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THERMODYNAMICS AND GALAXY CLUSTERING: NONLINEAR THEORY OF HIGH ORDER CORRELATIONS

WILLIAM C. SASLAW
Department of Astronomy, University of Virginia; Institute of Astronomy, Cambridge, England; and National Radio Astronomy Observatory,[1] Charlottesville

AND

ANDREW J. S. HAMILTON
Department of Astronomy, University of Virginia
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ABSTRACT

We develop a new theory for galaxy clustering in an expanding universe. It is based on the thermodynamics of gravitating systems and applies to the highly nonlinear regime of strong clustering. There are no free parameters in the simplest form of this theory. It predicts distribution functions of all orders, from voids to hundreds of galaxies.

Comparison of these predictions with the results of numerical N-body experiments shows substantial agreement. Comparison with the observed distribution of galaxies may determine whether it has unrelaxed structure that retains information from much earlier epochs of the universe.

Subject headings: cosmology — galaxies: clustering — hydrodynamics

I. INTRODUCTION, MAIN RESULT, AND A BASIC QUESTION

One of the fundamental challenges of gravitational many-body physics is to understand nonlinear clustering in an infinite gravitating system. The distribution of galaxies throughout the universe is the main manifestation of this process. However, high order clustering is sufficiently basic to be important in its own right.

This problem goes back to Newton (1692) who, in a famous letter to Bentley, surmised that the distribution of gravitating bodies in an infinite universe "could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered at great distances from one to another throughout all that infinite space."

There the problem rested until Jeans (1929), Lifshitz (1946), Bonnor (1957), and subsequently many others calculated the instability of a continuous gravitating *fluid* which was initially uniform throughout the universe. Van Albada (1960) seems to have been the first to return to Bentley's original question and calculate the linear behavior of a uniform distribution of *discrete* gravitating bodies. These investigations were all concerned with the development of the single-particle distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$. The fluid calculations are confined to the low-order moments of $f(\boldsymbol{v}, \boldsymbol{r}, t)$ suitable for a gravitating atomic gas. For discrete bodies, an analysis (Saslaw 1972) involving the two and three particle distribution function, and their related two-particle correlations, uncovered new more rapidly growing instabilities caused directly by the graininess of the gravitational field.

Most work on galaxy clustering during the past few years has been concerned with this two-particle correlation function. It is one of the most naturally defined properties of the galaxy distribution which can be observed fairly easily (see Peebles 1980 for a detailed review). Two-particle correlations are also amenable to analytic treatment in both the linear and nonlinear regimes, as well as to N-body simulations (summarized for example in Saslaw 1980*a*). Although the two-particle correlation function provides much useful information about N-body systems, it is usually thought of as just the first step in a hierarchy of correlation functions—the three, four, five, etc., particle correlations. These higher order correlations describe groups and larger clusters with many members. A chain of dynamical equations, the BBGKY hierarchy, relates the time evolution and spatial dependence of the nth order correlation function to that of the $(n + 1)$st function. This leads to a coupled series of nonlinear integro-differential equations. To provide a complete description, equivalent to Liouville's equation, requires as many equations as there are particles in the system. It is possible to solve this system for the early growth of the two-particle correlation function, starting from a completely uncorrelated state. But when the correlations become of order unity, the mathematical structure becomes too complicated to solve, even numerically. Some attempts, relying on gross simplifications, have been made to extend this approach to the three and higher order particle correlation functions. Since the difficulties of climbing to even the third rung of the BBGKY ladder have not been surmounted, this approach does not seem very promising.

And yet, the two-particle correlation function itself contains much information about larger scale clustering, for it evolves self-consistently with all the higher order correlation functions. The problem is how to extract this information about higher order correlations from the two-particle function, without having to solve the whole hierarchy (or indeed any of it!). In this paper we describe a new approach which shows how this may be done.

We work with a particularly simple measure of statistical

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clustering which is closely related to the standard correlation functions. This is the probability, $f(N)$, of finding N galaxies in a volume V. We show how to calculate the form of $f(N)$ analytically, and then compare it with the results of N-body experiments. Moreover, it is not too difficult to determine $f(N)$ observationally. The results can help to answer the question: How much structural information from the very early universe survives in the present distribution of galaxies?

Our theory is based on the well-known result (e.g., Hill 1956) that in a system whose members interact in pairs, as with the gravitational potential, the two-particle correlation function completely determines the thermodynamic properties. Among these properties are equations of state which include gravitation. These equations of state, in turn, determine the equilibrium fluctuations. From the fluctuations we extract distribution functions for the probability that a volume V of arbitrary shape contains just N galaxies. These relations provide the essence of our approach. The simplest form of the theory, which contains no free parameters, is compared with N-body simulations of clustering in an expanding universe. For low density, $\Omega = 0.1$, simulations the simple theory gives good agreement with the experimental distribution functions. For the high density $\Omega = 1.0$ simulations, the agreement is considerably better. More elaborate and profound versions of the theory are also possible, and remain to be explored.

The main result of the simple theory is that the probability of finding N galaxies in a volume V of arbitrary shape is

$$f(N) = e^{-\bar{N}(1-b)-Nb}\frac{\bar{N}(1-b)}{N!}[\bar{N}(1-b)+Nb]^{N-1}\,. \quad (1)$$

Here

$$\bar{N} = nV \quad (2)$$

is the average number expected in V with an average density n and

$$b = -\frac{W}{2K} \quad (3)$$

is the ratio of the correlation potential and kinetic energies of the system. When there is no gravitational force, $b = 0$. For $b = 0$, we recover the expected Poisson distribution. The case $N = 0$ gives the distribution of empty voids. For a hierarchical distribution in virial equilibrium $b \to 1$ and $\bar{N}(1-b) \to$ constant.

At this point, the reader may well ask why galaxy clustering in an expanding universe can usefully be described by an *equilibrium* thermodynamic theory. After all, the expanding universe in which clusters continually grow does not appear to be an equilibrium state. Moreover, it is widely thought that (Newtonian) gravitating systems do not have any nonsingular equilibrium state, because of their long-range, unshielded, attractive interaction.

Both these qualms exist, but for a variety of reasons they do not have strong consequences for infinite expanding systems which resemble our universe. Briefly, the expansion is not as troublesome as it might first seem because it is relatively slow. Regions of interest for galaxy clustering are sufficiently small that their early local relaxation times are somewhat faster than the Hubble expansion time for the universe. This is partly because the local grainy density in a clustering region is greater than for the average smoothed out universe. Thus it is possible to imagine thermodynamic "experiments" in which local parameters change but still satisfy the equilibrium thermodynamic relations. The situation is not as clear-cut as for an expanding gas of atoms (where the disparity of relaxation times is much larger). The expansion of the universe does make a difference to the thermodynamics of galaxy clustering, but it does not dominate (except on the largest scales where the important effect of expansion is essentially to subtract the mean gravitational field).

The tendency of gravitating systems to form clumps suggests that the ultimate state of a Newtonian system contains at least one point singularity and the system evolves until it reaches this singularity. At first sight this view seems reinforced by the infinite weighting of the phase space available *a priori* for this state. (Throughout, we are considering point particles. Hard cores would of course change the picture, but that is not the point here.) However, in practice there are energy and angular momentum barriers which render much of phase space inaccessible to the system for timescales long compared even with the expansion time. Here the expansion actually helps a bit by slowly decreasing the influence of unbound galaxies on one another. This discourages the formation of singular states.

More precisely, singular states are strongly inhibited by the fact that most galaxies develop significant angular momenta and energy relative to one another. Tightly bound binaries rarely form because they require three-body interactions. (Extended bodies may merge, but that process is not important here.) The contribution of most binaries to the large scale thermodynamics is small, and could even be removed by considering the binary as a "renormalized" point mass. The error this leads to is of the order of the binary's internal kinetic energy divided by the total kinetic energy. (It is worth mentioning that this argument would not apply to a finite gravitating system where even one tight binary can dominate the energy distribution after a long enough time.)

A related qualm about applying thermodynamics to gravitating systems arises over the intensive-extensive nature of different thermodynamic variables. This distinction is not actually fundamental to thermodynamics, which can be modified in cases where gravity changes the usual extensive nature of energy and entropy. The modifications arise in finite gravitating systems because the gravitational energy is not locally defined within the system; it involves global properties. For galaxy clustering, however, the situation is much simpler. The crucial point here is to realize that the gravitational contributions to thermodynamic variables involve integrals over the two-point correlation function and the range of these integrals is finite. This leads to an extensive thermodynamics on scales greater than the correlation length (Saslaw 1980*b*). Physically, scales which are uncorrelated do not interact gravitationally and their thermodynamics is extensive. On these uncorrelated scales, the long-range contribution by gravitational forces can be subtracted off. In an expanding universe resembling ours, the two-point correlation length is ~ 10 Mpc. On faster time scales than it takes this length to grow significantly, the correlations can be regarded as "frozen" into the thermodynamics.

These matters are discussed somewhat further in § II where

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the thermodynamics and equation of state are derived. In § III we show how to calculate the distribution probabilities and apply this to the probability of finding a void. This is especially important because it is independent of the way $b = -W/2K$ depends on thermodynamic variables. Voids are also especially important for observations (Aarseth and Saslaw 1982). Section IV uses the simplest version of the theory to derive the general probability distribution $f(N)$ of equation (1). The real test of any thermodynamic theory is its agreement with experiments, and § V describes the good agreement between $f(N)$ and the N-body simulations. We make some further remarks in § VI.

II. THERMODYNAMICS AND THE EQUATION OF STATE

We consider an infinite system of point galaxies which, for simplicity, all have the same mass m. The expanding background implicitly removes the mean gravitational field, but does not enter explicitly into the thermodynamics. Assume, for the reasons given in § I, that in a volume V containing N galaxies, the total internal energy U and the pressure P satisfy the normal equilibrium thermodynamic relations (e.g., Hill 1956):

$$U = \tfrac{3}{2}NT(1 - 2b) \tag{4}$$

and

$$P = \frac{NT}{V}(1 - b) . \tag{5}$$

The temperature (in energy units with Boltzmann's constant $k = 1$) is approximated by the random kinetic energy

$$K = \tfrac{3}{2}NT = \tfrac{1}{2}\sum_{i=1}^{N} m v_i^2 , \tag{6}$$

which is summed over the peculiar velocities of the N galaxies in V. We assume that velocity correlations do not substantially modify this relationship. This is reasonable because positional correlations contribute directly to W, but velocity correlations are less than the average random velocity contribution to T. The value of b measures the influence of the gravitational correlation energy, W. It is related to the two-particle correlation function, ξ, which generally depends on density and temperature by

$$b = -\frac{W}{2K} = \frac{2\pi G m^2 n}{3T}\int_0^\infty \xi(n, T, r) r dr . \tag{7}$$

Here $n = N/V$ is the average number density, and the integral is the product of ξ and the gravitational potential integrated over all space. In a universe of finite age, $\xi(r)$ is appreciable only over a limited region (unless there are very long range initial correlations) which may grow slowly with time. Thus the integral converges, and b is independent of scale over regions beyond which ξ is appreciable. The value of b depends on the form of $\xi(r)$, and it was previously shown that $\xi \sim r^{-2}$ over a finite range maximizes the gravitational entropy of clustering (Saslaw 1980*b*). Here, however, we are concerned primarily with b as a function of n and T. Therefore, we suppose the integral to be done and the result to give the quasi-equilibrium $b(n, T)$ at any particular time.

The resulting $b(n, T)$ contains information about clustering on all scales through the dependence of ξ on all the higher n-particle correlations in the full BBGKY hierarchy. Thus specifying $b(n, T)$ is a substitute for solving the whole coupled BBGKY hierarchy with given initial conditions. Of course, this would be of no use if we were to specify an incorrect or inconsistent $b(n, T)$. However, we find that the form of $b(n, T)$ is strongly constrained by simple physical arguments. These lead us to an expression which, if not unique, at least seems to be close to the correct one for galaxy clustering.

First $b(n, T)$ must depend only on a special combination of n and T. There are several ways to see this. The simplest is to note that b is a function of the dimensionless ratio of potential to kinetic energy. In an infinite, thermodynamically homogeneous system, the only characteristic length scale which enters the potential energy is the average galaxy separation $\bar{r} \approx n^{-1/3}$. So b should depend just on the ratio of the potential and kinetic energies of two typical galaxies at this separation: $Gm^2/n^{-1/3}T$. To eliminate fractional powers, we can cube this ratio and write

$$b = b(G^3 m^6 n T^{-3}) = b(nT^{-3}) . \tag{8}$$

This result also follows, more formally, from scale invariance. The value of b, and the consequent thermodynamics, is determined by averaging homogeneous systems. Again $\bar{r} = n^{-1/3}$ is the only length scale which can be formed from the thermodynamic variables which describe the system. All other lengths, including the correlation length, must be multiples of $\bar{r}$. We suppose there is a (possibly continuous) hierarchy of clustering. Homogeneity requires that b is invariant under the scale transformation $\bar{r} \to \lambda\bar{r}$. Physically this means that the value of b is the same for each level of the clustering hierarchy on scales larger than the two-particle correlations. Thus $n^{-1/3} \to \lambda n^{-1/3}$ or $n \to \lambda^{-3} n$ and $b \to \lambda^0 b = b$, this last equation being the expression of scale invariance. Since b is the ratio $-W/2T \propto Gm^2N^2/rNT$, scale invariance requires that $T \to \lambda^{-1}T$, because N refers to a very large constant volume and does not depend on the level of the clustering hierarchy. Physically, this scaling of temperature means that clusters at a higher level of the hierarchy have lower random velocities relative to each other than smaller clusters have relative to each other. (This should not be confused with the rescaling which would occur during a physical expansion of the entire system.) The scale invariance of b therefore requires that

$$b(\lambda^{-3} n, \lambda^{-1} T) = b(n, T) . \tag{9}$$

This is a functional equation whose solution is readily seen to be equation (8).

There is another, somewhat more subtle, constraint which again leads to the requirement $b = b(nT^{-3})$. This is the constraint that the entropy of a system in equilibrium (here assumed to apply to our quasi-equilibrium state) does not depend on the path by which the system reaches that state. In other words, the entropy must be a total differential. We develop this result now because it uses general thermodynamic formulae which will be important later on.

In a single-component thermodynamic system, the energy per particle (galaxy) satisfies a fundamental equation relating it to the specific entropy and volume:

$$u = u(s, v) , \tag{10}$$

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where

$$u(s, v) = N^{-1}U(S, V, N) \tag{11}$$

with

$$s = S/N \quad \text{and} \quad v = V/N . \tag{12}$$

As a result, $du = (\partial u/\partial s)_v ds + (\partial u/\partial v)_s dv$ with $(\partial u/\partial s)_v = (\partial u/\partial S)_{V,N} = (\partial U/\partial S)_{V,N} = T$ and similarly $(\partial u/\partial v)_s = -P$. This gives the standard relation $du = Tds - Pdv$. Substituting equations (4) and (5) for U and P, we find that the entropy is given by

$$d\left(\frac{S}{N}\right) = \frac{1}{T} d\left(\frac{U}{N}\right) + \frac{P}{T} d\left(\frac{V}{N}\right)$$
$$= d \ln (n^{-1}T^{3/2}) - 3db + bd \ln (nT^{-3}) . \tag{13}$$

In order that the entropy be a total differential, independent of how the system reached its state, we see again that equation (8) must be satisfied. Therefore we are led to the result $b = b(nT^{-3})$ from three basic related points of view.

To find the distribution of voids, we will see in § III that we just need the value b to be finite. It does not matter how b depends on n and T. For the probability distribution $f(N)$ with $N > 0$, however, we need to know the functional form of $b(nT^{-3})$. Physical constraints which we consider next also limit its nature severely.

For positive correlations and pressure, we see that the possible values of b lie between zero and one. Since b depends on the full quantity $G^3m^6nT^{-3}$, the value $b = 0$ applies where gravity is negligible, $G \to 0$, and so it must also apply when $n \to 0$. In general, thermodynamic variables are analytic functions of each other. Therefore a power series expansion of $b(G^3m^6nT^{-3})$ must start off linearly with its argument. This is consistent with the requirement of little correlation at high temperatures. Incidentally, this reasoning gives the Debye-Hückel theory when applied to dilute plasmas and electrolyte solutions. For gravity, however, we are more interested in the nonlinear terms, and here the virial theorem comes to our aid. At low temperatures and high densities, more strongly bound clusters form. These tend to contain more and more galaxies and to be close to virial equilibrium, suggesting $b \to 1$ as nT^{-3} becomes large. The simplest form of b which has these properties is

$$b = \frac{b_0 nT^{-3}}{1 + b_0 nT^{-3}}, \tag{14}$$

where $b_0(Gm^2)^{-3}$ is a positive dimensionless constant.

Additional arguments also suggest this form for b. Section IV shows that there is an analyticity requirement on b which derives physically from the discreteness of numbers of galaxies. This is related to the constraint that the distribution functions $f(N)$ which follow from b must all be positive. Equation (14) leads to the simplest form for these distribution functions. Finally, of course, the major test of equation (14) is whether it leads to distribution functions in agreement with N-body experiments. Section V shows that the results are perhaps surprisingly good for such a simple formula. We suspect that (14) may have other attractive properties, and regard it as a physically motivated solution which shortcuts the calculation of the BBGKY hierarchy. It is probably a first approximation to something better. Indeed, it is the first Padé approximant to a general function $b(nT^{-3})$.

This form of b has interesting thermodynamic implications. Consider, for example, the entropy it implies. Substituting (14) into (13) and integrating gives

$$\frac{S}{N} = \ln n^{-1}T^{3/2} - 3b - \ln (1 - b) + s_0 , \tag{15}$$

with s_0 an arbitrary constant. As expected, S is extensive. For small b and a given average density, the mildly correlated structure decreases S relative to its value for the random Maxwellian distribution with $b = 0$ (perfect gas case). Entropy decreases here because the clustering is assumed to be homogeneous and the theory, in this simple form, does not allow for entropy transfer between clustering galaxies and field galaxies, and for the changing velocity distribution.

Figures 1*a* and 1*b* show the relation between entropy and the degree of clustering, indicated by the value of b. These

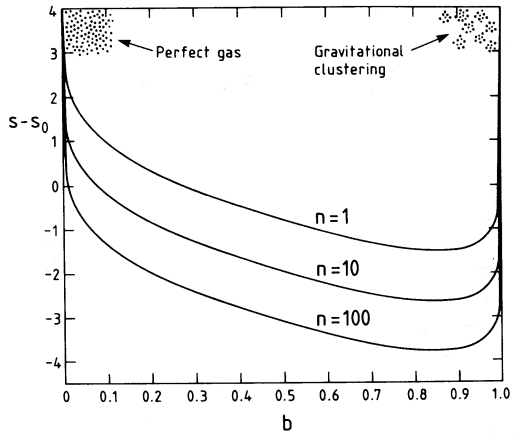

FIG. 1*a*

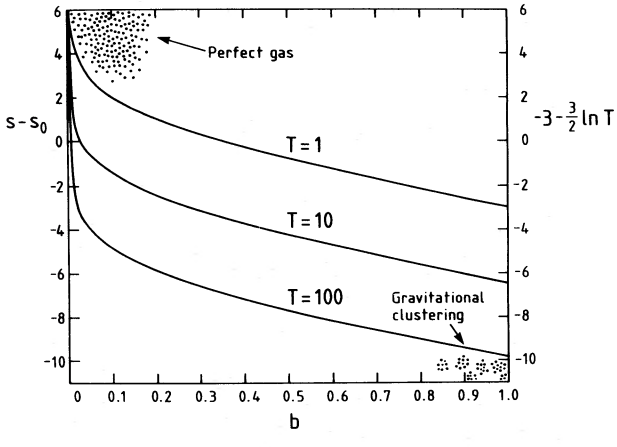

FIG. 1*b*

FIG. 1*a*.—Equi-density entropy contours, $s - s_0 = -\{\frac{1}{2} \ln n + 3b + \frac{1}{2} \ln [b(1 - b)]\}$, for an infinite gravitating system as a function of relative density, n, and the ratio, $b = -W/2K$, of gravitational to kinetic correlation energy.

FIG. 1*b*.—Isothermal energy contours, $s - s_0 = -(3b + \ln bT^{3/2})$, for an infinite gravitating system as a function of relative temperature, T, and the ratio, $b = -W/2K$, of gravitational to kinetic correlation energy.

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figures describe an ensemble of systems which need not be causally related. Unlike ordinary thermodynamics, equilibrium gravitational thermodynamics cannot be simply transformed into a nonequilibrium theory by quasi-static evolution. The reason is that as a system evolves, it clusters in an irreversible manner. This causes both the amplitude and scale length of the correlation function to change (unlike the case of a perfect gas where they remain zero). This evolution may also be expressed by the inclusion of new states in the partition function. As a result, there is nothing which corresponds to the reversible adiabatic or isothermal changes of a perfect gas. Thus evolutionary tracks should not generally be imposed on Figure 1*a* or 1*b* for $b \neq 0$.

As $b \to 1$, subclusters are present in virial equilibrium. Within each subcluster, the distribution is approximately a random Maxwellian. With most galaxies in virialized clusters, the contribution of the explicitly gravitational terms $S_{\rm grav} = -3b - \ln(1-b)$ to the entropy becomes positive, and as $b \to 1$, $S_{\rm grav} \to \infty$ with a logarithmic divergence. The change from negative to positive values of $S_{\rm grav}$ occurs at $b \approx 0.95$, which is very close to virial equilibrium, Figure 1*a*. If we look at systems with the same temperature, however, as $b \to 1$, $nT^{-3} \to \infty$ and the first term of (15),

$$\ln n^{-1}T^{3/2} = -\ln nT^{-3} - \tfrac{3}{2}\ln T\,, \tag{16}$$

diverges logarithmically to $-\infty$. To see which term wins, we substitute (16) and (14) into (15) to find

$$\lim_{b\to 1, T\to\infty} \frac{S}{N} \sim -\frac{3}{2}\ln T\,. \tag{17}$$

Therefore as gravitational clustering dominates and the average random velocities of the galaxies increase, the entropy isotherms diverge logarithmically toward $-\infty$, Figure 1*b*. This is the opposite extreme to a normal noninteracting gas where $S \to +\infty$ as $T \to \infty$ for $b = 0$. Both classical gases and gravitating systems tend toward "equilibrium" states of entropy extremes. The extremes may have different signs, depending on whether we consider an ensemble with the same average density, or with the same average temperature. Therefore the entropy extremum in an infinite gravitating system does not occur when every particle has fallen into a point singularity. It occurs when there is a hierarchy of clustering which has reached virial equilibrium on all scales. This seems in accord with the properties of N-body simulations, as well as with the structure of the universe.

III. THE DISTRIBUTION OF VOIDS

In this section we derive an analytic formula for the probability $f(0)$ of finding no galaxies in a randomly positioned volume V. Section IV derives $f(N)$ for all values of N, as well as the moments of the distributions. The reason for segregating $f(0)$ is that it does not depend on the form of $b(n, T)$ discussed in the last section.

We consider a large system described by equilibrium thermodynamics, at least as a first approximation. Collect an ensemble of subregions, all of the same volume, which are much smaller than the total volume. Both the number of galaxies and their total energy will vary among the members of the ensemble. This describes a reasonable way to sample the universe. Thermodynamically, each subregion can be regarded as being in contact with a large reservoir. Across the boundary, both particles (galaxies) and energy can flow. This is the grand canonical ensemble, characterized by a given chemical potential, $\mu \equiv (\partial U/\partial N)_{S,V}$ and temperature, $T \equiv (\partial U/\partial S)_{V,N}$. It describes systems with variable N and U.

From statistical thermodynamics we have the result that the probability of finding N particles in the energy state $U_j(N, V)$ is

$$P_{Nj}(V, T, \mu) = \frac{e^{-U_j/T + N\mu/T}}{Z_G(T, V, \mu)}\,, \tag{18}$$

where

$$Z_G(T, V, \mu) = \sum_{j,N} e^{-U_j/T + N\mu/T} \tag{19}$$

is the grand partition function (again in units with $k = 1$). Here we see why the states containing infinitesimally close binaries must be excluded, since they would make an infinite positive contribution to Z_G. As mentioned in the Introduction, this exclusion of binaries in the counting of states is reasonable because very few hard binaries actually form. The rate at which they would form in a finite nonexpanding spherical cluster is longer than the two-body relaxation time by a factor $\sim N$ and longer than the dynamical crossing time by a factor $\sim N^2$ (cf. Heggie 1980). For an expanding system, the rate is even slower because as the galaxies move apart their interaction becomes weaker. N-body experiments confirm these results.

Associated with the grand canonical ensemble is a potential, Ψ, whose derivatives give the thermodynamic properties of the ensemble:

$$\Psi\left(\frac{1}{T}, V, \frac{\mu}{T}\right) = \ln Z_G\,, \tag{20}$$

and

$$S = T\left(\frac{\partial\Psi}{\partial T}\right)_{V,\mu} + \Psi\,, \tag{21}$$

$$N = T\left(\frac{\partial\Psi}{\partial\mu}\right)_{V,T}\,, \tag{22}$$

for example. To find Ψ explicitly, note that the definition of entropy is

$$S = \sum_{j,N} P_{Nj}\ln P_{Nj}\,. \tag{23}$$

Substituting (18) into (23) readily gives

$$S = \frac{U}{T} - \frac{N}{T}\mu + \Psi\,. \tag{24}$$

Combining this with the thermodynamic Euler relation, $TS = U + PV - \mu N$, we see that

$$\Psi = \frac{PV}{T} = \bar{N}(1-b)\,, \tag{25}$$

where the last equation follows from (5), and N explicitly denotes the average number in a volume V. All this is independent of the function form of b.

The probability of finding a particular number, N, of

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particles in a volume V follows from summing (18) over all energy states:

$$f(N) = \sum_j P_{Nj} \,. \tag{26}$$

Noting that e^{Ψ} gives the normalization of the sum-over-states (19) suggests that we can calculate the $f(N)$ without explicitly performing the summation (26). To do this we define a generating function by averaging the quantity e^{qN}, where q is an arbitrary variable,

$$\begin{aligned}\langle e^{qN}\rangle &\equiv \sum_N e^{qN} f(N) \\ &= Z_G{}^{-1} \sum_{N,j} e^{-U_j/T + (\mu/T + q)N} \\ &= \exp\left[\Psi\left(\frac{\mu}{T} + q\right) - \Psi\left(\frac{\mu}{T}\right)\right], \end{aligned} \tag{27}$$

using (19) and (20) and regarding the arguments T^{-1} and V of Ψ as being understood (since they do not enter explicitly in the following analysis, and can be regarded as given). At this state it simplifies the algebra to regard $\Psi(\mu/T)$ as a function of $e^{\mu/T}$ rather than of μ/T and to write

$$z \equiv e^q \,. \tag{28}$$

Then (27) becomes

$$\begin{aligned}\sum_N f(N) z^N &= \exp\left[\Psi(ze^{\mu/T}) - \Psi(e^{\mu/T})\right] \\ &= e^{-\Psi} \exp\left[\Psi(ze^{\mu/T})\right]. \end{aligned} \tag{29}$$

Now expand the last exponential in (29) in a Taylor series around $z = 0$ and equate the coefficients of powers of z on both sides of the equation to find

$$f(N) = e^{-\Psi} \frac{e^{\mu N/T}}{N!} (e^{\Psi})_0^{(N)} \tag{30}$$

with

$$(e^{\Psi})_0^{(N)} = \left\{\left[\frac{d}{d(ze^{\mu/T})}\right]^N \exp\left[\Psi(ze^{\mu/T})\right]\right\}_{ze^{\mu/T}=0} . \tag{31}$$

This is the basic result.

For $N = 0$, we note (e.g., from eqs. [19] and [20] since $U = 0$ when $N = 0$) that $\Psi(0) = 0$ and so

$$f(0) = e^{-\Psi(\mu/T)} = e^{-\bar{N}(1-b)} = e^{-nV(1-b)} \tag{32}$$

using equation (25). In particular, for a spherical volume of radius r,

$$f(0) = \exp\left[-\tfrac{4}{3}\pi r^3 n(1-b)\right] \tag{33}$$

is the probability of finding the volume empty. This reduces to the standard result for a Poisson distribution as $b \to 0$ and gravity becomes of no consequence. It depends on the value of b, but not on the functional form of $b(n, T)$. As $b \to 1$, the galaxies become more and more clustered, so the probability of finding an empty void approaches unity in the idealized limit. This accords with our intuition, and also with the N-body simulations of § V.

IV. MANY-GALAXY DISTRIBUTION FUNCTIONS

The overall clustering of galaxies is characterized by the full set of distribution functions $f(N)$. To derive a closed analytic form for these we start with equation (30) and rewrite equation (31) as

$$(e^{\Psi})_0^{(N)} = \left\{\left(\frac{d}{de^{\mu/T}}\right)^N \exp\left[\Psi(e^{\mu/T})\right]\right\}_{e^{\mu/T}=0} . \tag{34}$$

Now

$$\frac{d}{de^{\mu/T}} = e^{-\mu/T} \frac{db}{d(\mu/T)} \frac{d}{db} \,. \tag{35}$$

To find $db/d(\mu/T)$ we use (25):

$$\frac{d\Psi}{d(\mu/T)} = \frac{d[Vn(1-b)]}{d(\mu/T)} = \frac{d[Vn(1-b)]/db}{d(\mu/T)/db} = nV \,, \tag{36}$$

where the last equality follows from (22) or (24), at constant V. Substituting (35) and (36) into (34) gives

$$(e^{\Psi})_0^{(N)} = \left[\left\{ne^{-\mu/T} \frac{db}{d[n(1-b)]} \frac{d}{db}\right\}^N e^{Vn(1-b)}\right]_{n=0} . \tag{37}$$

Here $n = n(b, T)$, and the complete operator in parentheses is applied N times. To find the chemical potential, we use equations (24), (25), (4), and (13) which show that in general

$$\frac{\mu(n, b)}{T} = \ln(nT^{-3/2}) - b - \int n^{-1} b\, dn \,. \tag{38a}$$

For b given by equation (14),

$$\frac{\mu}{T} = -b + \ln\left(\frac{b}{b_0 T^{3/2}}\right). \tag{38b}$$

Several features of equation (37) are especially interesting. First, the properties of the distributions for a gravitationally interacting system are determined by its infinitely diffuse limit, $n = 0$. This also occurs in the virial description of an imperfect gas. Second, the analytic nature of the expansion of (29) places constraints on the function $b(n)$. For example, (37) shows that b cannot be independent of n, for then all the $f(N)$ would be zero. More generally, one can show that b = constant does not lead to an analytic expansion. In addition, the form of $b(n)$ must be such that all the $f(N)$ are positive, smooth functions. Third, and most intriguing, we notice that there is a form of $b(n)$ which simplifies the operators in (37) enormously, namely $n(1-b) \propto b$. Remarkably, this is just the form (14) derived on physical grounds.

Using the relation $b(n)$ from equation (14) amounts to assuming all the information in a BBGKY hierarchy or summing a virial expansion. We do not know how close this comes to the exact solution, but § V will illustrate its usefulness. With (14) it is straightforward to calculate the first few forms of $(e^{\Psi})_0^{(N)}$:

$$(e^{\Psi})_{b=0}^{(0)} = 1 \,, \tag{39}$$

$$(e^{\Psi})_0^{(1)} = \frac{V}{b_0'} T^{3/2}\left(\frac{V}{b_0'} + 1\right)^0 \left.\frac{b_0' e^{(V/b_0' + 1)b}}{(1-b)^1}\right|_{b=0} = \frac{V}{b_0'} T^{3/2} b_0' \,, \tag{40}$$

$$\begin{aligned}(e^{\Psi})_0^{(2)} &= \frac{V}{b_0'} T^3 \left[\left(\frac{V}{b_0'} + 2\right) + (1-b)^{-1} - 1\right] \left.\frac{b_0'^2 e^{(V/b_0' + 2)b}}{(1-b)^2}\right|_{b=0} \\ &= \frac{V}{b_0'} T^3 b_0'^2 \left(\frac{V}{b_0'} + 2\right), \end{aligned} \tag{41}$$

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$$(e^{\Psi})_0^{(3)} = \frac{V}{b_0'} T^{9/2} b_0'^3 \left(\frac{V}{b_0'} + 3\right)^2 , \quad (42)$$

where $b_0' \equiv b_0 T^{-3}$.

From these expressions we surmise that generally

$$(e^{\Psi})_0^{(N)} = \frac{V}{b_0'} T^{3N/2} b_0'^N \left(\frac{V}{b_0'} + N\right)^{N-1} . \quad (43a)$$

Writing out the Taylor series explicitly,

$$e^{\Psi} = e^{Vb/b_0'} = \sum_{N=0}^{N} \frac{V}{b_0' N!} \left(\frac{V}{b_0'} + N\right)^{N-1} (be^{-b})^N . \quad (43b)$$

The appendix proves this result rigorously by induction.

Combining (43a) and (30) gives the distribution

$$f(N) = e^{-\Psi - bN} \frac{b^N}{N!} \frac{V}{b_0'} \left(\frac{V}{b_0'} + N\right)^{N-1} , \quad (44)$$

or, substituting (25) and (14),

$$f(N) = e^{-\bar{N}(1-b) - Nb} \frac{\bar{N}(1-b)}{N!} [\bar{N}(1-b) + Nb]^{N-1} . \quad (45)$$

Thus we have the distribution function for all values of N. It depends only on $\bar{N} = nV$ and b, both of which can be determined for any given collection of particles. So there are no free parameters in the basic theory. The theory does depend on the form used for $b(n^{-1}T^3)$, and equation (45) results from the simplest form which has physically reasonable behavior.

Special cases of equation (45) are particularly interesting. For $b = 0$, we recover the Poisson distribution; and for $b = 0$, $N = 0$, we find $f(0) = e^{-\bar{N}}$, which is the well known distribution for holes. For $b \to 1$ and $N \neq 0$, we see that $f(N) \to 0$ unless we consider increasingly large volumes such that $\bar{N}(1-b)$ remains nonzero. This is the virial limit which describes a hierarchy of clusters, all in virial equilibrium. Using (14), we see that this limit is

$$\text{virial limit} \quad \bar{N}(1-b) \to VT^3/b_0 \equiv N_{\text{vir}} . \quad (46)$$

Physically, we see that as $b \to 1$, more and more energy states become available to the system. As this happens, we must select ensembles with greater $\bar{N}$ in order to populate these added energy states and maintain the value of the grand canonical potential Ψ. Then equation (45) becomes

$$\lim_{b \to 1} f(N) = \exp(-N_{\text{vir}} - N) \frac{N_{\text{vir}}}{N!} (N_{\text{vir}} + N)^{N-1} . \quad (47)$$

Now if $N \ll N_{\text{vir}}$, we recover a Poisson distribution with mean value N_{vir}. This is as expected, since in this limit we are dealing with small numbers in regions mainly within virialized clusters. Indeed, for any value $b \neq 1$, in the limiting case of (45) with $Nb \ll \bar{N}(1-b)$, the distribution $f(N)$ is Poisson with a mean $\bar{N}(1-b)$.

By looking at $f(2)$, we can obtain a simple physical interpretation of b_0. Consider a spherical volume of radius r and ask for the value of $r = r_{\max}$ which maximizes $f(2)$. This is

$$r_{\max} = \left(\frac{3b_0}{4\pi b T^3}\right)^{1/3} [1 - b + (1+b^2)^{1/2}]^{1/3} \xrightarrow[b \to 1]{} \left(\frac{3\sqrt{2}\, b_0}{4\pi}\right)^{1/3} T^{-1} . \quad (48)$$

Thus at a given temperature and energy ratio, the value of b_0 determines the radius of the most probable volume which contains two particles. This is not the same as the most probable separation of the two particles, since the particles may be anywhere within the volume. Sometimes, for example, they will be at the edges of two different clusters and appear fairly close together in a larger volume which fortuitously includes them. So we are not surprised to find, using $n(1-b) = bT^3/b_0$, that in the Poisson case for $b = 0$ the value of $r_{\max}$ is $3^{1/3}$ times greater than the most probable nearest neighbor separation $(2\pi n)^{-1/3}$.

In addition to number fluctuations for a given volume, the energy in the volume will also fluctuate. It is possible to calculate energy probability distributions in a similar way. However, these do not appear to lead to simple analytic formulae. On the other hand, the moments of both the number and energy fluctuations can be calculated straightforwardly in the manner described by Callen (1960). For example,

$$\langle (N - \bar{N})^2 \rangle = \langle (\Delta N)^2 \rangle = \left. \frac{\partial \Psi}{\partial(\mu/T)} \right|_{T,V} , \quad (49)$$

$$\langle (U - \bar{U})^2 \rangle = \langle (\Delta U)^2 \rangle = - \left. \frac{\partial \Psi}{\partial(1/T)} \right|_{\mu/T,V} . \quad (50)$$

We have calculated some moments, complete up to the fourth order, and some examples are:

$$\langle \Delta N^2 \rangle = \frac{N}{(1-b)^2} ,$$

$$\langle \Delta N^3 \rangle = \frac{N}{(1-b)^4} (1 + 2b) ,$$

$$\langle \Delta N^4 \rangle = \frac{N}{(1-b)^6} (1 + 8b + 6b^2) + 3\langle \Delta N^2 \rangle^2 ,$$

$$\langle \Delta U^2 \rangle = \frac{3}{4} \frac{NT^2}{(1-b)^2} (5 - 20b + 34b^2 - 16b^3) ,$$

$$\langle \Delta N \Delta U \rangle = \frac{3}{2} \frac{NT}{(1-b)^2} (1 - 4b + 2b^2) ,$$

$$\langle \Delta N^2 \Delta U \rangle = \frac{3}{2} \frac{NT}{(1-b)^4} (1 - 6b + 2b^2) . \quad (51)$$

In each case we see that the fluctuations become very large, formally infinite, as $b \to 1$. This indicates the formation of bound virialized subclusters. The higher the moment, the more strongly it diverges, a result noted previously (Saslaw 1968).

V. COMPARISON WITH N-BODY SIMULATIONS

The "experimental tests" of any theory of gravitational clustering are its comparisons with N-body computer simulations. From this point of view, we have examined a series of 1000-body integrations of galaxy clustering, kindly made

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available to us by Dr. Sverre Aarseth. These experiments integrated the Newtonian equations of motion for 1000 galaxies in the background of an expanding universe. All galaxies had the same mass, normalized to unity, and were confined to a sphere whose radius was also normalized to unity. The gravitational potential, $\propto(r^2+\epsilon^2)^{-1/2}$, employed a small softening parameter to avoid direct collisions, as usual in these integrations (e.g., Aarseth, Gott, and Turner 1979). Values of ϵ were either 0.03 or 0.015 and remained constant in non-comoving coordinates. We examined both closed models with $\Omega \equiv \rho/\rho_{\rm crit} = 1$ as well as open models. The open models reached $\Omega \approx 0.1$ after an expansion factor of 32 times their initial radius. The initial states of all models had a Poisson spatial distribution and no velocity dispersion around the Hubble flow. Their final states approximated the degree of clustering presently observed in the universe; this requires an expansion factor of 8 for the closed model and 32 for the open one.

Figure 2 shows a two-dimensional projection for the final state of the closed model after expansion by a factor of 8. Its clustering is very pronounced and highly nonlinear. Correspondingly large voids have also developed.

To find the distribution functions $f(N)$, we first choose 10,000 spatial points uniformly at random throughout the inner 0.6 radius of the models (to avoid boundary effects). For each spatial point, we measured the distance to its first, second, third, and fourth nearest galaxies. This gives the probability of finding a given number of galaxies, 0–4, in a sphere of radius r. The analysis was done for the initial state, as well as for expansion factors of 2, 4, and 8 in the closed models and 2, 4, 8, 16, and 32 in the open models. In each case, the distribution functions were plotted for $N = 0, 1, 2, 3$, and 4, and a single value of b which gave the best fit, by eye, to equation (1) for all the $f(N)$ was determined. In each case

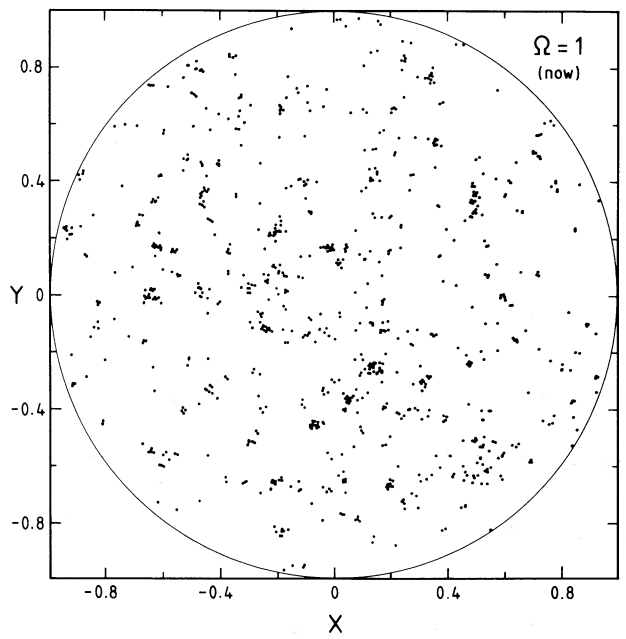

FIG. 2.—Two-dimensional projection of galaxy clustering in an N-body simulation of a closed universe with the presently observed form of the two-point correlation function.

the range of b's which gave acceptable fits varied by ± 0.05 around the value for the best fit. This is a rough measure of the uncertainty in determining b from the $f(N)$.

For each of the cases, we also calculate $b = -W/2K$ directly from the particle distribution, correcting for the softening parameter ϵ. The values of b calculated for different size regions in different positions within the sphere also varied around the mean value by about ± 0.05. This provides an estimate of the actual inhomogeneity of the thermodynamic parameter b, which should be constant in the idealized theory. Departures from the idealized theory occur at about the 10% level.

The difference between the two values of b, one found from the $f(N)$ and the other from $-W/2K$, give an estimate of the overall accuracy of the theory. The best fit between the experimental $f(N)$ and the theoretical $f(N)$ given by equation (1) provides another estimate of the theory's accuracy.

Figures 3 and 4 show the results for the closed, $\Omega = 1.0$, model at its present expansion factor of 8. Figure 3*a* is the distribution function $f(0)$ for the size of holes, Figure 3*b* is $f(1)$ for spheres containing $N = 1$ galaxy, etc. In each instance, the initial experimental Poisson distribution is plotted as a solid line reaching to the top of the graph, and the theoretical result for a Poisson distribution is the closely associated dotted line. The difference between these lines gives an idea of the statistical fluctuations of the initial sample. The final evolved distribution is the broader solid line, and the theoretical prediction, from equation (1), is the associated dotted line.

To get a concise picture of the much higher order distributions, we invert the procedure. Around each of the 10,000 randomly chosen spatial points, we draw three spheres of radius $r = 0.2$, 0.3, and 0.4. For each radius, we find the probability that it contains a given number of galaxies. These distributions, together with the theoretical comparisons of equation (1), are shown for the initial and final states in Figures 4*a*–4*c*. In Figures 4*b* and 4*c*, the experiments give somewhat higher frequencies than the theory predicts at small values of N. We are not sure if this is noise, or if it represents the formation of more small bound groups than equilibrium theories predict.

We have also examined the density moments up to fourth order, given for the theory by equation (51). There is good agreement with the simulations over intermediate scales $0.15 \lesssim R \lesssim 0.35$. Departures occur on smaller and larger scales for both the initial Poisson conditions and the evolved distributions. This is because the moments are especially sensitive to inhomogeneity on small scales and boundary effects on large scales.

All these curves for the clustered states in Figures 3 and 4 are fitted with the same value of $b = 0.65$. There is no other degree of freedom. The distribution functions are sufficiently close to the theoretical curves that fitting one of them would predict all the rest.

From the distribution functions we conclude that the theory provides a good description of nonlinear clustering. Examination of the other cases with expansion factors $\gtrsim 2$ in both the $\Omega = 0.1$ and $\Omega = 1.0$ models shows that the fits to equation (1) are just as good as those shown in Figures 3 and 4, with a different value of b characterizing each model.

The distribution functions $f(N)$ are closely related to the entire hierarchy of n-point correlation functions in the standard

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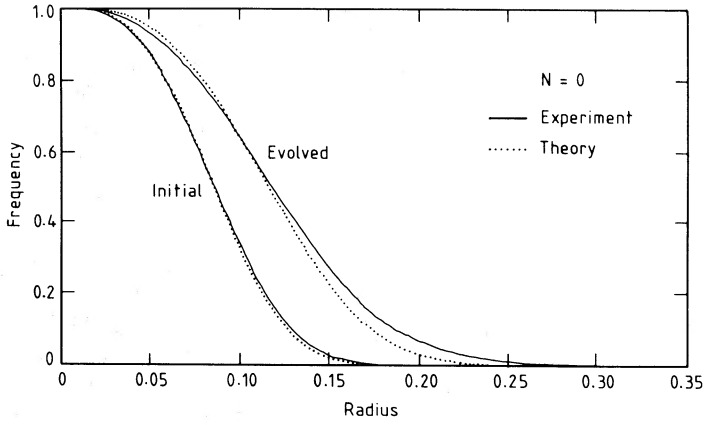

FIG. 3*a*

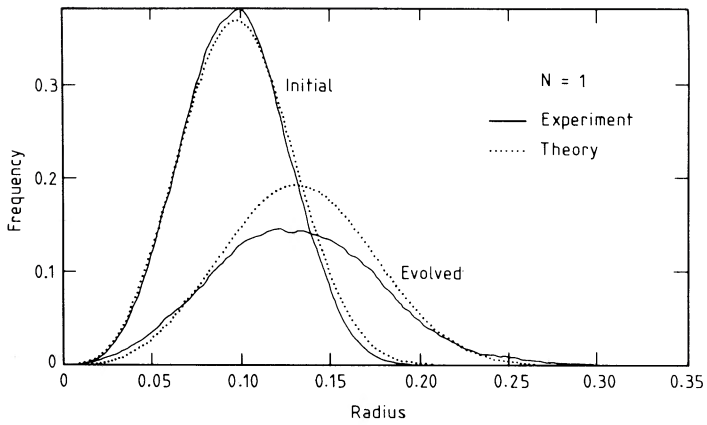

FIG. 3*b*

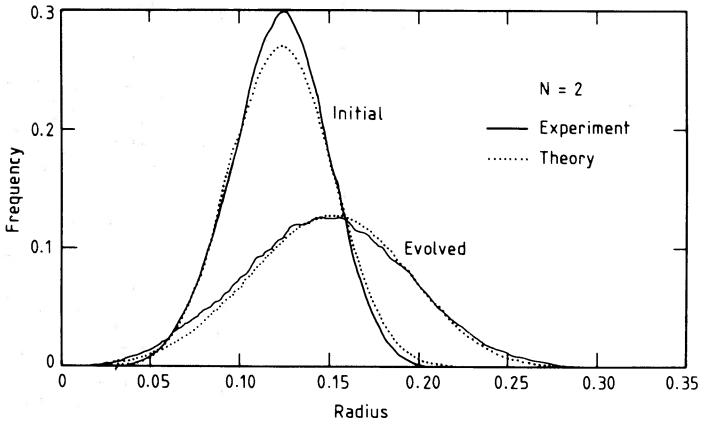

FIG. 3*c*

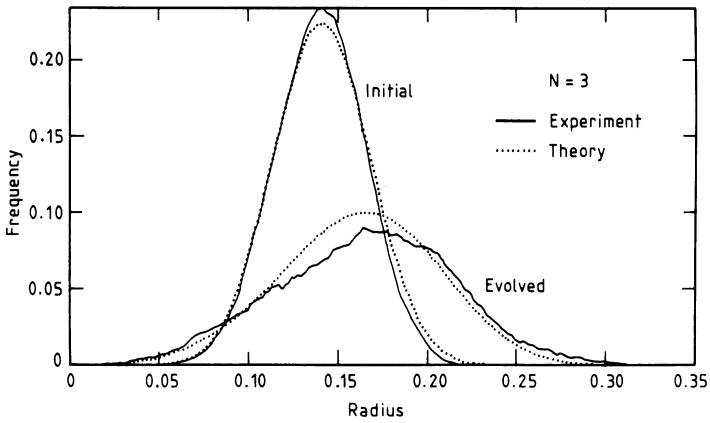

FIG. 3*d*

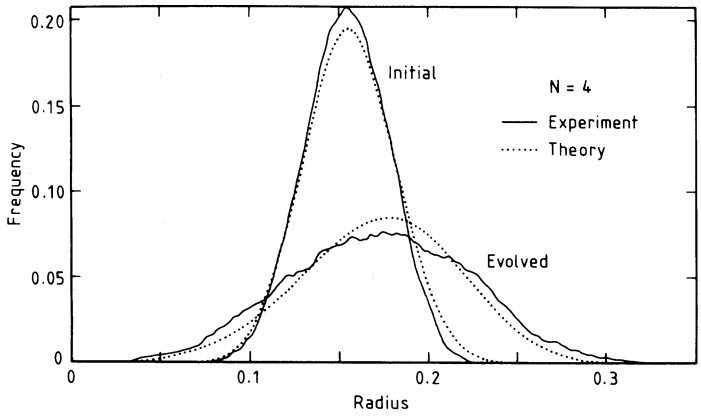

FIG. 3*e*

FIG. 3.—(*a*)–(*e*) Comparison between experimental and theoretical frequency distribution functions $f(N)$ for given $N = 0$, 1, 2, 3, and 4 for the closed model shown in Fig. 2. Both the initial Poisson distributions and the highly evolved nonlinear distributions are shown.

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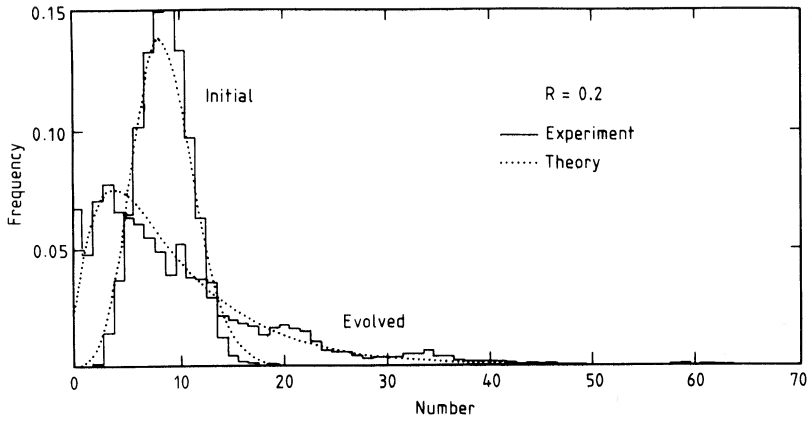

FIG. 4*a*

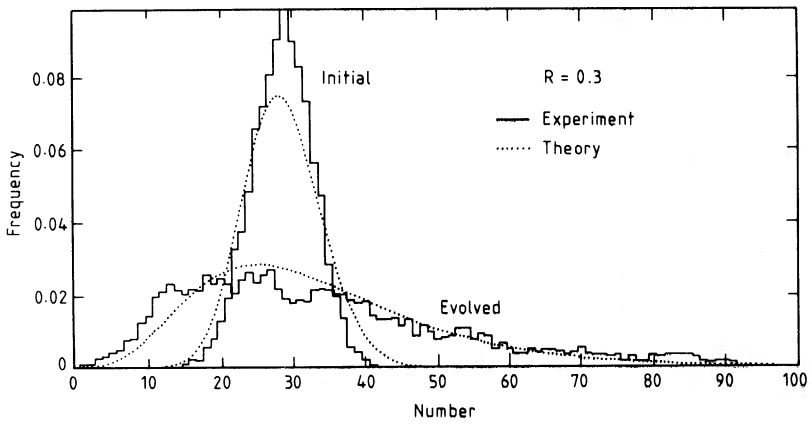

FIG. 4*b*

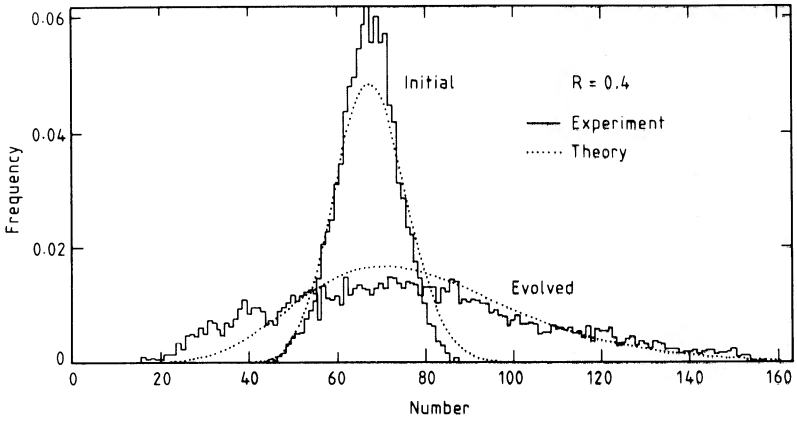

FIG. 4*c*

FIG. 4.—(*a*)–(*c*) Comparison between experimental and theoretical frequency distribution functions $f(N)$ at given radii $R = 0.2$, 0.3, and 0.4 for the closed model shown in Fig. 2. Both the initial Poisson distributions and the highly evolved nonlinear distributions are shown.

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formulation of clustering (White 1979). Given the correlation functions, it is possible to calculate the $f(N)$. However, one cannot generally find the correlation functions from the $f(N)$ without further positional information about the particles. Nevertheless, it is much easier to calculate the $f(N)$ than the correlations, especially in the nonlinear regime. Moreover, $f(N)$ is also easy to observe for many values of N.

In practice the determination of b from the N-body experiments is affected by several factors: (*a*) the use of a softening parameter; (*b*) the lack of homogeneity on the largest scale; (*c*) the finiteness of the simulations. Of these, the effect of a softening parameter is largest, and b must be corrected to take it into account. For example, at the termination of the open universe simulation with $\epsilon = 0.03$, the uncorrected value of $b = -W/2K$ is 1.23, in apparent contradiction to the requirement that b be less than its virial limit value of 1. This peculiar behavior was noted previously by Efstathiou and Eastwood (1981). The explanation for this apparent discrepancy is that the cosmic energy equation is modified in the presence of a softening parameter which remains constant in proper, rather than comoving, coordinates. The correlation potential energy is then no longer homogeneous of degree -1 in the cosmic scale factor. To allow for this, we determined from the simulations a corrected value of b, defined as $b = -W'/2K$ where

$$W' = -R\frac{\partial W}{\partial R} = W + \epsilon^2 \sum_{i>j} \frac{Gm_i m_j}{(R^2 x_{ij}{}^2 + \epsilon^2)^{3/2}}, \qquad (52)$$

with x_{ij} the comoving separation of two galaxies in an expanding background $R(t)$. Then the cosmic energy equation becomes

$$\frac{d}{dt}(K + W) + \frac{\dot{R}}{R}(2K + W') = 0 . \qquad (53)$$

The corrected value of b, which may differ from the uncorrected value by as much as 25%, has the property that it is always less than 1, the value it takes in the virial limit. It is this corrected value of b determined from the N-body experiments which is listed in Table 1.

A second difficulty with determining b from the simulations is caused by large scale inhomogeneities. For example, during the course of the integrations the temperature in the inner region (≤ 0.6 radius) of the simulations grew considerably hotter than the mean temperature, by almost 50% at the termination of the open model integration, and by 60% in the final closed model. On the other hand, W is similarly inhomogeneous, and the value of b determined from the inner region alone generally differed by $\lesssim 5\%$ from the mean value for the whole system. Because the simulations are sampled most heavily in the inner region, it is necessary to take this variation into account. In practice we took a simple average of the values of b determined from the system as a whole, and from the inner region alone.

A final difficulty in determining b from the simulations is the small numbers of particles compared to the thermodynamic limit. In particular, energy is not extensive on small scales because there are correlations across the boundary of any region. It is not possible to associate a unique energy with any prescribed subregion, except in the limit where the subregion becomes very large. This leads to uncertainties of a few percent in correlation energy of the inner region of radius 0.6. The procedure we adopted, which is not unique, was to calculate the correlation energy across the boundary and ascribe precisely half this energy to the inner region.

We turn now to compare the fitted value of b with the value $-W/2K$ computed directly from the distribution of galaxies in the experiments. For the case of the closed universe shown in Figures 2–4, this computed value of b is 0.74 ± 0.05. The range in this value comes from the uncertainties mentioned previously. This agrees quite well with the fitted value of 0.65 ± 0.05. Some of the difference between $-W/2K$ and the fitted value of b seems caused by the softening parameter ϵ. To check this, we examined two 1000-body integrations for an open model which had exactly the same detailed initial conditions except for their different constant values of $\epsilon = 0.03$ and 0.015. Because of the increased computer time, the model with the smaller softening parameter was integrated only to an expansion factor of 16. Table 1 shows the resulting values of b (corrected for the value of ϵ in the $-W/2K$ computation). After moderate evolution, the value of ϵ influences $b = -W/2K$ at about the 10% level. It does not affect the fitted value of b significantly.

Two important results emerge from Table 1. First, the values of $b_{\rm fit}$ and $b = -W/2K$ are closer in the closed models than in the open ones. Thus, the results for the closed models are more self-consistent in the context of our theory. This seems to be mainly a reflection of the fact that clustering occurs faster and involves more galaxies more strongly in the closed models. One indication is that the velocity dispersion of galaxies in closed models does not depend as strongly on the degree of their local clustering as in open models (Saslaw and Aarseth 1982). Thus the closed models are more "isothermal" over larger scales, as a first approximation. This makes them

TABLE 1

VALUES OF b IN OPEN AND CLOSED MODELS

EXPANSION FACTOR	OPEN		CLOSED	
	$b_{\rm fit} \pm 0.05$	$b = -W/2K \pm 0.05$, $\epsilon = 0.03/0.015$	$b_{\rm fit} \pm 0.05$	$b = -W/2K \pm 0.05$, $\epsilon = 0.03$
1	0	...	0	...
2	0.05	0.02/0.30	0.10	0.12
4	0.30	0.56/0.61	0.40	0.58
8	0.48	0.70/0.82	0.65	0.74
16	0.58	0.84/0.85		
32	0.65	0.97/ ...		

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more suitable for a thermodynamic description. Therefore in the difference between $b_{\rm fit}$ and $b = -W/2K$ we are probably seeing some nonequilibrium effects of the expansion.

Second, since $b_{\rm fit} \approx 0.65$ for both the open and closed models, spatial clustering is not a sensitive test for Ω, as pointed out previously (Saslaw 1980*a*, *b*; Aarseth and Saslaw 1982). In fact, the value $b = -W/2K = 0.74$ we find for the closed model agrees with the value $\frac{3}{4}$ found previously in N-body integrations (Miyoshi and Kihara 1975). This value also characterizes the system during the linear growth regime (Saslaw 1980*b*).

VI. FURTHER REMARKS AND CONCLUSIONS

We have developed a relatively simple analytical theory to calculate the probability $f(N)$ of finding N gravitating bodies in a volume V of arbitrary shape. Previous theories, such as the BBGKY hierarchy, could be solve only for the two-point correlation function and only in the linear regime. Our new approach complements these earlier theories in that it works for high order correlations, and in the very nonlinear regime. It is based on the equilibrium thermodynamics of gravitating systems, regarded as an asymptotic state toward which these systems evolve.

Gravitational N-body experiments give good agreement with the theory. This shows that even though an equilibrium theory may not explain the ultimate fate of galaxy clustering, it does provide a good description over extremely long time scales when the correlations are "frozen out" by the expansion of the universe. Furthermore, the experiments show that, as a first approximation, it is reasonable to characterize the peculiar velocities of galaxies by a single average temperature, even though their velocity distributions differ among clustered and unclustered regions. As long as the thermodynamics is averaged over scales greater than the scale of the two-point correlation function, the inhomogeneities combined with nonequilibrium effects of the expansion appear to give only small errors in the theoretical distribution functions $f(N)$. Thus the equilibrium theory seems to explain most of the results. From the theoretical viewpoint, the next steps are to try to understand this conclusion more deeply, and to begin to explore nonequilibrium theories.

The result that $f(N)$ does not depend strongly on the expansion of the universe, or the value of Ω, provides a powerful observational test of the general simple theory of gravitational clustering. The important observational question is how initial conditions affect the observed distribution. Equation (1) for $f(N)$ involves the presumption that structure on all scales has relaxed, or that initially the galaxies were randomly distributed. If there exists large scale unrelaxed structure now, equation (1) should not apply to it. So, if equation (1) does not fit the observed universe, it would be strong evidence, independent of Ω, for special initial conditions within the gravitational clustering theory, or for some completely different theory of galaxy clustering. In this way we may hope to understand how galaxies were distributed soon after they formed, and whether that distribution retained information from still earlier epochs of the universe.

It is a pleasure to thank Sverre Aarseth for allowing us to use his N-body programs and for his generous help and advice on the numerical experiments. We are happy to thank our colleagues at Cambridge, expecially Donald Lynden-Bell and Shogo Inagaki, for stimulating discussions. A. J. S. H. thanks the Institute of Astronomy for its hospitality during the summer of 1982.

APPENDIX

INDUCTIVE PROOF OF EQUATION (1)

The purpose of this Appendix is to derive the equality

$$\sum = \sum_{i=0}^{\infty} \frac{v(v+i)^{i-1}}{i!} (be^{-b})^i = e^{vb} \tag{A1}$$

which may be used to prove equation (44) from equation (37) by induction (with $v/b_0' \equiv v$ and $i = N$).

Note (A1) is trivially true for $v = 0$. In (A1) rewrite

$$(be^{-b})^i = b^i e^{-ib} = \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} b^{i+j}$$

and

$$\frac{(v+i)^{i-1}}{i!} = \begin{cases} v^{-1} & (i=0) \\ \sum_{k=0}^{i-1} \dfrac{v^k i^{i-k-2}}{k!\,(i-k-1)!} & (i \geq 1)\,. \end{cases}$$

Then

$$\begin{aligned} \sum &= 1 + \sum_{j,k=0}^{\infty} \sum_{i=k+1}^{\infty} \frac{v^{k+1}}{k!} \frac{i^{i-k-2-j}}{(i-k-1)!} \frac{(-1)^j}{j!} b^{i+j} \\ &= 1 + \sum_{i,j,k=0}^{\infty} \frac{v^{k+1}}{k!} \frac{(i+k+1)^{i+j+1}}{i!} \frac{(-1)^j}{j!} b^{i+j+k+1} \\ &= \sum_{i,j,k=0}^{\infty} \frac{(bv)^k}{k!} \frac{k(i+k)^{i+j-1}}{i!j!} (-1)^j b^{i+j}\,. \end{aligned} \tag{A2}$$

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A necessary and sufficient condition that (A2) equal e^{vb}, for arbitrary v, is that

$$\sum_{i+j=l} \frac{k(i+k)^{i+j-1}}{i!j!}(-1)^j = \begin{cases} 1 & \text{if} \quad l=0 \\ 0 & \text{if} \quad l \geq 1 \end{cases}. \tag{A3}$$

(A3) is trivially true for $l = 0$. For $l \geq 1$, (A3) is

$$\sum_{i=0}^{l} \frac{k(i+k)^{l-1}}{i!(l-i)!}(-1)^{l-i} = \sum_{i=0}^{l} \sum_{j=0}^{l-1} \frac{k^{l-j} i^j (-1)^{l-i}(l-1)!}{i!(l-i)!(l-j-1)!j!}. \tag{A4}$$

A necessary and sufficient condition that (A4) equal zero for arbitrary k is that

$$\sum_{i=0}^{l} \frac{i^j(-1)^i}{i!(l-i)!} = 0 \qquad \text{for all} \quad j = 0, \ldots, l-1 . \tag{A5}$$

The binomial theorem implies (A5) for $j = 0$. To prove (A5) for $j > 0$, suppose inductively that (A5) is true for all values of j between 0 and $j - 1$. In particular the inductive hypothesis allows one to suppose that

$$\sum_{i=0}^{l} \frac{P(i^{j-1})(-1)^i}{i!(l-i)!} = 0$$

for any polynomial P of i of order $j - 1$ or less. Then (A5) can be written

$$\begin{aligned} \sum_{i=0}^{l} \frac{i^j(-1)^i}{i!(l-i)!} &= \sum_{i=0}^{l} \frac{[i^j + P(i^{j-1})](-1)^i}{i!(l-i)!} \\ &= \sum_{i=0}^{l} \frac{i(i-1)\cdots(i-j+1)(-1)^i}{i!(l-i)!} \\ &= \sum_{i=j}^{l} \frac{(-1)^i}{(i-j)!(l-i)!} \\ &= \sum_{i=0}^{l-j} \frac{(-1)^{i+j}}{i!(l-j-i)!} \\ &= 0 \qquad \text{if} \quad j < l , \end{aligned} \tag{A6}$$

the last step of which follows from the binomial theorem as long as $j < l$. It follows that (A5) is true for all $j = 0, \ldots, l - 1$, as required. Consequently (A4), (A3), and (A2) are also true, and the equality (A1) is thereby proved.

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A. J. S. Hamilton and W. C. Saslaw: Department of Astronomy, P.O. Box 3818, University Station, University of Virginia, Charlottesville, VA 22903

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