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Search for Pulsation Mode Resonances in RR Lyrae Star Models

by

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ABSTRACT

A linear adiabatic and non-adiabatic survey of pulsation periods (up to the fourth overtone) was performed in the parameter ranges relevant for double-mode RR Lyrae stars. It was shown that among the lowest order resonances between the frequencies of the normal modes the closest is the 2:1 resonance between the fundamental and third overtone. It is argued that proper inclusion of non-adiabaticity may further improve the situation for this resonance. However, there is a problem with satisfying simultaneously constrains imposed by observed periods and resonance condition.

As a by-product of our survey it was shown, that systematic differences exist between the different pulsational codes, causing uncertainty of 10 – 20 percent in mass determination.

1. Introduction

It has been known for several decades that there are some stars in the Cepheid instability strip whose light variation can be reasonably explained only if we assume that they pulsate simultaneously in two modes. The first group of these stars (called *beat Cepheids*) have fundamental periods between 2 and 6 days and first overtone periods such that their ratio is $0.697 \leqslant P_1/P_0 \leqslant 0.711$. Except for their double-mode pulsations, beat Cepheids are indistinguishable from the monomode Cepheids. Their total number is now 11 (Stobie, 1977), or 12, if we include CO Aur (Antonello and Mantegazza, 1984) which, however, shows a discordant period ratio of 0.797.

The second group of double-mode pulsators (called *RR_s stars, dwarf Cepheids* or *AI Vel stars*) have fundamental periods between 0.05 and 0.22 days and period ratios $0.768 \leqslant P_1/P_0 \leqslant 0.778$. There are now altogether 7 such stars known (Fitch and Szeidl, 1976). There are strong controversies between different investigators on the physical properties and evolutionary status of these stars (see e.g. Petersen, 1978, Simon, 1979 and Breger, 1979). They are in many respects very similar to other δ Scuti stars, but their pulsational characteristics may indicate that they belong to the stars of Population II.

The only known star showing triple-mode radial pulsation is *AC And.* The physical parameters and evolution of this star is also controversial (see e.g. Fitch and Szeidl, 1976 and Petersen, 1978).

It is well known that the very promising method of the mass and radius determination *via* the $P_0 - P_1/P_0$ calibration diagram (introduced by Petersen, 1973 and hereafter referred as *Petersen-diagram*) has led to one of the most severe problem in the pulsation theory, i.e. beat Cepheid mass discrepancy. These facts and the failure of modelling sustained double-mode pulsation (Hodson and Cox, 1976, however Stellingwerf, 1975b) suggest that it would be extremely useful to have some new type of multimode radial pulsators for testing pulsation theory.

The new group of double-mode pulsators emerges from the RR Lyrae stars. Until 1981, only *AQ Leo* was known as a *double-mode RR Lyrae star* (Jerzykiewicz and Wenzel, 1977). By a period analysis of the photographic observations of the globular cluster *M15* (Sandage, Katem and Sandage, 1981) Cox, Hodson and Clancy (1981), Cox (1982) and Cox, Hodson and Clancy (1983, hereafter CHC) have shown that 10 RR Lyrae stars pulsate in two modes with the same periods than that of the field star AQ Leo, i.e. $P_0 = 0\overset{d}{.}55 \pm 0\overset{d}{.}03$, $P_1/P_0 = 0.746 \pm 0.001$. Further studies in this and other globular clusters now seem to reveal more double-mode RR Lyrae stars (for references see CHC). Importance of these stars lies in the fact that owing to their cluster membership we have a better chance to decipher their physical parameters.

As shown by Cox, King and Hodson (1980) and by CHC, double-mode RR Lyrae stars seem to fit well the linear pulsation and evolution theories. Non-linear hydrodynamic models for these stars are not yet available. Before attempting these laborous, complicated computations it seems quite useful to make a detailed description of their linear modal characteristics. In addition, in the light of the possible importance of resonances in double-mode pulsation (Dziembowski and Kovács, 1984), it is hoped that linear theory might help to restrict the parameter ranges for non-linear computations.

We deal with the accuracy of the mass determination with the aid of Petersen-diagram in the next section. Our results on the resonances encountered in these stars are given in Section 3. Finally, we speculate on the significance of our results for further non-linear studies.

2. Mass determination by use of Petersen-diagram

Inaccuracies in the mass determination introduced by different effects have been discussed elsewhere (*e.g.* Petersen, 1978). He concludes that in the case of Cepheids with periods longer than 2 days the uncertainty in mass determination varies from 4 to 9 percent. For dwarf Cepheids, however, the situation is much worse as, according to their period ratios, they lie close to the unfavourable upper part of the corresponding Petersen-diagram. An error of 100 percent is not uncommon in this region.

For the double-mode RR Lyrae stars, errors in the mass determination might be similar to those of the beat Cepheids, because by decreasing heavy element content, the P_0-P/P_0 curves become separated in the relevant region of the diagram (compare Figs. 2 and 7 of Petersen, 1978). To clarify the problem we studied further the influence of various effects on the Petersen-diagram for RR Lyrae stars.

We adopted Dziembowski's code for the computation of the static model envelopes and full linear (adiabatic and non-adiabatic) modal characteristics. The details of model construction are given by Dziembowski (1977). All models consisted of about 400 mass shells down to some 3 percent of the radius. At the edge of the core the temperature was not higher than 10^7K and the envelope contained about 10 percent of the total mass. Experience showed that such a depth of the envelope was sufficient to obtain accurate periods and growth rates for all modes up to the fourth overtone (see Section 4 for some details of the tests).

Masses, luminosities and effective temperatures were changed in the ranges $0.55 \leqslant M/M_\odot \leqslant 0.85$, $50 \leqslant L/L_\odot \leqslant 80$ and $3.81 \leqslant \log T_e(\mathrm{K}) \leqslant 3.87$ with steps of 0.1, 10 and 0.02 respectively. Beside of this standard series we chose the models of $M/M_\odot = 0.65$, α (ratio of the mixing length to pressure scale height) $= 0.0$, $X = 0.7$, $Y = 0.299$ with finer steps, i.e. 5 and 0.01 in the above luminosity and effective temperature ranges, respectively. Except for the red edge, the above ranges cover the possible parameters relevant for RR Lyrae stars.

Studying the sensitivity of the periods and period ratios to change in chemical composition, α, position of models on the HR-diagram and including non-adiabaticity, we got essentially the same result as Petersen (1978). In Figs. 1 and 2 we show the Petersen-diagrams calculated with $\alpha = 0$, $\alpha = 1$ for adiabatic models and for the non-adiabatic ones for the model series with $X = 0.7$, $Y = 0.299$. The effect of increasing heavy element content is shown in Fig. 3. Except for the last figure, we do not recognize any systematic shifts in the calibration lines. Calculations made with different light element contents $(X, Y) = (0.65, 0.349)$, $(0.7, 0.299)$, $(0.75, 0.249)$ yielded smaller than 0.001 non-systematic changes in the period ratios. Position effect caused a broadening smaller than 0.002 in the period ratio coordinate of the calibration lines. Summing up all these effects we may say that the maximum broadening of the constant mass lines corresponding to the adiabatic, $\alpha = 0$, $X = 0.7$, $Y = 0.299$ case is smaller than 0.002 in the period ratio coordinate and that periods themselves are estimated with an accuracy better than one percent. The broadening of the constant mass lines results in an inaccuracy in the mass

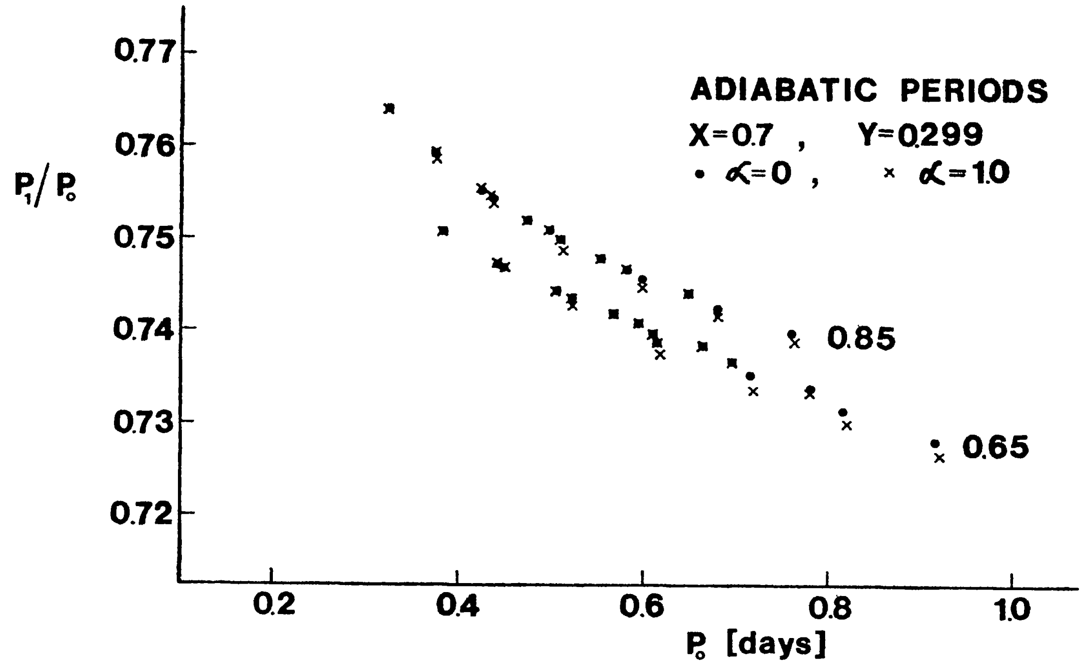

Fig. 1. Comparison of the Petersen-diagrams with and without convection. Periods of our model series with standard steps of parameters (described in the text) are plotted. Only two different series with masses $0.85 M_\odot$ and $0.65 M_\odot$ are shown to avoid confusion.

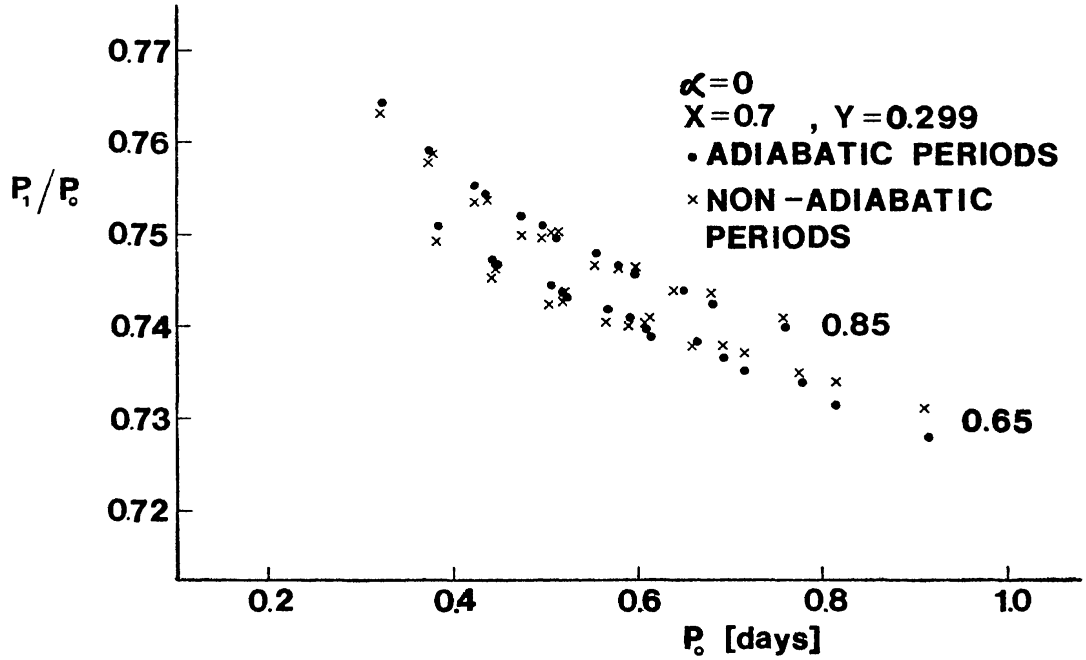

Fig. 2. Same as in Fig. 1, but for adiabatic and non-adiabatic periods.

determination for RR Lyrae stars of about $\pm 0.03\ M_\odot$, which corresponds to the relative error for beat Cepheids given by Petersen (1978).

Now there is another source of inaccuracy, i.e. model dependence. Though it was shown by previous studies (see Petersen, 1978 and references therein) that the beat Cepheid mass anomaly is quite model indepe-

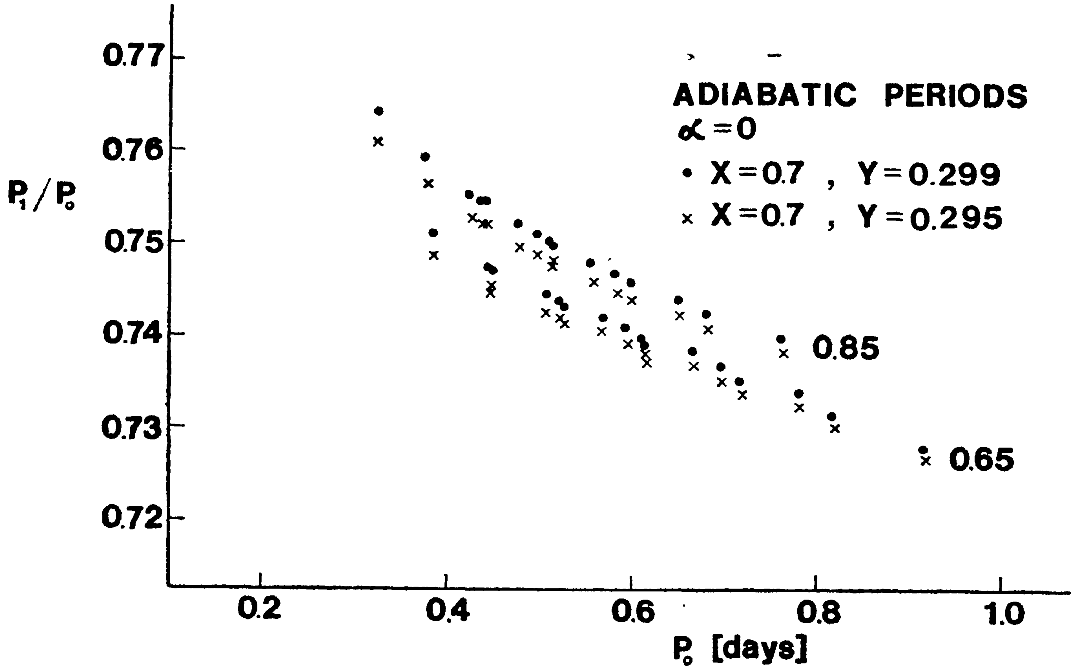

Fig. 3. Same as in Fig. 1, but for different chemical compositions.

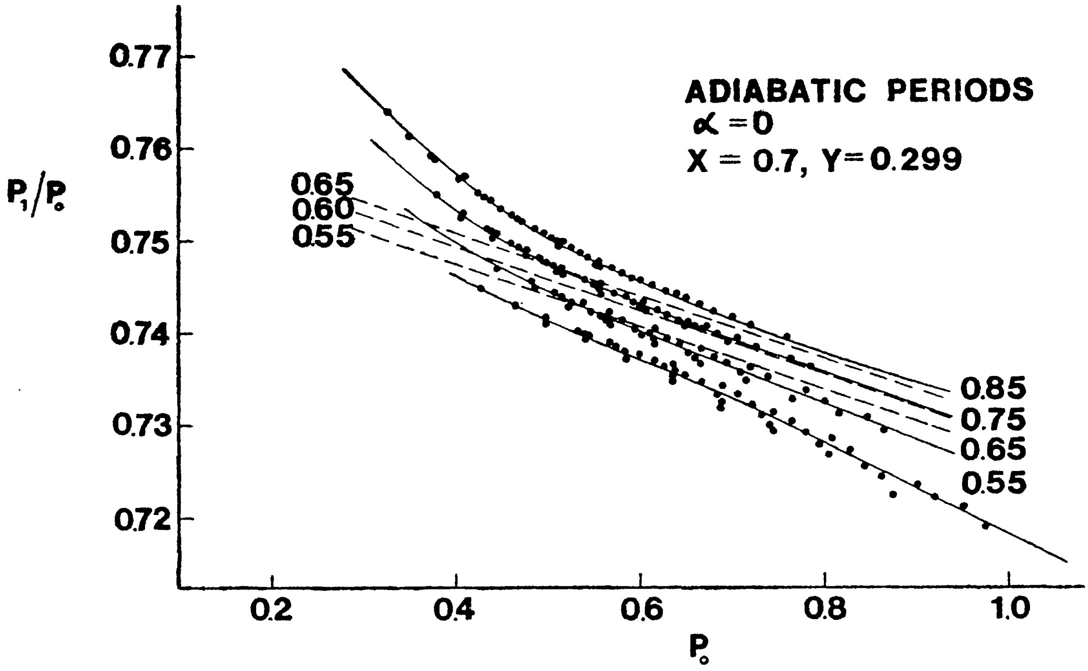

Fig. 4. Comparison of the Petersen-diagrams of CHC (dashed lines) and ours (dots with continuous lines drawn by free-hand). Numbers at the lines denote masses in Solar units. Models with standard steps of parameters (described in the text) were used in the parabolic interpolation to get series with steps 5 and 0.01 in luminosity and log T_e(K) respectively.

ndent, (see however Simon, 1982) it is not sure that small errors do not occur at the level claimed by the inner accuracy of our model construction. Fig. 4 compares our Petersen-diagram obtained for adiabatic periods with $\alpha = 0$, $X = 0.7$, $Y = 0.299$ with that given by CHC. Systematic shift of $0.002 - 0.003$ is observed in the direction of the period ratio coordinate for each constant mass line, giving an average mass of $(0.7 - 0.75)$ $M_\odot$ for double-mode RR Lyrae stars instead of 0.65 $M_\odot$ as given by CHC. It is not known whether this discrepancy is merely due to the different treatment of opacities (i.e. use of Stellingwerf's interpolation formula for the King's mixtures, Stellingwerf, 1975a, b in Dziembowski's code and use of the original tables by CHC). CHC claim that their period ratios for the model series with $M = 0.58\ M_\odot$ agree within 0.001 with those of Stellingwerf (1975a); however, their line for this mass is systematically above the corresponding Stellingwerf's values (see Fig. 2 of Cox, King and Hodson, 1980). Though the effect is smaller than in our case, it is clearly noticeable and might be accounted for by opacity effects. In addition, Jerzykiewicz and Wenzel (1977) get a mass of $0.70\ M_\odot \pm 0.02\ M_\odot$ for AQ Leo using Stellingwerf's (1975a) period fitting formula.

All these facts suggest some systematic difference between the different codes and may invalidate our optimistic estimation of the errors in the mass determination for double-mode RR Lyrae stars. The total inaccuracy of the mass determination due to opacity treatment and/or model construction effects can now be estimated to 10–20 percent.

3. Survey of the resonances

One of the most intriguing and still unsolved problems of the pulsation theory is what causes the observed stable multimode pulsation. The view that double-mode pulsation is simply a transient state from one kind of single-mode pulsation to another has met the problem of the observed stability and large number of double-mode pulsators. The idea of the existence of stable double-mode states has not been confirmed yet by hydrodynamical calculations. Hodson and Cox (1976) have failed to confirm the validity of the mixed-mode model of Stellingwerf (1975b).

It is not known what is the reason for failure in finding stable models of double-mode pulsation. If it is caused by a wrong guess of the model parameters, then any information which can restrict the parameter space is highly valuable. In this respect the resonance hypothesis might give some help.

Dziembowski and Kovács (1984) showed by means of a simplified analysis that stable double-mode pulsators may exist in the "single-mode only" regions if one of the excited modes is in a close 2 : 1 resonance with a higher order damped mode. According to their formalism, Simon's-type three-mode resonances (Simon, 1979) promote single-mode states rather than double-mode ones.

Resonance surveys made up to now were restricted only to beat and dwarf Cepheids and focused the attention mainly on the three-mode resonances (Simon, 1979 and Petersen, 1979), and the bump Cepheids with the 2 : 1 resonances (Simon and Schmidt, 1976 and Simon, 1977). These surveys have shown that the appropriate resonance condition is always nearly satisfied. However, Simon (1979) remarks: "... those models whose interaction frequency was resonant (i.e. models for which $d_3 = 0$) also exhibited two other near resonances, invididually involving the excited modes, and that their period ratios always fell within the following ranges: $0.51 \leqslant P_2/P_0 \leqslant 0.53$ and $0.476 \leqslant P_4/P_1 \leqslant 0.488$. What role, if any, these additional resonances might play in the maintenance of double-mode pulsations is a question that must be left for the future." A similar observation was also made by Petersen (1979).

So, it might be useful to compare the proximities of various resonances in order to determine which one might play the leading role in promoting double-mode pulsation.

By use of the adiabatic periods for our standard series with $\alpha = 0$, $X = 0.7$, $Y = 0.299$ we got by parabolic interpolation a series of model periods with much finer steps in the mass, luminosity and $\log T_e(K)$ (steps of 0.05, 2.5 and 0.005, respectively). Then, models with $0\overset{d}{.}51 \leqslant P_0 \leqslant 0\overset{d}{.}59$, $0.744 \leqslant P_1/P_0 \leqslant 0.748$ were selected and their frequency distances were

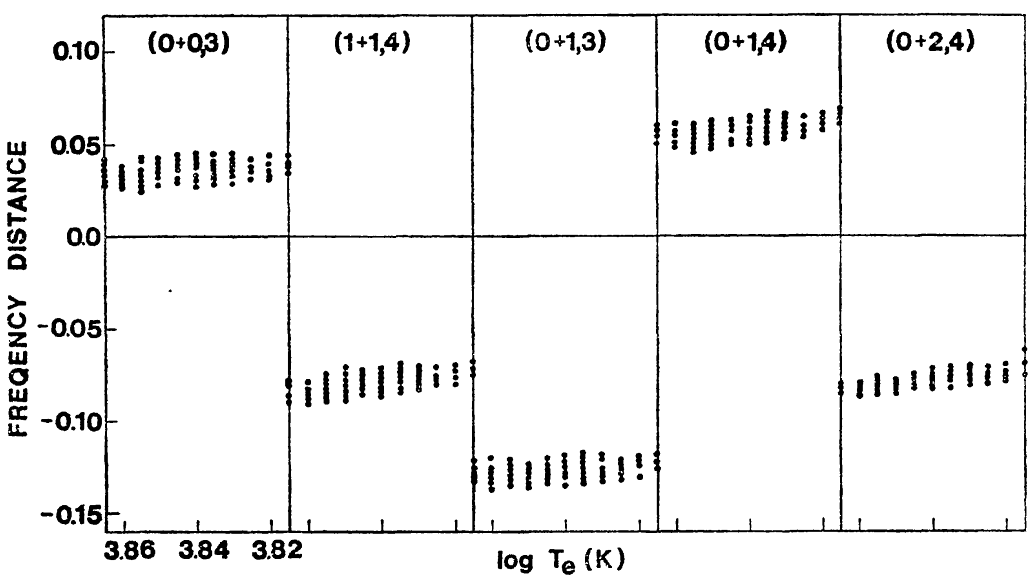

Fig. 5. Adiabatic frequency distances versus log T_e(K) plots for our fine interpolated model series with $0\overset{d}{.}51 \leqslant P_0 \leqslant 0\overset{d}{.}59$, $0.744 \leqslant P_1/P_0 \leqslant 0.748$, $\alpha = 0$, $X = 0.7$, $Y = 0.299$. The type of resonance is shown in the upper part of the figure. The log T_e(K) scale is the same in all boxes as indicated in the first one.

plotted versus effective temperature. Our result is shown in Fig. 5. We used the same notation for the frequency distances as Petersen (1979), i.e.

$$d(i+j,\,k) = 1 - \frac{f_i+f_j}{f_k},$$

where f_i denotes the frequency of the i-th mode.

It is seen that models with appropriate periods and period ratios (dots) display the closest resonance between the fundamental and third overtone. Beside this two-mode resonance, there is a three-mode one between the fundamental, first and fourth overtones which has similar d value. It is also seen that there is no preferred temperature range for the models. A plot on the HR-diagram (Fig. 6), however, shows that there is a well-marked dependence of the allowed luminosities on the effective temperature.

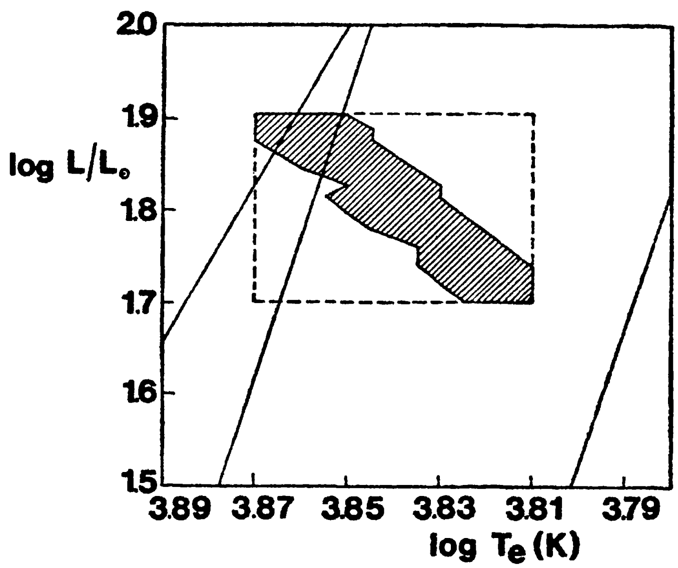

Fig. 6. HR-diagram for adiabatic models with $\alpha = 0$, $X = 0.7$, $Y = 0.299$, $0\overset{d}{.}51 < P_0 < 0\overset{d}{.}59$, $0.744 < P_1/P_0 < 0.748$ (hatched area). Dashed lines show the area tested. Red and blue edges (continuous lines) were taken from Fig. 4 of CHC.

Another representation of our results is shown in Fig. 7. Here, similarly as Simon (1979) and Petersen (1979) did, we plotted the period ratio as a function of various frequency distances. In order to avoid disturbing overlap between the curves of the different resonances the (0+1, 4), (0+2, 4) resonances were not plotted. Some of these curves show remarkably small dispersion, giving a better than ± 0.003, ± 0.002 correspondence between the frequency distances and period ratios respectively.

It is clear, that the requirement of perfect resonance of any kind gives quite wrong period ratio. In contrast to Cepheids (Simon, 1979) in double-mode RR Lyrae stars we always encounter only approximate resonances. This problem will be discussed further in the next section.

So far, we examined only adiabatic periods with one chemical composition and neglected convection. Now, the question is how the frequency distances change by allowing some variations of the above parameters. This problem was studied in the same manner as that of the period ratios. In summary, except for non-adiabaticity, none of the parameter changes in the ranges mentioned in Section 2 resulted in systematic or significant (> 0.001) changes in the frequency distances. Non-adiabaticity, however, caused a decrease of the fundamental and first overtone periods and an increase of the periods of higher-order modes. This effect resulted in con-

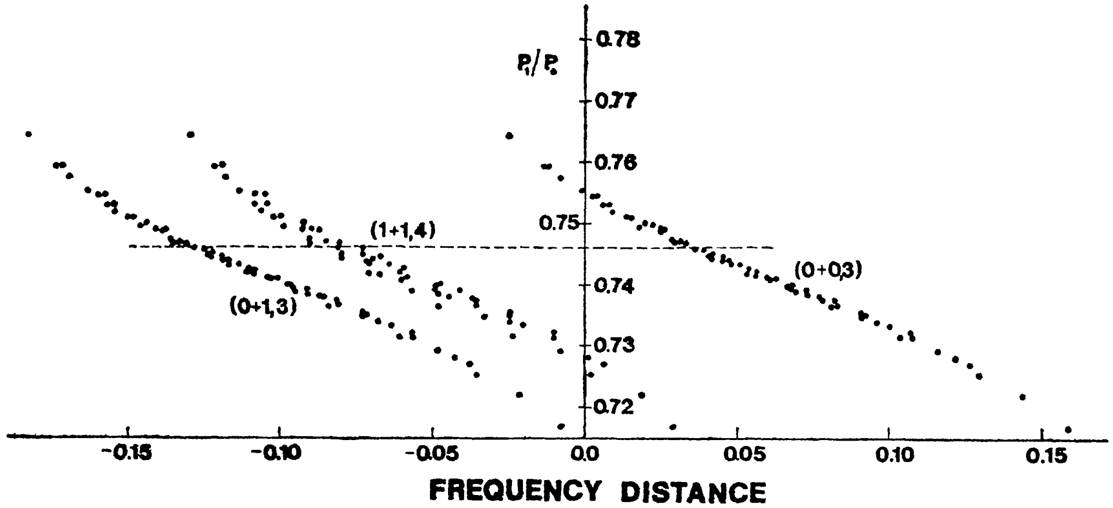

Fig. 7. Period ratio versus frequency distances plot. Adiabatic periods for our standard series with $a = 0$, $X = 0.7$, $Y = 0.299$ were used. The period ratio of the double-mode RR Lyrae stars is shown by dashed line.

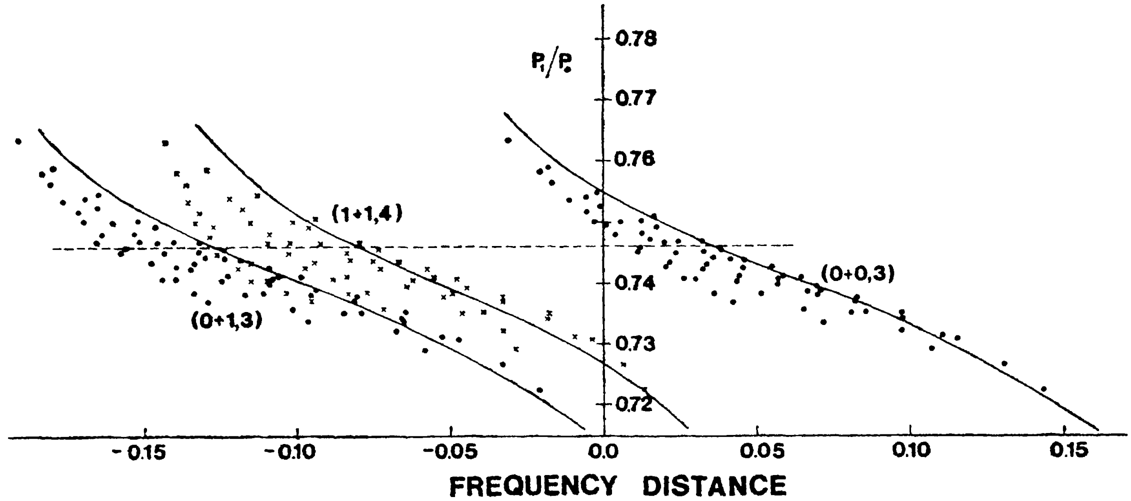

Fig. 8. Same as in Fig. 7, but for non-adiabatic periods. Dots denote the $(0+0,3)$ and $(0+1,3)$ resonances, crosses are for the $(1+1,4)$ one. For comparison, the free-hand drawn version of the adiabatic result shown in Fig. 7 is also plotted.

siderable changes in the frequency distances, while leaving the period ratios practically unaltered. We compare the adiabatic and non-adiabatic frequency distances with the aid of the period ratio versus frequency distances plot in Fig. 8. Again, to avoid confusion the (0+1, 4), (0+2, 4) resonances were not plotted. It is seen that non-adiabaticity causes a significant broadening of the lines and the (0+0, 3) resonance becomes clearly the closest one. The (0+1, 4) and (0+2, 4) resonances behave in a similar manner than the (0+0, 3) and (1+1, 4) ones respectively, however, the (0+1, 4) resonance remains always weaker than the (0+0, 3) one.

Finally, we give the parameters of some representative models with proper periods and small $d(0+0, 3)$ values in Table 1.

Table 1

Representative models of double-mode RR Lyrae stars with small $d(0+0,3)$ values. M and L are measured in Solar units, periods are given in days. Consecutive numbers in each column denote the corresponding adiabatic, non-adiabatic values and growth rates

Model No.	M	L	log T_e(K)	P_0	P_1	P_2	P_3	P_4	P_1/P_0	d(0+0,3)
1.	0.75	60.	3.83	0.5546	0.4132	0.3293	0.2658	0.2217	0.745	0.041
				0.5537	0.4126	0.3305	0.2676	0.2236	0.745	0.033
				+0.015	+0.062	+0.023	−0.084	−0.189		
2.	0.75	70.	3.85	0.5396	0.4025	0.3216	0.2602	0.2176	0.747	0.036
				0.5384	0.4010	0.3225	0.2627	0.2210	0.745	0.024
				+0.003	+0.038	+0.003	−0.141	−0.289		
3.	0.85	70.	3.83	0.5809	0.4338	0.3472	0.2809	0.2345	0.747	0.033
				0.5800	0.4329	0.3483	0.2827	0.2363	0.746	0.025
				+0.012	+0.053	+0.031	−0.070	−0.176		
4.	0.85	80.	3.85	0.5557	0.4157	0.3337	0.2706	0.2265	0.748	0.026
				0.5546	0.4141	0.3343	0.2730	0.2297	0.747	0.016
				+0.002	+0.032	+0.013	−0.115	−0.262		

4. Discussion

It is clear that the present survey did not result in a significant restriction of the range of parameters for double-mode pulsation. Relying only on periods and period ratios we got a relatively large area in the HR-diagram occupied by appropriate models. Of course, it is possible to restrict this area further by using the observed luminosities (as actually CHC did), but one should be aware of large observational errors involved.

Another way of restricting parameter ranges is to make use of the resonance hypothesis and select models fitting the desired resonance condition in the best way. In the case of Cepheids, this method has not resulted in a parameter restriction, as all models with appropriate period ratios showed a rather close (0+1, 3) resonance (Simon, 1979). If in the case of double-mode RR Lyrae stars we turn to the probably more important 2 : 1 resonance between one of the excited mode and a higher order overtone we might restrict the parameter ranges. However, due to the conflict between the observed periods and the avoided crossing of resonance center and the presence of other resonances of similar proximity, this process can be questioned. In the light of these, one may resist to attribute any importance to the resonance hypothesis. It is clear that our linear pulsation models cannot be as wrong as the requirement of perfect resonance (of any kind) would demand. It is possible, however, that the mechanism of resonant mode coupling requires non-perfect resonance (inclusion of non-adiabatic resonant coupling for example, may cause such an effect). Though we must wait for the final conclusion until thorough hydrodynamical modelling, it is thought that the resonance playing the leading role in double-mode pulsation comes from the closest ones. In this frame, the present survey clearly supports the resonance hypothesis.

It was seen in the previous section that the (0+0, 3) and (0+1, 4) resonances are most likely. Relying upon the result of Dziembowski and Kovács (1984) and treating the two resonances separately, we must consider only the (0+0, 3) one because three-mode resonances are unimportant in maintaining double-mode pulsation. In this frame we may speculate on the predictions of the above two-mode resonance.

Because the fundamental mode is involved in the resonance, it is expected that the star is located near to the non-resonant "either-or" → → "fundamental only" transition. If the positions of Stellingwerf's nonlinear transition lines given for 0.578 $M_\odot$ do not depend too much on stellar masses, a star with log $L/L_\odot = 1.78$, log $T_e(\mathrm{K}) = 3.84$ (i.e. with the medium parameters for double-mode RR Lyrae stars given by CHC) might be really close to this transition, actually much closer than to the other one. In contrast to the mode switching hypothesis, sustained double-mode pulsation with two-mode resonance does not require evolution in a certain direction. It is necesasary only to have a close resonance with a damped higher order mode and to satisfy some other not very restricting conditions (for the details see Dziembowski and Kovács, 1984). The required proximity of the resonance depends mainly on how far the star is from the transition line (the closer to the transition line the weaker is the resonance required).

It is seen that our guess of the possible importance of the (0+0, 3) resonance is not in contradiction with the behaviour suggested by nonlinear results. There may, of course, exist several effects which may help or hinder the work of this resonance. Non-adiabaticity helps, as it was shown in the previous section. Presence of three-mode resonance might hinder, as it promotes single-mode states. Other effects (non-adiabatic resonant coupling, multimode resonant and non-resonant interaction) not included in the resonance theory yet might modify but not invalidate the suggested importance of the 2 : 1 resonance.

Finally, it seems to be worthwhile to remark that a proper inclusion of resonances in the non-linear hydrodynamical computations require the consideration of sufficiently deep envelopes. Though our $M = 0.65\,M_{\odot}$, $L = 60L_{\odot}$, $\log T_e(\mathrm{K}) = 3.84$, $\alpha = 0$, $X = 0.7$, $Y = 0.299$ model yielded not very discordant periods when we lowered the envelope mass from 10 to 2 percent of the total mass (period ratios decreased by 0.0005, frequency distances increased by 0.004), this was not the case with Stellingwerf's (1975a) mixed-mode model (model 2.6 in his series). Again, decreasing the fractional envelope mass from 10 to 1 percent, the period ratio increased by 0.003, while frequency distances decreased by 0.03–0.04. During this, the $(1+1, 4)$ resonance went through zero, while the others remained in absolute value greater than 0.02 and, the $(0+0, 3)$ resonance greater than 0.11. As the star is at the very cool end of the "fundamental only" region, the possible double-mode state promoted by the 2 : 1 resonance involving the first overtone is unstable. The $(0+0, 3)$ resonance is also unable to maintain double-mode pulsation because of its very large d value. Our guess for the failure to repeat Stellingwerf's results is that his model was either started in the tiny region of the occasionally existing stable three-mode equilibrium state in the three-mode resonance case or slided by chance to this state (see Dziembowski and Kovács, 1984).

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REFERENCES

Antonello, E. and Mantegazza, L., 1984, *Astron. Astrophys.*, **133**, 52.

Breger, M., 1979, *Publ. Astr. Soc. Pacific*, **91**, 5.

Cox, A. N., 1982, *Pulsation in Classical and Cataclysmic Variable Stars*, p. 157. Papers presented at a conference held in Boulder, Colorado, June 1–4, 1982, eds. Cox, J. P. and Hansen, C. J.

Cox, A. N., Hodson, S. W. and Clancy, S. P., 1981, *Astrophysical Parameters for Globular Clusters*, p. 337. IAU Coll. No. 68.

– 1983, *Ap. J.*, **266**, 94.

Cox, A. N., King, D. S. and Hodson, S. W., 1980, *Ap. J.*, **236**, 219.

Dziembowski, W., 1977, *Acta Astr.*, **27**, 95.
Dziembowski, W. and Kovács, G., 1984, *M. N. R. A. S.*, **206**, 497.
Fitch, W. S. and Szeidl, B., 1976, *Ap. J.*, **203**, 616.
Hodson, S. W., and Cox, A. N., 1976, *Proceedings of the Solar and Stellar Pulsation Conference*, eds. Cox, A. N. and Deupree, R. G., LASL: LA-6544c, p. 202.
Jerzykiewicz, M. and Wenzel, W., 1977, *Acta Astr.*, **27**, 35.
Petersen, J. O., 1973, *Astron. Astrophys.*, **27**, 89.
— 1978, *ibid.* **62**, 205.
— 1979, *ibid.*, **80**, 53.
Sandage, A., Katem, B. and Sandage, M., 1981, *Ap. J. Suppl.*, **46**, 41.
Simon, N. R., 1977, *Ap. J.*, **217**, 160.
— 1979, *Astron. Astrophys.*, **75**, 140.
— 1982, *Ap. J. Letters*, **260**, L87.
Simon, N. R. and Schmidt, E. G., 1976, *Ap. J.*, **205**, 162.
Stellingwerf, R. F., 1975a, *Ap. J.*, **195**, 441.
— 1975b, *ibid.*, **199**, 705.
Stobie, R. S., 1977, *M. N. R. A. S.*, **180**, 631.