

Low-Frequency, One-Armed Oscillations of Keplerian Gaseous Disks

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Abstract

It is shown that in geometrically thin, nearly Keplerian disks, one-armed, low-frequency waves are present in the disk plane. In the case of the standard accretion disks, the waves may be trapped in the inner part of the disks. The normal modes of the trapped oscillations are examined by the WKB method. In general, the frequency ω of the oscillations is lower than the angular velocity Ω of the disk rotation by the factor $(c_s/\Omega r)^2$, i.e., $\omega = B\Omega(c_s/\Omega r)^2$, where c_s is the sound velocity in the disk, r is the radius where the oscillations are trapped, and B is a numerical factor. In the cases considered, the numerical factor B is much less than 0.1. A brief comparison of the calculated periods of oscillations with the periods of spectrum variations observed in some shell stars is made.

Key words: Keplerian disks; Low-frequency oscillations; One-armed oscillations; Trapping.

1. Introduction

Disk systems exist in various astronomical objects, e.g., quasars, active galactic nuclei, X-ray binaries, cataclysmic variables, Be stars and so on. To know wave phenomena in these disk systems is important to understand dynamical properties of the systems. If waves are present in disks, they will become one of the causes of light variations of the systems and also become one of the possible processes of angular momentum redistribution in disks. In most of the systems quoted above, the disks are nonself-gravitating. Furthermore, except for the case of a very hot gas, the disks will be nearly Keplerian. Hence, in this paper we shall examine what kinds of wave phenomena can be expected in non-relativistic, nearly Keplerian, thin disks.

Before discussing the problem, we should remember the nonaxisymmetric wave phenomena in self-gravitating disks, and discuss how the wave phenomena in nonself-gravitating disks differ from those in self-gravitating disks. Wave phenomena in self-gravitating disks have been examined extensively in relation to the spiral structure of disk-like galaxies. One of the important findings of these studies is that there exist unstable normal wave modes (e.g., Lau and Bertin 1978; Lin and Lau 1979). This comes from the fact that Mark's (1974) three-wave interaction mechanism takes place at the corotation radius. In non-self-gravitating disks, however, this wave amplification process cannot be expected, because the waves cannot propagate in the region near the corotation radius. This difference comes from the following facts. In self-gravitating *stellar* disks, the frequency of oscillations seen from an observer rotating with the disk, i.e., $|\omega - m\Omega|$, is smaller than the epicyclic frequency κ , i.e., $(\omega - m\Omega)^2 < \kappa^2$. This is because the self-gravity acts so as to decrease the restoring

force due to rotation. Here, ω is the frequency of the oscillations seen from the inertial frame, $\Omega(r)$ is the angular frequency of disk rotation at radius r , and m is the number of arms. The relation $(\omega - m\Omega)^2 < \kappa^2$ means that the region where waves can propagate is the region bounded by the radii of the inner Lindblad resonance ($\omega/m = \Omega - \kappa/m$) and the outer Lindblad resonance ($\omega/m = \Omega + \kappa/m$).

On the other hand, in nonself-gravitating *gaseous* disks, we have $(\omega - m\Omega)^2 > \kappa^2$, because the self-gravity is absent and the gaseous pressure always acts so as to increase the restoring force of oscillation. Because of this, the regions where waves can propagate are inside the radius of the inner Lindblad resonance and outside the radius of the outer Lindblad resonance.

In addition to the above difference of propagation regions, there is a difference of the problem of wave absorption at the Lindblad resonances. As is known in the density wave theory, waves are absorbed at the Lindblad resonances (Toomre 1969; Mark 1971; Lynden-Bell and Kalnajs 1972). This wave absorption is unfavorable to the maintenance of waves. In the case of self-gravitating disk-like galaxies, the wave absorption at the inner Lindblad resonance can be avoided by the absence of the resonance itself or by a large Q -value [Toomre's (1964) Q value] in the bulge region (e.g., Lin and Lau 1979). On the other hand, in the nonself-gravitating disks, there is no such convenient situation as long as the gas temperature is so low that the disk is geometrically thin.

Furthermore we should consider that in nonself-gravitating, geometrically thin disks, the waves will have very short wavelengths in the radial direction for the following reasons, even when they are in a region not close to the radii of the Lindblad resonances. This comes from the fact that the disk rotation is differential. Since the disk rotation is differential, the difference between the wave frequency and the rotational frequency becomes of the order of the rotational frequency as the waves propagate radially, even if they are equal at some radius. This frequency difference must be balanced by the pressure restoring force. The difference can be balanced only by wavelength becoming very short, since the sound velocity is much smaller than the rotational velocity in thin disks. This suggests that the wave damping due to the phase mixing will not be negligible even in the region not close to the resonances.

For the above reasons, a large-scale nonaxisymmetric structure will not be expected in general in nonself-gravitating geometrically thin disks. However, there is an exception. This is one-armed waves ($m=1$). In nonrelativistic Keplerian disks, $(\omega - m\Omega)^2 = \kappa^2$ is realized for $\omega=0$ when $m=1$. This has the following implications.

Let us consider an one-armed small amplitude perturbation which stands still in the inertial frame. If an observer rotating with the disk at radius r sees the perturbation, he notices a periodic time variation with frequency $\Omega(r)$. This frequency is just equal to the epicyclic frequency $\kappa(r)$ since $\kappa = \Omega$ in nonrelativistic Keplerian disks. This means that one-armed perturbations are just neutral throughout the whole region of the disk, if the effects of pressure force are neglected.

In the framework of the inertial system, the above-mentioned neutrality of one-armed perturbations can be explained in the following way. Let us consider a bunch of particles with the same angular momentum. It is assumed that they were at the same position A at an initial epoch and started to turn around the disk center at the same time with various velocities slightly different from the velocity of the circular orbit of that angular momentum. After one turn around the disk center, they gather again at the initial point A at the same epoch. If many particles exist and a steady distribution of particles along orbits is realized, a density enhancement occurs at the point A, which is a fixed point in the inertial frame. The above consideration can be made for particles with various angular momenta.

From the above considerations, we find that unlike the waves with $m \neq 1$, whether or not one-armed perturbations can have the wave nature and where this occurs cannot be determined by the examination only of the disk rotation. We must examine the effects of the pressure and density distributions in order to know the net amount of the restoring force acting on the perturbations. It is also evident at this early stage that waves with $m=1$, if they exist, have frequencies much lower than the epicyclic frequency.

The purpose of this paper is to show the possibility that one-armed low frequency waves which are trapped in the central region of the disk exist. The presence of the trapping nature is important to anticipate that the waves considered are really excited and maintained in the disk. If there is no trapping, the waves will be propagated away and will be damped finally.

2. Basic Equations Describing Perturbations

We consider a geometrically thin, nonself-gravitating steady disk which is surrounding a central object of mass M . Small amplitude, nonaxisymmetric perturbations are superposed on such a disk. The perturbations are assumed to be inviscid and adiabatic. The unperturbed disk may have steady radial flows (i.e., accretion flow or wind), but their effects on perturbed motions are neglected in this paper.

The cylindrical coordinates (r, φ, z) whose origin is the disk center is introduced to describe the perturbations. The z -axis is perpendicular to the disk plane. The equation of continuity describing small-amplitude perturbations is then

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) \rho_1 + \frac{1}{r} \frac{\partial}{\partial r} (\rho_0 r u_r) + \frac{\partial}{r \partial \varphi} (\rho_0 u_\varphi) + \frac{\partial}{\partial z} (\rho_0 u_z) = 0, \quad (2.1)$$

where $\rho_0(r, z)$ is the density in the unperturbed state and ρ_1 is the perturbation over ρ_0 . The velocity associated with the perturbation is (u_r, u_φ, u_z) . The quantity Ω is the angular velocity of disk rotation in the unperturbed state. This Ω is close to the Keplerian one, $\Omega_K(r) = (GM/r^3)^{1/2}$, but deviates slightly by the presence of the pressure force [see, e.g., equation (2.15)].

The equations of motion describing perturbed motions are

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) u_r - 2\Omega u_\varphi = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial r} + \frac{\rho_1}{\rho_0^2} \frac{\partial p_0}{\partial r}, \quad (2.2)$$

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) u_\varphi + \frac{\kappa^2}{2\Omega} u_r = -\frac{1}{\rho_0} \frac{\partial p_1}{r \partial \varphi}, \quad (2.3)$$

and

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right) u_z = -\frac{1}{\rho_0} \frac{\partial p_1}{\partial z} + \frac{\rho_1}{\rho_0^2} \frac{\partial p_0}{\partial z}, \quad (2.4)$$

where p_1 is the perturbation over the unperturbed pressure p_0 , and κ is the epicyclic frequency defined by

$$\kappa^2 = 2\Omega \left(2\Omega + r \frac{d\Omega}{dr} \right). \quad (2.5)$$

The final equation to be discussed is the adiabatic relation, for which some comments

will be necessary. We assume that in the unperturbed disk the density is stratified in the vertical direction so that the convective neutrality holds. That is, $p_0 \propto \rho_0^\gamma$ in the vertical direction, where γ is the ratio of specific heats. Then the equation describing small amplitude, adiabatic perturbations is

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}\right)(p_1 - c_s^2 \rho_1) - \left(1 - \frac{\Gamma}{\gamma}\right) c_s^2 \frac{\partial \rho_0}{\partial r} u_r = 0, \quad (2.6)$$

where c_s is the sound velocity defined by

$$c_s = (\gamma p_0 / \rho_0)^{1/2}, \quad (2.7)$$

and Γ is a quantity characterizing the density-pressure relation in the horizontal direction in the unperturbed state:

$$\Gamma = \frac{\partial \ln p_0 / \partial r}{\partial \ln \rho_0 / \partial r}. \quad (2.8)$$

In general, Γ is not equal to γ , because the relation $p_0 \propto \rho_0^\gamma$ is not assumed in the horizontal direction.

Equations (2.1) to (2.4) and (2.6) are basic equations describing perturbations in the present problem. The quantities describing perturbations are ρ_1 , p_1 , u_r , u_φ , and u_z .

The basic equations describing perturbations are cyclic in the variables t and φ , and hence we may seek solutions in the form of normal modes

$$\rho_1 \propto \exp[i(\omega t - m\varphi)], \quad (2.9)$$

with similar expressions for the other variables. The positive integer m denotes the number of arms. Eliminating u_r , u_φ , u_z , and ρ_1 from equations (2.1) to (2.4) and (2.6), we obtain the following equation for the pressure perturbation p_1 :

$$\rho_0 \left(\frac{\partial^2}{\partial r^2} + S^* \frac{\partial}{\partial r} + Q^* \right) \left(\frac{p_1}{\rho_0} \right) + (\omega - m\Omega) T^* \frac{\partial}{\partial z} \left[\frac{\rho_0}{\omega - m\Omega} \left(\frac{\partial}{\partial z} + U^* \right) \left(\frac{p_1}{\rho_0} \right) \right] = 0. \quad (2.10)$$

The derivation of this equation is given in the Appendix with expressions for $S^*(r, z)$, $Q^*(r, z)$, $T^*(r, z)$, and $U^*(r, z)$.

In disk oscillations, the vertical motion and the horizontal motion are coupled to each other in general. Because of this, we cannot treat only the relations among vertically integrated quantities even if the disk is thin, unlike the cases of steady accretion disks. This makes the situation somewhat complicated. That is, equation (2.10) is a partial differential equation. Discussion of this equation is important and will be made in a subsequent paper. However, we shall be satisfied here with the following approximate treatment of equation (2.10). Equation (2.10) is integrated in the vertical direction in order to eliminate the last term of the z -derivative. Then, as long as the radial variation of the effective thickness of the disk is slow, the resulting equation can be approximated as

$$\left[\frac{d^2}{dr^2} + S(r) \frac{d}{dr} + Q(r) \right] \frac{P_1}{\sigma_0} = 0. \quad (2.11)$$

This is an ordinary differential equation of P_1/σ_0 , where $P_1(r)$ is the pressure perturbation integrated over the vertical direction, and $\sigma_0(r)$ is the surface density in the unperturbed state, obtained by integrating ρ_0 over the vertical direction. Furthermore, $S(r)$ and $Q(r)$ in equation (2.11) are those obtained by replacing ρ_0 in equations (A5) and (A6) by σ_0 :

$$rS(r) = -\frac{d}{d \ln r} \ln \left| \frac{\kappa^2 - (\omega - m\Omega)^2(1+R)}{\sigma_0 r} \right| \quad (2.12)$$

and

$$\begin{aligned} r^2 Q(r) = & \frac{2m\Omega}{\omega - m\Omega} \frac{d}{d \ln r} \ln \left| \frac{\kappa^2 - (\omega - m\Omega)^2(1+R)}{\sigma_0 \Omega} \right| - m^2(1+R) - \frac{r^2}{c_s^2} [\kappa^2 - (\omega - m\Omega)^2] \\ & + \left(1 - \frac{\Gamma}{\gamma}\right) \frac{d \ln \sigma_0}{d \ln r} \left[\frac{4m\Omega}{\omega - m\Omega} - \frac{d}{d \ln r} \ln |\kappa^2 - (\omega - m\Omega)^2(1+R)| \right], \end{aligned} \quad (2.13)$$

where

$$R(r) = -\frac{\Gamma}{\gamma} \left(1 - \frac{\Gamma}{\gamma}\right) \frac{c_s^2}{(\omega - m\Omega)^2 r^2} \left(\frac{d \ln \sigma_0}{d \ln r} \right)^2. \quad (2.14)$$

In the above equations, the angular velocity Ω of the disk rotation has been regarded as a function of r . Its deviation from the Keplerian one is given by the radial force balance in the unperturbed state:

$$-\frac{1}{\sigma_0} \frac{dP_0}{dr} + \Omega^2 r - \Omega_K^2 r = 0, \quad (2.15)$$

where $P_0(r)$ is the vertically integrated pressure in the unperturbed state. The quantity Γ is also regarded as a function of r and defined by the equation obtained by replacing ρ_0 and p_0 in equation (2.8) by σ_0 and P_0 , respectively.

Equation (2.11) is nothing but what can be obtained from equations describing those motions which are completely restricted in the disk plane. Such motions have been treated in the study of galactic density waves when the gas analogue of spiral galaxies is considered (e.g., Lin and Lau 1979). In this sense, it is worthwhile to compare equation (2.11) with the equations in such study. In the limit of $\Gamma = \gamma$ equation (2.11) is reduced to the equation obtained from equation (B6) of Lau and Bertin (1978) by neglecting the potential perturbation. Furthermore, equation (2.11) is identical to the equation of the adiabatic limit of equation (2.9) of Kato and Inagaki (1978).

If the dependent variables in equation (2.11) is changed so that the term of the first order derivative with respect to r is eliminated, we have

$$\frac{d^2 u}{dr^2} + k^2 u = 0, \quad (2.16)$$

where

$$u = \frac{P_1}{\sigma_0} \left[\frac{\sigma_0 r}{|\kappa^2 - (\omega - m\Omega)^2(1+R)|} \right]^{1/2}, \quad (2.17)$$

and

$$\begin{aligned} k^2 = & -\frac{1}{c_s^2} [\kappa^2 - (\omega - m\Omega)^2] + \frac{1}{r^2} \frac{2m\Omega}{\omega - m\Omega} \frac{d}{d \ln r} \ln \left| \frac{\kappa^2 - (\omega - m\Omega)^2(1+R)}{\sigma_0 \Omega} \right| \\ & - \frac{m^2}{r^2} (1+R) - \left[\frac{\sigma_0 r}{|\kappa^2 - (\omega - m\Omega)^2(1+R)|} \right]^{-1/2} \frac{d^2}{dr^2} \left[\frac{\sigma_0 r}{|\kappa^2 - (\omega - m\Omega)^2(1+R)|} \right]^{1/2} \\ & + \frac{1}{r^2} \left(1 - \frac{\Gamma}{\gamma}\right) \frac{d \ln \sigma_0}{d \ln r} \left[\frac{4m\Omega}{\omega - m\Omega} - \frac{d}{d \ln r} \ln |\kappa^2 - (\omega - m\Omega)^2(1+R)| \right] \\ & + \frac{1}{r^2} \frac{d}{d \ln r} \left[\left(1 - \frac{\Gamma}{\gamma}\right) \frac{d \ln \sigma_0}{d \ln r} \right]. \end{aligned} \quad (2.18)$$

3. One-Armed Waves

3.1. Wave Equation for One-Armed Waves

Hereafter we consider the case of one arm, i.e., $m=1$. In this case $[\kappa+(\omega-m\Omega)(1+R)^{1/2}]/\kappa$ is a small quantity compared with unity, because the main terms in the numerator are canceled out as $\kappa=\Omega$ in the Keplerian limit. From equations (2.5) and (2.15) we know that

$$\Omega \cong \Omega_K \left(1 + \frac{1}{2\Omega_K^2 r^2} \frac{r}{\sigma_0} \frac{dP_0}{dr} \right) \quad (3.1)$$

and

$$\kappa \cong \Omega_K \left[1 + \frac{1}{2\Omega_K^2 r^2} \frac{d}{dr} \left(\frac{r^3}{\sigma_0} \frac{dP_0}{dr} \right) \right]. \quad (3.2)$$

Hence we have

$$\begin{aligned} \varepsilon(r, \omega) &\equiv \frac{\kappa + (\omega - m\Omega)(1+R)^{1/2}}{\kappa} \\ &= \frac{\omega}{\Omega_K} + \frac{1}{2\Omega_K^2 r^3} \left[\frac{d}{d \ln r} + \left(1 - \frac{\Gamma}{\gamma} \right) \frac{d \ln \sigma_0}{d \ln r} \right] \left(\frac{r^2}{\sigma_0} \frac{dP_0}{dr} \right). \end{aligned} \quad (3.3)$$

Substitution of equations (3.1)–(3.3) into equation (2.18) gives, after some lengthy manipulation,

$$k^2 r^2 = -2\varepsilon + A_0 + A_{-1}\varepsilon^{-1} + A_{-2}\varepsilon^{-2}, \quad (3.4)$$

where

$$\begin{aligned} A_0 &= -\frac{21}{16} + \frac{1}{4} \left(5 - \frac{d \ln \sigma_0}{d \ln r} - 2 \frac{d}{d \ln r} \right) \frac{d \ln \sigma_0}{d \ln r} \\ &\quad + \left(-\frac{5}{2} + \frac{\Gamma}{\gamma} \frac{d \ln \sigma_0}{d \ln r} + \frac{d}{d \ln r} \right) \left(1 - \frac{\Gamma}{\gamma} \right) \frac{d \ln \sigma_0}{d \ln r}, \end{aligned} \quad (3.5)$$

$$A_{-1} = -\frac{1}{4} \left(1 - 2 \frac{\Gamma}{\gamma} \right) f \frac{d \ln \sigma_0}{d \ln r} + \frac{r^4}{4c_s^2} \frac{d}{dr} \left(\frac{f c_s^2}{r^3} \right), \quad (3.6)$$

$$A_{-2} = -\frac{3}{16} f^2, \quad (3.7)$$

and

$$\varepsilon = \frac{\Omega_K^2 r^2}{c_s^2} \varepsilon(r; \omega). \quad (3.8)$$

In equations (3.6) and (3.7), f denotes

$$f = \frac{r^{3/2}}{c_s^2} \frac{d}{dr} \left\{ \frac{1}{r^{3/2}} \left[\frac{d}{d \ln r} + \left(1 - \frac{\Gamma}{\gamma} \right) \frac{d \ln \sigma_0}{d \ln r} \right] \left(\frac{r^2}{\sigma_0} \frac{dP_0}{dr} \right) \right\}. \quad (3.9)$$

3.2. Adopted Disk Models

Our main interest here is to know in what region in the disk the value of k^2 is positive.

To examine this problem, we shall specify models of disks. The models adopted are the standard accretion disks with the so-called α -viscosity (Shakura and Sunyaev 1973); the $r\varphi$ -component of the viscous stress tensor, $t_{r\varphi}$, is proportional to gas pressure p_g as $t_{r\varphi} = \alpha p_g$ in the case where the gas pressure p_g dominates over the radiation pressure p_r , while in the case of $p_r \gg p_g$, we adopt $t_{r\varphi} = \alpha(p_r p_g)^{1/2}$, following Ichimaru (1976) and Hoshi (1977). The disks are assumed to be optically thick with electron scattering. The inner boundary condition adopted tentatively is that the disk has a free surface ($t_{r\varphi} = 0$) at $r = r_{in}$.

Then, in the case of $p_g \gg p_r$, we have (Shakura and Sunyaev 1973)

$$P_0 \propto (r/r_{in})^{-3/2} [1 - (r/r_{in})^{-1/2}] \quad (3.10)$$

and

$$\sigma_0 \propto (r/r_{in})^{-3/5} [1 - (r/r_{in})^{-1/2}]^{3/5}. \quad (3.11)$$

On the other hand, in the case of $p_r \gg p_g$, we have (e.g., Shibazaki 1978)

$$P_0 \propto (r/r_{in})^{-8/3} [1 - (r/r_{in})^{-1/2}]^{8/9} \quad (3.12)$$

and

$$\sigma_0 \propto (r/r_{in})^{1/3} [1 - (r/r_{in})^{-1/2}]^{-1/9}. \quad (3.13)$$

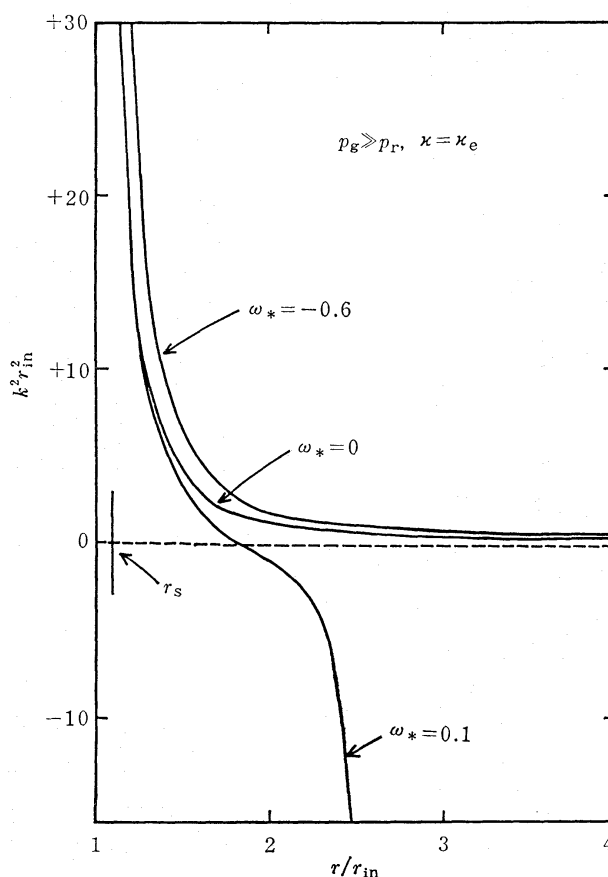


Fig. 1. Functional forms of $k^2(r)$ for some values of parameter ω_* . The unperturbed disk adopted is the standard accretion disk where the gas pressure p_g dominates over the radiation pressure p_r and the opacity comes from the electron scattering.

The above standard accretion disk models are not realistic near the inner edge r_{in} . That is, by the adoption of free boundary condition, the inner edge r_{in} is a kind of singular point and some physical quantities (e.g., the accretion velocity in the case of $p_g \gg p_r$, and the surface density in the case of $p_r \gg p_g$) become infinite there. This unrealistic behavior near r_{in} has been removed by considering the disk structure beyond the radius of marginally stable circular orbit in the case when the central object is a black hole (Stoeger 1980; Kato et al. 1982; Paczyński and Bisnovatyi-Kogan 1982; Muchotrzeb and Paczyński 1982; Muchotrzeb 1982).

In this paper, however, we are interested in a nonrelativistic disk around a central object whose radius is r_s . The central object is assumed to be rotating more slowly than the breakup speed at the equator. Then at the boundary between the central object and the disk, there appears a viscous boundary layer where the disk rotation slows down inwards from the Keplerian one to that of the surface of the central object (e.g., Lynden-Bell and Pringle 1974). In this region the formulation made in this paper breaks down since the deviation from the Keplerian motion is not small. Detailed discussion on this layer is beyond the scope of this paper (some discussion related to this point will be made in the last section). Hence, we shall be satisfied here by simply adopting a truncated standard disk model. That is, we assume that the disk structure obeys equations (3.10) and (3.11) or (3.12) and (3.13) until a radius r_s slightly larger than r_{in} and there the disk has a rigid surface of the central object. The value $r_s/r_{\text{in}} (> 1)$ is a parameter.

Substituting equations (3.10) and (3.11) or equations (3.12) and (3.13) into expressions for A_0 to A_{-2} , we obtain the values of $k^2 r_{\text{in}}^2$ as a function of r/r_{in} with a dimensionless

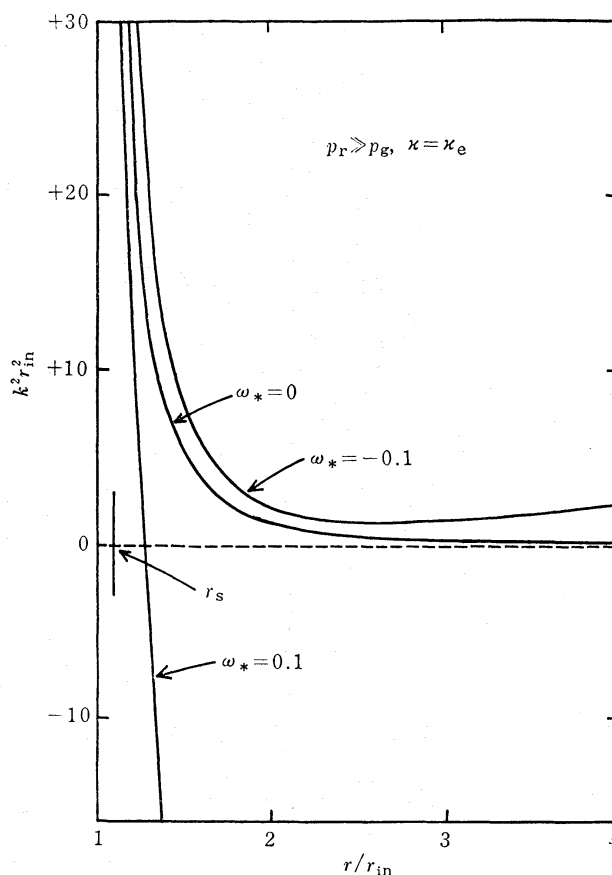


Fig. 2. Same as figure 1, except that $p_r \gg p_g$.

parameter ω_* defined by

$$\frac{\omega}{\Omega_{\text{in}}} = \omega_* \left(\frac{c_s^2}{\Omega^2 r^2} \right)_{\text{in}}. \quad (3.14)$$

Here the subscript “in” means the values at $r=r_{\text{in}}$. Of course, the disk is truncated at r_s which is larger than r_{in} , as mentioned before. For convenience, however, all quantities have been normalized by quantities that the disk would have at $r=r_s$ if there were no truncation. The parameter ω_* denotes the magnitude of frequency of waves. Functional forms of $k^2 r_{\text{in}}^2$ are shown in figures 1 and 2 for some values of parameter ω_* . Figure 1 is for the case of $p_g \gg p_r$ and figure 2 is for the case of $p_r \gg p_g$.

4. Wave Trapping and Normal Modes of Oscillations

An important result obtained in the previous section is that for a range of parameter ω_* , $k^2(r)$ is positive in the inner region of the disks, but becomes negative in the outer region. This means that when there is a rigid inner boundary of the disks [in this paper we assume a rigid inner boundary at $r=r_s$ ($>r_{\text{in}}$) as mentioned before], waves are trapped in the inner region of $k^2 > 0$. The purpose of this section is to determine the value of ω_* for trapped normal modes of oscillations. Estimate of ω_* by the WKB method will be sufficient for our present purpose, because we are interested only in a rough magnitude of ω_* .

Let us denote the turning point (the zero point) of k^2 by r_t . Near the point, k^2 can be approximated as $k^2 = -a^2(r-r_t)$, where a is a constant. Hence the solution of equation (2.16) near $r=r_t$ is

$$u \sim (r_t - r)^{1/2} J_{\pm 1/3} \left[\frac{2}{3} a(r_t - r)^{3/2} \right]. \quad (4.1)$$

On the other hand, in the region of $r_s \leq r \ll r_t$, the solution of equation (2.16) is written approximately as

$$u \sim k^{-1/2} \exp \left[\pm i \int k dr \right]. \quad (4.2)$$

A proper form of u which tends to equations (4.1) and (4.2) in the both limits is (e.g., Morse and Feshbach 1953)

$$u \cong \left(\frac{w}{k} \right)^{1/2} [A J_{1/3}(w) + B J_{-1/3}(w)], \quad (4.3)$$

where

$$w = \int_r^{r_t} k dr. \quad (4.4)$$

Here, A and B are arbitrary constants to be determined from the boundary conditions.

One of the boundary conditions we should take is that u tends to zero at $r=\infty$. This implies $A=B$ (e.g., Morse and Feshbach 1953). The other condition we adopt here is $u=0$ at $r=r_s$. Combining these conditions, we have

$$J_{1/3}(w_s) + J_{-1/3}(w_s) = 0, \quad (4.5)$$

where w_s is defined by

Table 1. Frequency ω_* and trapped radius r_t .

r_s/r_{in}	$p_g \gg p_r$		$p_r \gg p_g$	
	ω_*	r_t/r_{in}	ω_*	r_t/r_{in}
1.05	0.060	2.1	0.014	1.8
1.10	0.035	2.6	0.0059	2.2
1.20	0.017	4.1	0.0009	3.2

$$w_s = \int_{r_s}^{r_t} k dr .$$

(4.6)

Solutions of equation (4.5) are a discrete set of w_s . From the lowest (positive) value, they are $w_s=2.38, 5.51, \dots$. Combining this with equation (4.6) we have a discrete set of dimensionless frequencies ω_* . The value of ω_* corresponding to $w_s=2.38$ is the frequency of the fundamental mode of oscillations. The values of ω_* and r_t for the fundamental mode are shown in table 1 for some values of r_s . Table 1 shows that as the inner boundary r_s of the disk leaves r_{in} , the frequency of the fundamental mode decreases and the trapping region becomes wider.

5. Discussion

In nearly Keplerian disks, one-armed perturbations have a particular position different from the other types of perturbations. That is, in such disks one-armed perturbations are neutral in the lowest order approximations. Because of this, to know the net value of restoring force acting on the perturbation, the effects of the pressure and density distributions on the unperturbed disks must be examined carefully. An important result obtained at this early stage is that if one-armed perturbations have a wavy characteristics, the wave frequency is quite low. If we consider that $k^2 \cong [\kappa^2 - (\omega - m\Omega)^2]/c_s^2 \cong \omega\Omega/c_s^2$ and $kr \cong 1$ for the fundamental mode, we know that the order of the frequency of one-armed oscillation is

$$\omega \cong \Omega \left(\frac{c_s}{\Omega r} \right)^2 .$$

(5.1)

The results in the previous sections imply that the numerical factor which should multiply the right-hand side is much less than 10^{-1} in the cases considered.

In this paper we have adopted the standard accretion disk models in order to show examples. Two cases were considered: (1) the case where the gas pressure p_g dominates over the radiation pressure p_r , and (2) the opposite case. In these two cases, the surface density distribution is rather different. That is, in the case of $p_g \gg p_r$, it has a maximum at $r/r_{in}=9/4$, while in the case of $p_r \gg p_g$, it has a minimum at $r/r_{in}=49/36$. In spite of this difference, results shown in figures 1 and 2 and in table 1 are qualitatively the same in both cases. This similarity comes from the fact that the difference of density distribution has only minor effects on k^2 . The major contribution to the form of k comes from the pressure distribution through ϵ . The pressure distribution is quite similar in both cases, because of the adoption of the α -models.

Here a comment on the boundary condition at the inner edge of the disk will be necessary. At the boundary between the disk and the central object, there appears a viscous boundary layer where the disk rotation changes abruptly from the Keplerian one

to that of the surface of the central object. Without examining in detail the wave motions in this viscous boundary layer, we have simply adopted, in section 3, truncated models of the standard accretion disks and assumed a rigid boundary at $r=r_s(>r_{in})$. An important question is how the above simple procedure to treat the inner boundary of the disk is valid. In other words, our question is whether there is the possibility that a resonance (the Lindblad resonance) appears in this boundary layer and the waves are absorbed there before reflected back outwards.

This question is examined here by comparing the values of $\kappa^2 - (\omega - m\Omega)^2$ and $\kappa^2 - (\omega - m\Omega)^2(1+R)$. In the case of $p_g \gg p_r$, from the standard models we have $R > 0$ and $\varepsilon < 0$ in the region near $r=r_{in}$. This means that $\kappa^2 - (\omega - m\Omega)^2(1+R) < \kappa^2 - (\omega - m\Omega)^2 < 0$. In the boundary layer the value of Ω does not increase inwards as much as expected from the Keplerian motion. This implies that κ becomes larger than Ω as we proceed inwards in the layer. Hence, if other quantities are fixed, both values of $\kappa^2 - (\omega - m\Omega)^2(1+R)$ and $\kappa^2 - (\omega - m\Omega)^2$ increase inwards, but the latter will become zero and positive earlier than the former, because $\kappa^2 - (\omega - m\Omega)^2(1+R) < \kappa^2 - (\omega - m\Omega)^2$. If this is the case, the wave propagation region ($k^2 > 0$) will be terminated at a radius outside the radius where the resonance [$\kappa^2 - (\omega - m\Omega)^2(1+R) = 0$] appears. In such cases, the waves running from the outside towards the inner edge of the disk will be reflected back outward without being absorbed much, and the inner boundary condition adopted in section 4 will be qualitatively adequate.

We should emphasize here that in this paper we have approximated the two-dimensional wave equation [equation (2.10)] by the one-dimensional one [equation (2.11)], by integrating it in the vertical direction. This procedure is not rigorous in general. Hence, it is of importance to examine how our present results in this paper are modified when the wave equation is examined fully without introducing the integration process. This examination is of particular interest in relation to the wave behavior (i.e., wave reflection) in the region near the inner boundary, because in that region the disk thickness will change rapidly and the one-dimensional equation cannot represent well the real situation.

A point to be noticed here is the adiabaticity of oscillation. As was shown, the frequency of trapped oscillations is lower than Ω by more than $(c_s/\Omega r)^2$. In accretion disks with α -type viscosity, the thermal time scale is of the order of $1/\alpha\Omega$. Hence, the assumption that the oscillations are adiabatic is valid only when α is smaller than $(c_s/\Omega r)^2$. Otherwise, oscillations will occur under a balance between heating and cooling. Our next problem is thus to examine how our conclusions in this paper are modified in such cases.

Finally comparison with observations is made briefly. In many Be stars, the so-called V/R variations (time variations of the ratio of violet and red emission components) of the order of 7 yr have been observed (Copeland and Heard 1963; Hirata and Hubert-Delplace 1981). These variations are often interpreted as a result of nonaxisymmetric structure of the shell. As typical values, we take here 100 km s^{-1} and 20 km s^{-1} as the rotational velocity of the disks and the sound velocity on the disks, respectively. The radius of the shell is taken to be 10^{12} cm . Then we have $\Omega(c_s/\Omega r)^2 \sim 4 \times 10^{-7} \text{ s}^{-1}$. Hence the period τ of one-armed oscillation is estimated as $\tau = (2\pi/\Omega)[(c_s/\Omega r)^2 \omega_*]^{-1} \sim 0.5/\omega_*$ yr. This value of τ is somewhat larger than 10 yr if ω_* given in table 1 is adopted. However, there are many uncertainties in our estimate of ω_* . For example, the assumption that the oscillations are adiabatic is obviously irrelevant to this case. In addition, we should consider that the value of ω_* will depend strongly on the models adopted. Furthermore, we should remember that although we are treating small amplitude wave motions, the real phenomena will be highly nonlinear. Hence, the above difference between observed and theoretical figures should not be taken seriously.

Another example is periodic ($\sim 100 \text{ d}$) and quasi-periodic ($\sim$ a few years) light varia-

tions observed in some QSOs and Seyfert galaxies (e.g., Ozernoy and Usov 1977). The central objects of these systems will be black holes. Hence, direct application of our present results will be unsuitable, since the epicyclic frequency κ is not close to the angular velocity Ω of the disk rotation. Furthermore, whether or not the trapping of oscillations occurs has not been examined yet in the cases of black holes. However, it will be still interesting to compare the values obtained from direct application of our results with the above-mentioned observed periods. The Schwarzschild radius r_g for mass M is $\sim 3 \times 10^5 (M/M_\odot)$ cm. If the relativistic effects are neglected, the Keplerian frequency Ω of disk rotation is $\sim 7.2 \times 10^4 (M/M_\odot)^{-1} (r/r_g)^{-3/2} \text{ s}^{-1}$. For the sound velocity of 10^3 km s^{-1} , we have $\Omega(c_s/\Omega r)^2 \sim 1.5 (M/M_\odot)^{-1} (r/r_g)^{-1/2} \text{ s}^{-1}$. Hence, the period of one-armed oscillation is $\tau \sim 4.2 \omega_*^{-1} (M/M_\odot) (r/r_g)^{1/2} \text{ s}$. If we take $M/M_\odot = 10^6$, $r/r_g = 10$, we have $\tau \sim 1.5 \times 10^7 \omega_*^{-1} \text{ s}$, which is 5 yr for $\omega_* = 0.1$.

Note added in proof: It was pointed out by Mr. R. Matsumoto that equation (2.15) is not rigorous. The gravitational force in the radial direction changes slightly with height within the disk. This effect is neglected in equation (2.15), although it is not negligible. If this effect is taken into account in equation (2.15), some changes are necessary in subsequent equations, although final results of this paper have no qualitative change.

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Appendix. Derivation of Equation (2.10) and Expressions S^* , Q^* , T^* , and U^*

From equation (2.6) we have

$$\rho_1 = \frac{1}{c_s^2} p_1 + i \frac{(1 - \Gamma/\gamma)}{(\omega - m\Omega)r} \frac{\partial \ln \rho_0}{\partial \ln r}. \quad (\text{A1})$$

This is substituted into equation (2.2). After this, u_r or u_ϕ is eliminated from equations (2.2) and (2.3) to obtain

$$[(\omega - m\Omega)^2(1 + R^*) - \kappa^2] \rho_0 u_r = i(\omega - m\Omega) \left(\frac{\partial}{\partial r} - \frac{\Gamma}{\gamma} \frac{\partial \ln \rho_0}{\partial r} \right) p_1 - i \frac{2m\Omega}{r} p_1 \quad (\text{A2})$$

and

$$[(\omega - m\Omega)^2(1 + R^*) - \kappa^2] \rho_0 u_\phi = -\frac{\kappa^2}{2\Omega} \left(\frac{\partial}{\partial r} - \frac{\Gamma}{\gamma} \frac{\partial \ln \rho_0}{\partial r} \right) p_1 + \frac{m(\omega - m\Omega)}{r} (1 + R^*) p_1, \quad (\text{A3})$$

where

$$R^*(r, z) = -\frac{\Gamma}{\gamma} \left(1 - \frac{\Gamma}{\gamma} \right) \frac{c_s^2}{(\omega - m\Omega)^2 r^2} \left(\frac{\partial \ln \rho_0}{\partial \ln r} \right)^2. \quad (\text{A4})$$

Substitution of the above three equations and equation (2.4) into equation (2.1) gives finally an equation of p_1 , which is equation (2.10). In this equation S^* , Q^* , T^* , and U^* are given, respectively, by

$$rS^*(r, z) = -\frac{d}{d \ln r} \ln \left| \frac{\kappa^2 - (\omega - m\Omega)^2(1 + R^*)}{\rho_0 r} \right|, \quad (\text{A5})$$

$$\begin{aligned} r^2 Q^*(r, z) = & \frac{2m\Omega}{\omega - m\Omega} \frac{d}{d \ln r} \ln \left| \frac{\kappa^2 - (\omega - m\Omega)^2(1 + R^*)}{\rho_0 \Omega} \right| - \frac{r^2}{c_s^2} [\kappa^2 - (\omega - m\Omega)^2] \\ & + \left(1 - \frac{\Gamma}{\gamma} \right) \frac{d \ln \rho_0}{d \ln r} \left[\frac{4m\Omega}{\omega - m\Omega} - \frac{d}{d \ln r} \ln |\kappa^2 - (\omega - m\Omega)^2(1 + R^*)| \right], \end{aligned} \quad (\text{A6})$$

$$T^*(r, z) = \frac{(\omega - m\Omega)^2(1 + R^*) - \kappa^2}{(\omega - m\Omega)^2}, \quad (\text{A7})$$

and

$$U^*(r, z) = \frac{c_s^2}{\omega - m\Omega} \left(1 - \frac{\Gamma}{\gamma} \right) \frac{\partial \ln \rho_0}{\partial r} \frac{\partial \ln \rho_0}{\partial z} \frac{(\omega - m\Omega) \left[\frac{\partial}{\partial r} + \left(1 - \frac{\Gamma}{\gamma} \right) \frac{\partial \ln \rho_0}{\partial r} \right] - \frac{2m\Omega}{r}}{(\omega - m\Omega)^2(1 + R^*) - \kappa^2}. \quad (\text{A8})$$

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