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MILGROM'S REVISION OF NEWTON'S LAWS: DYNAMICAL AND COSMOLOGICAL CONSEQUENCES

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ABSTRACT

Milgrom's recent revision of Newtonian dynamics was introduced to eliminate the inference that large quantities of invisible mass exist in galaxies. I show by simple examples that a Milgrom acceleration, in the form presented so far, implies other far-reaching changes in dynamics. The momentum of an isolated system is not conserved, and the usual theorem for center-of-mass motion of any system does not hold. Naive applications require extreme caution. The model fails to provide a complete description of particle dynamics and should be thought of as a revision of Kepler's laws rather than Newton's.

The Milgrom acceleration also implies fundamental changes in cosmology. A quasi-Newtonian calculation adapted from Newtonian cosmology suggests that a "Milgrom universe" will recollapse even if the classical closure parameter Ω is $\ll 1$. The solution, however, fails to satisfy the cosmological principle. I examine reasons for the breakdown of this calculation. A new theory of gravitation will be needed before the behavior of a Milgrom universe can be predicted.

Subject headings: cosmology — galaxies: clustering — galaxies: internal motions — stars: stellar dynamics

I. MILGROM ACCELERATION

The astronomical evidence (Faber and Gallagher 1979; Davis *et al.* 1980) for large amounts of "invisible mass" (essentially, mass not contained in luminous stars) in and around galaxies comes almost entirely from applications of Newton's second law to galaxies and galaxy systems. The accelerations in these systems are much smaller than those for which the law has been tested in the laboratory or in the solar system.

In three unorthodox papers, Milgrom (1983*a*, *b*, *c*) has proposed to do away with invisible mass by altering the second law. *Inter alia*, he proposes that, at least with respect to gravitational forces, Newton's $a = F_N/m$ should be replaced, in the low-acceleration limit, by

$$a \approx \sqrt{\left(a_0 \frac{F_N}{m}\right)} \equiv \sqrt{(a_0 g_N)} \ . \tag{1}$$

Here g_N is the Newtonian acceleration calculated from the mass distribution in the usual way, and a_0 is a new physical constant having dimensions of acceleration. Law (1) is assumed to apply when the true acceleration a is $\ll a_0$. Milgrom finds that he can explain the flat rotation curves of galaxies and large virial velocities in clusters without adding invisible mass. By equation (1), small accelerations due to a given galaxy mass are larger than they would be in Newtonian theory. He can also avoid smaller scale observational limits on non-Newtonian forces, e.g., from solar system observations. All this is possible provided

$$a_0 \approx 8 \times 10^{-8} h^2 \text{ cm s}^{-2} \approx (\tfrac{4}{5}h)cH_0 \ , \tag{2}$$

where H_0 is the Hubble constant, and h is the dimensionless Hubble constant $H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$, of order unity. The numerical value 8×10^{-8} in equation (2) is roughly equal to the Newtonian acceleration occurring in the inner parts of a galaxy. In the outer parts, then, $a < a_0$, and Milgrom's law (1) comes into play.

The last form of equation (2), relating a_0 to cH_0, is interesting. A "cosmic acceleration" cH_0 would reduce a speed c to speed zero in a Hubble time. From Milgrom's point of view, relation (2) is fortuitous and adds one more to the list of "numerical coincidences" in cosmology (Bondi 1960, § 7.1). It suggests vaguely a Machian basis for the Milgrom acceleration and the constant a_0 and is therefore regarded as an asset to the model.

II. DYNAMICAL PROBLEMS

Without dismissing Milgrom's ideas out of hand, I wish to point out that a dynamical law of type (1) has other consequences (some undesirable) in addition to the consequences sought by Milgrom. Note at once that the accelerations a given to a test particle by two or more attracting bodies acting jointly do not add linearly; the Newtonian accelerations g_N do add linearly, so their square roots cannot do so. Next consider the dynamics of some simple multiparticle systems. Assume that equation (1) gives the correct dynamics for particles with low acceleration. Consider a system consisting of two particles only, with masses m_1 and m_2, interacting gravitationally. Let them be placed at rest on the x-axis, with $x_2 - x_1 \equiv r > 0$. Let m_1, m_2 be small enough so that equation (1) applies. Set $F_N = Gm_1m_2/r^2$. Note that the masses are constant, for law (1) is intended to apply only in the nonrelativistic limit. Differentiating the total momentum $p \equiv p_1 + p_2$ and using equation (1), we find

$$\dot{p} \equiv \frac{dp}{dt} = (a_0 F_N)^{1/2}(m_1{}^{1/2} - m_2{}^{1/2}) \ . \tag{3}$$

When $m_1 \neq m_2$, p for this isolated system is not conserved. This is obvious from equation (1), because the two accelerations are not inversely proportional to the masses as in Newtonian dynamics. Except for the special case $m_1 \approx m_2$, $|\dot{p}|$ is of the same order as the larger of $|\dot{p}_1|$ and $|\dot{p}_2|$.

Consider a second simple case: a system S consisting of two particles m_1 and m_2, S being placed in the gravitational field of a third and larger body m_3. Place these objects at rest on the x-axis, with $x_2 - x_1 \equiv r > 0$ and $x_3 - x_2 \equiv R > 0$. Let m_1 and m_2 be small enough so that the interaction of m_1 and m_2 can be neglected, and let R be large enough so that law (1) describes the motion of m_1 and m_2 in the field of m_3; i.e., the accelerations of m_1 and m_2, respectively, in the field of m_3 are given by equation (1), with

$$\frac{F_N}{m} = \frac{F_{Ni}}{m_i} = g_{Ni} \quad (i = 1, 2)\,, \tag{4}$$

where g_{Ni} is the Newtonian acceleration produced by m_3 at particle i. The center of mass of S is

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\,. \tag{5}$$

Differentiating and using equation (1), we find that the acceleration of the center of mass is

$$\ddot{x}_{CM} = \frac{\sqrt{a_0}}{m_1 + m_2}\,(m_1\sqrt{g_{N1}} + m_2\sqrt{g_{N2}})\,. \tag{6}$$

Let us ask whether the usual Newtonian theorem on motion of the center of mass is valid for Milgrom accelerations. The external forces on system S are F_{N1} and F_{N2}. Therefore, the center-of-mass theorem applied to law (1) would say

$$\begin{aligned}\ddot{x}_{CM} &= \sqrt{\left(a_0\,\frac{F_{N1} + F_{N2}}{m_1 + m_2}\right)} \\ &= \sqrt{\left(\frac{a_0}{m_1 + m_2}\right)}\sqrt{(m_1 g_{N1} + m_2 g_{N2})}\,.\end{aligned} \tag{7}$$

A little algebra shows that expressions (6) and (7) are equal only if $g_{N1} = g_{N2}$. In general, then, if individual test particles in an external gravitational field obey law (1), their center of mass obeys the same law only if the external field is uniform. Equation (3) shows in addition that even in the case of zero external field, the center-of-mass theorem fails if the particles have active gravitational mass too large to be neglected.

Astronomical data describe multiparticle systems. Milgrom therefore assumed explicitly that law (1) applies to the center of mass. But the examples above show that if law (1) applies to individual particles, it cannot in general apply to the center of mass. It can, however, apply to the center of mass in the special case of a system of one or more test particles (particles having negligible active mass) moving in a uniform external field. The galaxy systems studied by Milgrom are in general bound by their own active mass, and it is not altogether clear that a test-particle approximation is justified. If we simply postulate with Milgrom that law (1) applies generally to the center of mass, then it cannot apply in general to individual particles.

Milgrom dynamics is therefore incomplete at present and gives no clear prescription for particle motions. Naive applications require extreme caution. More or less extensive changes in many-body dynamics are implied and cannot be predicted without a more complete theory. At present Milgrom's law must be thought of as a phenomenological modification of Kepler's laws rather than a systematic modification of Newtonian dynamics. Milgrom (1983*a*) is aware of drawbacks in the theory and discussed at some length another problem, namely, an apparent violation of the principle of equivalence (a freely falling elevator is not equivalent to an unaccelerated inertial frame). A more complete theory is in course of development (Bekenstein and Milgrom 1984).

III. COSMOLOGICAL IMPLICATIONS

Despite these difficulties, it is of some interest to speculate on the cosmological consequences of a Milgrom-type dynamics. Consider the classical Friedmann universes (Friedmann 1922; Rindler 1977, §§ 9.9–9.11), i.e., relativistic universes of zero pressure, and set the cosmological constant $\Lambda = 0$. The classical "closure parameter" is

$$\Omega \equiv \frac{\rho}{\rho_{cl}} = \frac{8\pi G\rho}{3{H_0}^2}\,, \tag{8}$$

where ρ is the universal mass density. If $\Omega < 1$, the universe is open and expands forever. If law (1) holds, Milgrom shows that most, if not all, galaxy velocity data can be interpreted with mass-to-luminosity ratios for galaxies equal to roughly 1–10 in solar units. (This M/L ratio measures all forms of mass which are clumped with the galaxies; uniformly distributed mass is, as always, excluded, but there is no evidence for its presence.) A ratio $M/L \sim 1$–4 is roughly characteristic of stellar matter (Faber and Gallagher 1979), so there is no longer strong evidence for substantial quantities of mass other than that contained in stars. The mean M/L required to give $\Omega = 1$ (Davis *et al.* 1980; Felten 1977) is much larger, $\sim 1400h$. We conclude that $\Omega \ll 1$.

However, we cannot conclude that the universe will expand forever, because a Milgrom acceleration implies extensive changes in cosmology. To show this, I adapt a familiar argument (McCrea and Milne 1934; Bondi 1960, §§ 9.1–9.3; Peebles 1971, § I*b*) from "Newtonian cosmology." Consider a comoving sphere of radius $r(t)$ in the uniform cosmological fluid. Within a finite, spherically symmetric system, however large, Gauss's law holds in the Newtonian case and tells us that the acceleration of a point on the surface of the sphere depends only upon the mass within. We have

$$g_N = \ddot{r} = -\frac{G}{r^2}\frac{4\pi {r_0}^3\rho}{3} = -\frac{\Omega {H_0}^2 {r_0}^3}{2r^2}\,, \tag{9}$$

where ρ is the *present* density (a constant), and r_0 is the present radius of our chosen sphere. Integration of this equation gives the exact Friedmann equation for the universal scale factor $R(t)$, from which we conclude that a zero-pressure universe with $\Lambda = 0$ recollapses only for $\Omega > 1$.

It is easy to adapt this argument to a Milgrom acceleration, because equation (1) depends only upon g_N. In place of equation (9), we obtain, for Milgrom acceleration,

$$\ddot{r} = -\left(a_0\,\frac{\Omega {H_0}^2 {r_0}^3}{2r^2}\right)^{1/2}\,, \tag{10}$$

assuming for the moment that $|\ddot{r}| \ll a_0$, so that equation (1) applies. Integrating equation (10) once, we find that

$$(\dot{r})^2 = (\dot{r})_0^{\,2} - (2\Omega {H_0}^2 a_0 {r_0}^3)^{1/2} \ln (r/r_0)\,. \tag{11}$$

For any assumed value of the present expansion velocity $(\dot{r})_0$, $\dot{r}(t)$ necessarily has a zero; i.e., there is a maximum radius and a turnaround. Introducing the usual notation $r \equiv r_0 R$ and imposing the initial condition $H_0 = (\dot{r}/r)_0$, we find that the scale factor at turnaround is

$$R_{max} \equiv r_{max} {r_0}^{-1} = \exp \mu^2\,, \tag{12}$$

where

$$\mu \equiv \left(\frac{H_0{}^2 r_0}{2\Omega a_0}\right)^{1/4} . \tag{13}$$

Substituting $z \equiv R_{\max}/R$ and integrating again, we find that the turnaround occurs after a time

$$\Delta t \equiv t_{\max} - t_0 = \frac{\mu}{H_0} R_{\max} \int_1^{R_{\max}} \frac{dz}{z^2(\ln z)^{1/2}} . \tag{14}$$

The integral is an incomplete gamma function $\gamma(\frac{1}{2}, \ln R_{\max})$ and is equal to $\pi^{1/2}\mathrm{erf}\,\mu$ (Abramowitz and Stegun 1964). Therefore,

$$\Delta t \approx \pi^{1/2}\mu H_0{}^{-1} R_{\max} \qquad \text{if} \qquad R_{\max} \gg 1 . \tag{15}$$

Having carried the Milgrom constant a_0 along explicitly, let us now set $a_0 \sim cH_0$ as in Milgrom's theory. For a cosmological turnaround we set $r_0 \sim r_{\mathrm{H}} \equiv c/H_0$. (Note that, for $\Omega \ll 1$ and $a_0 \sim cH_0$, the acceleration [10] is and remains $\ll a_0$ for all values of $r_0 \lesssim r_{\mathrm{H}}$, so law [1] should in fact apply. We have neglected any corrections to law [1] associated with $v \to c$; these corrections should not dominate for $r_0 \lesssim r_{\mathrm{H}}$ and should certainly be small for $r_0 \ll r_{\mathrm{H}}$.) Then $\mu \sim (2\Omega)^{-1/4}$. The turnaround time depends on Ω, i.e., on the assumed present density ρ. For $\Omega = 10^{-2}$ we find $\mu \sim 2.7$, $R_{\max} \sim 1200$, and $\Delta t \sim 5 \times 10^{13}$ yr. For $\Omega = 10^{-3}$ the numbers are $\mu \sim 4.7$, $R_{\max} \sim 5 \times 10^9$, and $\Delta t \sim 4 \times 10^{20}$ yr. Thus it seems that Milgrom universes of all densities recollapse, although the time required is very long for $\Omega \ll 1$. This result is the cosmic analog of Milgrom's (1983*a*) observation that the effective Milgrom potential of a point mass is logarithmic at large distances.

This derivation is a fraud, although it points the way to further work. Note the presence of r_0 in equation (13). There should be no need to set r_0 equal to r_{H} or to any other specific value. In the analogous Newtonian derivation, r_0 drops out as soon as the initial-value condition $H_0 = (\dot{r}/r)_0$ is applied. The presence of r_0 in equation (13) shows that $R_{\max}$ and Δt are different for spheres of different initial size. Therefore, the "universe" described by equations (10)–(15) does not admit a universal scale factor $R(t)$ and, by the theorem of Robertson and Walker (Rindler 1977, § 9.5), cannot be homogeneous and isotropic. To improve this situation we might entertain the obvious possibility, suggested by Milgrom (1983*c*), that the constant a_0 is variable on a cosmic time scale. For example, we might set $a_0 \sim c\dot{r}/r$ instead of $a_0 \sim cH_0$. It may be evident, however, that this does not help. The parameter r_0 still fails to drop out of the equations, and we are still stuck with a universe which does not satisfy the cosmological principle.

To understand this paradox we must think about the logical basis of "Newtonian cosmology." There is an extensive literature on this (e.g., Layzer 1954; McCrea 1954; Raychaudhuri 1979, §§ 2.1–2.3), centering on the objection that the Newtonian potential is infinite in an infinite sea of mass. Two schools of thought have emerged to explain the remarkable success of the Newtonian calculation. The first school (Heckmann and Schücking 1959; Bondi 1960, §§ 9.1–9.3; Rindler 1977, §§ 9.2, 9.8) points out that the Newtonian argument leading to equation (9) above can be carried out within any finite sphere of mass, however large, and that the solution in an infinite sea must be the limit of the finite-sphere solutions and must therefore be the same. One might object that an infinite sea is also the limit of a finite cube or ellipsoid, and that these shapes certainly will give different solutions. The answer is that these solutions, unlike that in the spherical case, are not homogeneous and isotropic. It is argued that requiring the cosmological principle to be satisfied makes the large but finite sphere a unique and correct Newtonian model for cosmology. Thus in Newtonian cosmology the cosmological principle plays the role of, or replaces, a boundary condition. This justification cannot be extended to the "Milgrom universe" developed in equations (10)–(15). These equations describe a large, finite spherical universe, but observers within this sphere do not find its motion isotropic and homogeneous. It appears that a solution for a finite sphere satisfying the cosmological principle and equation (1) cannot be obtained.

The second school of thought on Newtonian cosmology (Callan, Dicke, and Peebles 1965; Peebles 1971, § I*b*; Weinberg 1972, § 15.1) appeals to Birkhoff's theorem in general relativity (Lemaître 1931; Bonnor 1962). Birkhoff's theorem, the relativistic analog of Gauss's law, implies that the four-dimensional curvature is zero inside a concentric spherical cavity in a spherically symmetric mass distribution. This means that the gravitational field at the surface of a uniform sphere depends only upon the mass within the sphere and may be calculated by the Newtonian approximation when the sphere is small enough. The well-known weak-field linearity of general relativity is used implicitly in reaching this conclusion. This argument provides the strongest justification for the Newtonian calculation, although it is not a Newtonian argument. Once again we find that the argument breaks down when applied to a quasi-Newtonian calculation with a Milgrom acceleration. In a Milgrom universe, a new theory of gravitation will have to be found, which reduces to a Milgrom acceleration in certain limits, for general relativity does not. Birkhoff's theorem may not be valid in such a theory of gravitation. Weak-field linearity will certainly fail, because equation (1) is nonlinear, as noted earlier.

The failure of Birkhoff's theorem would not be a fatal objection to a gravitational theory; in fact, some might see it as an advantage. Birkhoff's theorem, which in effect limits the connection between local and global phenomena, limits general relativity's ability to account for numerical coincidences in cosmology. Relationships such as Milgrom's $a_0 \sim cH_0$ might more readily be explained in the absence of Birkhoff's theorem. It is clear, however, that cosmology will be in a state of confusion if a Milgrom-type acceleration is verified. Naive intuition suggests that all universes may indeed recollapse because of the long-range character of the force, but we do not know how to prove this. A new theory of gravitation will be needed. This difficult task should perhaps not be undertaken unless there is clear evidence that the Milgrom acceleration law holds on the scale of galaxies. Dressler and Lecar (1983) suggest that Milgrom's law does not explain the velocity data adequately.

Note added in manuscript 1984 June 6.—Finzi (1963*a*, *b*) proposed a gravitational force varying like $r^{-n}(n < 2)$ for r larger than a characteristic distance r_0. This is related but not identical to Milgrom's idea. Tohline (1983, 1984) showed that for $n = 1$ such a force can stabilize the disks of galaxies. Yabushita (1964) had objected that large random cosmic accelerations arise from such a force, a difficulty which will be absent or much reduced in Milgrom's theory. I am grateful to V. C. Rubin and J. E. Tohline for providing these references. Milgrom (1983*b*) argued that a force law of the type favored by Finzi and Tohline can be ruled out by the Tully-Fisher relationship.

It is of some interest that the "cosmology" of equations (10)–(15) possesses a natural length scale r_c. This may be defined as the present radius of the shell which is now turning around, i.e., the largest shell which has been perturbed substantially out of the Hubble flow and has begun to recollapse. Smaller shells are further advanced in recollapse; larger shells still have the Hubble motion. We may estimate r_c very roughly by setting $\Delta t = H_0^{-1}$ in equation (15) and solving equations (15) and (13) for r_0. The result, for $a_0 \sim cH_0$, is $r_c \sim 10^3\ \Omega h^{-1}$ Mpc, i.e., 10 Mpc for $\Omega = 10^{-2}$. A better estimate, done retrospectively rather than prospectively, gives $r_c \sim 140h^{-1}$ Mpc for $\Omega = 10^{-2}$ and $r_c \sim 30h^{-1}$ Mpc for $\Omega = 10^{-3}$. (A more accurate calculation cannot be done until the new law of acceleration is specified for accelerations $\sim a_0$.) It is curious that these estimates agree roughly with the largest scale of observed inhomogeneities in the universe. Note that this simple cosmology does not describe a universe with random lumps; what it describes is a single condensation developing around the observer. But if a new theory of gravitation, taking proper account of the influence of distant matter, could nevertheless preserve this natural length scale of the simplest Milgrom cosmology, it might have application to inhomogeneities in the real universe.

Bekenstein and Milgrom (1984) propose a new and more complicated classical potential equation for gravitational acceleration to replace the Poisson equation. Their equation reduces to Milgrom's law (1) above in cases of high symmetry. Their theory, however, preserves the law $F = ma$, essentially by making it a definition of force. In this way they avoid the difficulties discussed in § II above. On the other hand, the calculation of force becomes more difficult. The force can no longer be obtained from Newton's law of gravitation, but only by solving the potential equation.

The discussion of cosmology in § III above applies fully to the classical Bekenstein-Milgrom potential equation, and one can conclude that the equation possesses no homogeneous isotropic dynamical solution. In an appendix, Bekenstein and Milgrom sketch a new gravitational theory to replace general relativity. It will be of interest to see whether these equations possess any cosmologically useful solution.

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