\pi\sigma_0$: a circularly symmetric surface mass distribution deflects light by an angle $\alpha(\Theta) = -[4m(\Theta)/\Theta^2 D_d]\Theta$, where $m(\Theta)$ is the mass contained within the impact angle $\Theta = |\Theta|$. For the circular disk of uniform density, we have $m(\Theta) = \pi\Theta^2\sigma_0$, so its deflection angle is $\alpha(\Theta) = -(4\pi\sigma_0/D_d)\Theta$. If $\sigma_0 = \sigma_f$, where $\sigma_f \equiv (D_d/4\pi)(D_s/D_{ds})$, then the disk's focal length is $D_d D_{ds}/D_s$. Such a distribution focuses at the observer's location all rays emanating from a source at $\Theta_s = 0$.

We define ε by $\sigma_0 \equiv (1 - \varepsilon)\sigma_f$ (Paper II), whence transformation (4) corresponds to:

$$\alpha(\Theta) \rightarrow \varepsilon \alpha(\Theta) - (1 - \varepsilon)(D_s/D_{ds})\Theta. \quad (5)$$

Finally, we move the source position as follows:

$$\Theta_s \rightarrow \varepsilon \Theta_s. \quad (6)$$

Applying transformations (5) and (6) to equations (1) and (2), we find $\Theta_{i,j} \rightarrow \Theta_{i,j}$, $[M_{i,j}] \rightarrow \varepsilon[M_{i,j}]$, $[M_{ij}] \rightarrow [M_{ij}]$, and $L_{i,j} \rightarrow \varepsilon^{-2} L_{i,j}$. Applying transformation (5) to equation (3), we find $\Delta\tau_{ij} \rightarrow \varepsilon \Delta\tau_{ij}$. Evaluation of equation (3) for the first term on the right side of transformation (5) yields $\varepsilon \Delta\tau_{ij}$; evaluation of equation (3) for the second term yields zero (Paper II).

We now consider two special cases to illustrate the effect of transformations (4)–(6). First, we assume that $\sigma(\Theta)$ describes a circularly symmetric surface mass distribution with surface mass density decreasing monotonically with increasing Θ and with $\sigma(0) > \sigma_f$. If the source is collinear with the observer and such a lens, then the observer sees a ring image. Equation (1) yields the radius Θ_r of the ring: $4m(\Theta_r)/\Theta_r D_d = (D_s/D_{ds})\Theta_r$. The mean surface mass density within Θ_r is just σ_f : $m(\Theta_r)/\pi\Theta_r^2 = \sigma_f$. Transformation (4) then leaves the mean surface mass density σ_f within the ring invariant and therefore the ring's diameter remains unchanged. Next, note that if we have no lens, i.e., if $\sigma(\Theta) = 0$, then transformation (5) corresponds to the placement at D_d of a magnifying lens centered at $\Theta = 0$ with absolute magnification ε^{-2} . Transformation (6) then leaves Θ , the magnified source position, equal to Θ_s .

Under the total transformation, the properties of the images transform as $\Theta_{i,j} \rightarrow \Theta_{i,j}$, $[M_{ij}] \rightarrow [M_{ij}]$, $L_{i,j} \rightarrow (h/\epsilon)^2 L_{i,j}$, and $\Delta\tau_{ij} \rightarrow (\epsilon/h)\Delta\tau_{ij}$; the surface mass distribution and bend angle transform, respectively, as $\sigma(\Theta) \rightarrow (\epsilon/h)\sigma(\Theta) + (1-\epsilon)h^{-1}\sigma_f$ and $\alpha(\Theta) \rightarrow \epsilon\alpha(\Theta) - (1-\epsilon)(D_s/D_{ds})\Theta$. For $\epsilon > 1$, this transformation subtracts a uniform mass distribution from $(\epsilon/h)\sigma(\Theta)$. Thus, the value of ϵ must be bounded: if σ_m is the minimum value of the untransformed $\sigma(\Theta)$, we must have $\epsilon \leq \sigma_f/(\sigma_f - \sigma_m)$ for the transformed $\sigma(\Theta)$ to remain nonnegative for all Θ , and, of course, $\epsilon > 0$, by hypothesis. For $\epsilon \neq 1$, all image properties are then unchanged if we scale H_0 such that $h = \epsilon$; bounds on h follow: $0 < h \leq \sigma_f/(\sigma_f - \sigma_m)$.

In Paper II, we will consider models that do not contain explicitly a uniform surface mass density but nonetheless allow a range of parameter combinations under which the bending law transforms as in transformation (5).

III. APPLICATION TO Q0957+561A,B

a) A Galaxy-Cluster Bending Law for Q0957+561

Consider a two-component model for the lens of Q0957+561 where the components approximate, respectively, the light bending due to the elliptical galaxy G1 and that due to the cluster. Let the surface mass distribution of G1 be that of a singular isothermal sphere (SIS), for which $\sigma_g(\Theta) = (\sigma_g\Theta_g/2)\Theta^{-1}$ with σ_g being the surface mass density at $\Theta = \Theta_g/2$, and Θ_g being chosen only for convenience (Chandrasekhar 1967, p. 157). Thus the light bending $\alpha_g(\Theta)$ due to G1 is $\alpha_g(\Theta) = -\alpha_g\hat{\Theta}$, a constant bend angle, where $\alpha_g = 4\pi\Theta_g\sigma_g/D_d$ and $\hat{\Theta} = \Theta/|\Theta|$ is a unit vector (on the sky) pointing outward from the center of the galaxy.

For the region spanned by the image positions, we approximate the light bending $\alpha_c(\Theta)$, due to the visible and dark matter in the cluster, with the leading two terms in a Taylor series (Gorenstein *et al.* 1983):

$$\alpha_c(\Theta) = \alpha_0 + [Q]\Theta, \quad (7)$$

where we define $[Q]$ as the matrix of partial derivatives of the components of $\alpha_c(\Theta)$ taken with respect to the (Cartesian) components of Θ and evaluated at the center of the galaxy ($\Theta = 0$); α_0 is the light bending at that origin due to the cluster.

Since gravitational light-bending follows the laws of geometrical optics (Misner, Thorne, and Wheeler 1973, p. 570), a general bending law must be expressible as the gradient of a scalar potential. Two consequences follow: $\text{curl } \alpha(\Theta) = 0$ and $\text{div } \alpha(\Theta) = -8\pi\sigma(\Theta)/D_d$ (Paper II). Because $\text{curl } \alpha(\Theta) = 0$, the matrix $[Q]$ in equation (7) must be symmetric when expressed in Cartesian coordinates; we define $[Q]$ as:

$$[Q] = Q_S[I] + Q_A[\omega(\gamma)], \quad (8)$$

where Q_S , Q_A , and γ are scalar parameters, and where the elements of the matrix $[\omega(\gamma)]$ are $[\omega(\gamma)]_{xx} = -[\omega(\gamma)]_{yy} = -\cos 2\gamma$ and $[\omega(\gamma)]_{xy} = [\omega(\gamma)]_{yx} = \sin 2\gamma$. From equations (7) and (8), we have $\text{div } \alpha_c(\Theta) = 2Q_S$; thus, $Q_S = -4\pi\sigma_c(0)/D_d$, where $\sigma_c(0)$ is the surface mass density of the

cluster at the center of the galaxy. Both the $Q_S\Theta$ and the $Q_A[\omega(\gamma)]\Theta$ terms contribute zero relative delay (Paper II). For a mass distribution composed of a circular disk centered at the origin with uniform surface mass density $\sigma_c(0)$ and a point mass m lying along position angle γ at a distance Θ_m from the origin, the light bending within the disk can be described approximately by equations (7) and (8) with $Q_S = -4\pi\sigma_c(0)/D_d$ and $Q_A = -4m/\Theta_m^2 D_d$.

b) Model Parameters for Q0957+561

We use the expressions for $\alpha_g(\Theta)$ and $\alpha_c(\Theta)$ to account for the properties of the A and B images of the quasar Q0957+561. Based on the discussion of § II b, we exclude the α_0 and the $Q_S\Theta$ terms from $\alpha_c(\Theta)$, and obtain the model bending law

$$\alpha(\Theta) = -\alpha_g\hat{\Theta} + Q_A[\omega(\gamma)]\Theta, \quad (9)$$

a function of the three parameters α_g , Q_A , and γ .

According to Stockton (1980), the x, y coordinates of the (optical core of the) B image with respect to that of G1 is $\Theta_B = (-0''.19 \pm 0''.03, -1''.00 \pm 0''.03)$; according to Young *et al.* (1981), the coordinates of the (optical core of the) A image with respect to that of the B image is $\Theta_A - \Theta_B = (-1''.23 \pm 0''.01, 6''.05 \pm 0''.01)$ and the ratio of image magnifications $\mathcal{M}_B/\mathcal{M}_A$ is -0.77 ± 0.03 for the images' optical emission lines. Since there is no reliable value for H_0 , we arbitrarily take $H_0 = 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The redshifts of G1 (0.36 ± 0.01 ; Young *et al.* 1981) and the quasar (1.4136 ± 0.0002 ; Weymann *et al.* 1979) then yield values of D_d and D_s/D_{ds} for our adopted (simplified) cosmology, that range from 689 to 629 Mpc and from 1.651 to 1.666, respectively, for values of the mean mass density of the universe Ω , expressed in units of the critical density, that range from 0 to 1.

From equations (1) and (9), we can obtain two expressions that relate α_g , Q_A , and γ to the values for Θ_A and Θ_B . Equations (2) and (9) yield an expression for $\det[M_{BA}]$, thereby providing a relation between α_g , Q_A , and γ and $\mathcal{M}_B/\mathcal{M}_A$. These three relations are sufficient to determine the values of the three parameters; the measured values for Θ_B , $\Theta_A - \Theta_B$, and $\mathcal{M}_B/\mathcal{M}_A$ yield $\alpha_g = +3''.4 \pm 0''.1$, $Q_A = 0.61 \pm 0.04$, and $\gamma = 70^\circ 5 \pm 0^\circ 5$, and a value for Θ_s of $(-0''.09 \pm 0''.01, 1''.27 \pm 0''.05)$, all for $\Omega = 0$ (Paper II). From equations (2) and (3), we find the model values $\mathcal{M}_A = 1.630 \pm 0.006$, $\mathcal{M}_B = -1.26 \pm 0.05$, and, for the assumed value for H_0 , $\Delta\tau_{BA} = (0.98 \pm 0.03) \text{ yr}$, with temporal variations in the quasar's luminosity appearing first in the A image. The errors given for these parameters are the standard errors that follow from the measurement standard errors and the model. A lens specified by these values of α_g , Q_A , and γ can yield one, two, or four images of a source depending on its position. We have verified that this lens forms only single images of the extended radio components located northeast and southwest of the A image (Greenfield, Burke, and Roberts 1980). Note, also, that because an SIS has a singular surface mass distribution, a third image of Q0957+561 would appear at the center of the lens with zero magnification.

From § II b, we find that scaling H_0 by h , scaling $\alpha(\Theta)$ by ϵ , and adding to the lens model a disk of uniform surface

mass density $\sigma_0 = (1 - \epsilon)h^{-1}\sigma_f$, where $\sigma_f \approx 4.4 \times 10^{10} M_\odot \text{ arcsec}^{-2}$ for the Q0957+561 system, leaves Θ_A , Θ_B , and $\mathcal{M}_B/\mathcal{M}_A$ unchanged. However, these transformations scale the model values of the respective image magnifications and the images' relative time delay. Thus, for the *transformed* model, the predicted values of these quantities are $\mathcal{M}_A = (1.630 \pm 0.006)\epsilon^{-2}$, $\mathcal{M}_B = (-1.26 \pm 0.05)\epsilon^{-2}$, and $\Delta\tau_{BA} = (0.98 \pm 0.03)(\epsilon/h) \text{ yr}$. Given a value for $\Delta\tau_{BA}$, we see that bounds on h follow: $0 < h \leq (0.98 \pm 0.03)(\Delta\tau_{BA})^{-1}$, since ϵ is restricted to $0 < \epsilon \leq 1$ by the requirements that the SIS mass remain positive and the transformed value of $\sigma(\Theta)$ remain nonnegative for all Θ .

c) Discussion

The one-dimensional velocity dispersion v of an elliptical galaxy is an observable quantity that can determine the light-deflection angle $\epsilon\alpha_g$ due to the galaxy. For an SIS, $v^2 = \epsilon\alpha_g/4\pi$ (Young *et al.* 1980); thus $v = (344 \pm 5)\epsilon^{1/2} \text{ km s}^{-1}$ in the case of G1. Consequently, a measurement of v can determine the value of ϵ , and a value for $\Delta\tau_{BA}$ can then determine h , all, of course, within the confines of the model described in § III.

Borgeest and Refsdal (1984) have explored the relation between parameters that describe two particular galaxy-cluster lens models and the resulting relative time delays between images. They concluded that a model appropriate for the case of Q0957+561 may yield an upper limit on the product $\Delta\tau_{BA}H_0$. It follows from our analysis in § IIb that such a bound always arises when a uniform surface mass density can be added to an arbitrary lens.

Paper II will explore more complex models for G1 to investigate further the properties of the expected third image (Young *et al.* 1980, 1981; Greenfield 1981) and to determine any bounds on h that other lens models may yield. Paper II will also explore the implications of transformations (5) and (6) for the estimation of cosmologically relevant parameters from gravitational lens systems.

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REFERENCES

- Borgeest, U., and Refsdal, S. 1984, *Astr. Ap.*, in press.
 Chandrasekhar, S. 1967, *An Introduction to the Study of Stellar Structure* (New York: Dover).
 Cooke, J. H., and Kantowski, R. 1975, *Ap. J. (Letters)*, **195**, L11.
 Gorenstein, M. V., Shapiro, I. I., Cohen, N. L., Bonometti, R. J., Falco, E. E., and Rogers, A. E. E. 1983, in *Quasars and Gravitational Lenses, Proceedings of the 24th Liège International Astrophysical Colloquium*, (Liège, Belgium: Université de Liège, Institut d'Astrophysique), p. 178.
 Gorenstein, M. V., *et al.* 1984, *Ap. J.*, **287**, 538.
 Gorenstein, M. V., *et al.* 1985, in preparation (Paper II).
 Greenfield, P. E. 1981, Ph.D. thesis, Massachusetts Institute of Technology.
 Greenfield, P. E., Burke, B. F., and Roberts, D. 1980, *Nature*, **286**, 865.
 Gunn, J. E. 1967, *Ap. J.*, **147**, 61.
 Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, *Gravitation* (San Francisco: Freeman).
 Refsdal, S. 1964, *M.N.R.A.S.*, **128**, 307.
 ———. 1966, *M.N.R.A.S.*, **132**, 101.
 Stockton, A. 1980, *Ap. J. (Letters)*, **242**, L141.
 Walsh, D., Carswell, R. F., and Weymann, R. J. 1979, *Nature*, **279**, 381.
 Weymann, R. J., Chaffee, F. C., Davis, M., Carleton, N. P., Walsh, D., and Carswell, R. F. 1979, *Ap. J. (Letters)*, **233**, L43.
 Young, P., Gunn, J. E., Kristian, J., Oke, J. B., and Westphal, J. A. 1980, *Ap. J.*, **241**, 507.
 ———. 1981, *Ap. J.*, **244**, 736.

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