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A MODIFICATION OF THE NEWTONIAN DYNAMICS: IMPLICATIONS FOR GALAXIES[1]

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ABSTRACT

I use a modified form of the Newtonian dynamics (inertia and/or gravity) to describe the motion of bodies in the gravitational fields of galaxies, *assuming that galaxies contain no hidden mass*, with the following main results.

1. The Keplerian, circular velocity around a finite galaxy becomes independent of r at large radii, thus resulting in asymptotically flat velocity curves.
2. The asymptotic circular velocity (V_∞) is determined only by the total mass of the galaxy (M): $V_\infty^4 = a_0 GM$, where a_0 is an acceleration constant appearing in the modified dynamics. This relation is consistent with the observed Tully-Fisher relation if one uses a luminosity parameter which is proportional to the observable mass.
3. The discrepancy between the dynamically determined Oort density in the solar neighborhood and the density of observed matter disappears.
4. The rotation curve of a galaxy can remain flat down to very small radii, as observed, only if the galaxy's average surface density Σ falls in some narrow range of values which agrees with the Fish and Freeman laws. For smaller values of Σ, the velocity rises more slowly to the asymptotic value.
5. The value of the acceleration constant, a_0, determined in a few independent ways is approximately $2\times10^{-8}(H_0/50 \text{ km s}^{-1} \text{ Mpc}^{-1})^2 \text{ cm s}^{-2}$, which is of the order of $CH_0 = 5\times10^{-8}(H_0/50 \text{ km s}^{-1} \text{ Mpc}^{-1}) \text{ cm s}^{-2}$.

The main predictions are:

1. Rotation curves calculated on the basis of the *observed* mass distribution and the modified dynamics should agree with the observed velocity curves.
2. The $V_\infty^4 = a_0 GM$ relation should hold exactly.
3. An analog of the Oort discrepancy should exist in all galaxies and become more severe with increasing r in a predictable way.

Subject headings: galaxies: internal motions — galaxies: stellar content — galaxies: structure — stars: stellar dynamics

I. INTRODUCTION

In an accompanying paper (Milgrom 1983*a*, hereafter Paper I) I suggest that a certain modification of the Newtonian laws of dynamics (inertia and/or gravity) eliminates the need to assume the existence of hidden mass in galaxies and galaxy systems. The basic assumptions of the modified dynamics are: (*a*) Standard dynamics breaks down in the limit of small accelerations; (*b*) In the limit of small accelerations, the acceleration of a test particle, in a gravitating system, is given by $(a/a_0)\boldsymbol{a} \approx \boldsymbol{g}_N$, where $\boldsymbol{g}_N$ is the conventional gravitational acceleration and a_0 is a constant with the dimensions of acceleration; (*c*) The transition from the Newtonian regime to the low acceleration asymptotic regime is determined by the acceleration constant a_0 (in the sense that the transition occurs within a range of accelerations of order a_0 around a_0).

I discuss in Paper I questions of principle concerning the proposed modification. In particular, it is pointed out that the modification can be one of the law of inertia, in which case it has to be implemented, whenever the acceleration is small, for whatever combination of forces produces it. Alternatively, it may turn out to be a modification of gravity alone. Within each interpretation there are still different formulations which can be built on the basic assumptions. *It is important to realize (see § IX) that all the qualitative results of this paper do not depend on the exact formulation or interpretation of the modified dynamics. They result only from the above basic assumptions.*

In the present paper I discuss the implication of the proposed modification for galaxies. In testing the above ideas, I assume that no hidden mass exists besides that

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in stars, ISM, and IGM, which can be observed directly. It is, of course, possible that some unseen mass exists, but I would argue that it does not play the major role in determining the dynamics in galaxies and galaxy systems.

I use a nonrelativistic formulation whereby the modified dynamics, in a gravitational field, takes the form

$$\mu(a/a_0)\boldsymbol{a}=\boldsymbol{g}_N, \quad (1)$$

where $\boldsymbol{g}_N$ is the conventional gravitational acceleration, assumed to have the usual dependence on its sources and their distribution in space, and $\boldsymbol{a}$ is the acceleration of a particle ($a=|\boldsymbol{a}|$) with respect to some fundamental frame (determined, say, by distant matter in the universe). The acceleration $\boldsymbol{g}_N$ is assumed to be a *static* field in this frame. The quantity μ is a function of the ratio a/a_0. For $a/a_0 \gg 1$, $\mu \approx 1$ (with the usual choice of the gravitational constant G). For $x \ll 1$, $\mu(x) \propto x$, in which case we can choose a_0 so that $\mu(x) \approx x$ for $x \ll 1$.

In Milgrom (1983*b*, hereafter Paper III) I show that this modification results in masses for galaxy systems (binaries, clusters, etc.) which are consistent with all being due to the observed stars and gas.

In the following sections I discuss what I think are the more straightforward (to derive and to test) implications, for galaxies, of modifying the dynamics in this way. In § II, I consider the rotation curves of idealized disk galaxies. In § III, I discuss Tully-Fisher relations. Section IV is devoted to motions perpendicular to the plane and the relevance to the Oort limit. Section V deals with our own Galaxy. In § VI, I discuss two aspects of galaxy structure, the existence of a maximum average surface brightness (the Freeman and Fish laws) and the particular distributions of the surface brightness galaxies have. In § VII, I briefly discuss elliptical galaxies. In § VIII I list the main predictions related to observations of galaxies. Section IX is a discussion.

II. THE VELOCITY CURVE

The observed velocity curves of disk galaxies provide perhaps the major constraint on theories which propose that there is no hidden mass in appreciable quantities. If the observed mass is all there is, the expected rotation curves can be calculated on the basis of the observed mass distribution and the theory, and compared with the observed curves. (Some uncertainty still remains in deducing mass distribution from light distribution.)

The prominent feature of the rotation curves seems to be their asymptotic flatness, evidenced by the many curves given by Krumm and Salpeter (1977), Salpeter (1978), Bosma (1978, 1981*a*, *b*) and by Rubin, Ford, and Thonnard (1978, 1980, 1982). Various aspects of this feature are discussed by Faber and Gallagher (1979, hereafter FG) with further references.

With the proposed modification of the dynamics, the asymptotic flatness of the rotation curve of a finite massive body, is ensured by the required linearity of $\mu(x)$ for small values of the argument, or from assumption (b) of the modified dynamics (and indeed it is the asymptotic flatness which motivated this requirement).

At large galactic radii we can put $g_N \approx MGr^{-2}$ and with $a=V^2/r$ we get:

$$V^4(r) \approx MGa_0. \quad (2)$$

Here M is the total mass of the galaxy.

Calculation of detailed velocity curves for actual galaxies are important for the following reasons: (1) Testing the theory by comparing calculated curves for the observed mass distribution of galaxies with their observed velocity curves; (2) Obtaining information on the function $\mu(x)$ via such comparisons; (3) It appears that the flatness of the rotation curves has more to it than the fact that they are asymptotically flat. The velocity remains, in many cases, approximately constant down to radii well within the observed mass distribution. I find that there is a tight link between this fact and the question of why galaxy disks tend to have certain profiles and a preferred value of the surface density. To understand this link I consider theoretical velocity curves for various mass distributions. I discuss this matter in more detail in § VI.

In this section I calculate rotation curves, on the basis of the modified dynamics, for an idealized model galaxy made up of a planar flat disk and a spherical bulge, assuming purely circular motions in the disk.

Let $g_N(r)$ be the conventional radial gravitational acceleration at radius r in the disk's plane. Then the rotational velocity for equilibrium circular orbits is given by (from eq. [1]).

$$[V^2(r)/r]\mu(V^2(r)/ra_0)=g_N(r). \quad (3)$$

Let M_d, M_s, and M be, respectively, the mass in the disk, the sphere, and the total mass ($M=M_d+M_s$). Let h be some characteristic length scale in the galaxy which we shall use as unit length, i.e. define $s \equiv r/h$. As we saw (eq. [2]), for $r\to\infty, V(r)\to V_\infty=(MGa_0)^{1/4}$. Define $v(s)\equiv V(sh)/V_\infty$.

It is true in general that $g_N(r)$ can be written in the form

$$g_N(r)=MGr^{-2}\gamma(s,t_1,\ldots,t_n), \quad (4)$$

where γ depends on the mass distribution and t_i are dimensionless parameters (such as the fraction of mass in the various components, the characterizing length parameters in units of h, etc.). $\gamma(s)\xrightarrow{s\to\infty} 1$.

Equation (3) can then be written in a dimensionless form

$$s^{-1}v^2\xi\mu(s^{-1}v^2\xi)=\xi^2s^{-2}\gamma(s,t_1,\ldots,t_n), \quad (5)$$

where ξ is given by

$$\xi \equiv \left(MG/a_0 h^2\right)^{1/2} = V_\infty^2/a_0 h. \tag{6}$$

For a given density distribution (or equivalently γ), one gets a family of scaled velocity curves $v(s, \xi)$ spanned by the parameter ξ (by solving $x\mu(x) = \xi^2 s^{-2}\gamma$ for $x = s^{-1}v^2\xi$). Note that since $\mu(x) \overset{x \ll 1}{\approx} x, v(s) \xrightarrow{s \to \infty} 1$. In the limit of very small ξ, $v(s)$ becomes independent of ξ, and we obtain the limiting velocity curve:

$$v(s, \xi \ll 1) \approx \left[\gamma(s, t_1, \ldots, t_n)\right]^{1/4}. \tag{7}$$

The parameter ξ measures the "typical" acceleration in the galaxy in units of a_0. If mass is replaced by luminosity, ξ defines some average surface brightness of the galaxy.

I shall not attempt here the computation of the velocity curve of specific galaxies, but rather give the results of a parametric study.

For the sake of concreteness I shall assume that the disk has an exponential surface mass density law $\Sigma(r) \propto e^{-r/h_d}$, and use a deprojected de Vaucouleurs density distribution $\rho_v(r/r_e)$ for the spheroid. In fact I use the approximated density laws for small and large radii given by Young (1976) at radii smaller and larger, respectively, than the radius at which they coincide. The parameter r_e is the effective radius (containing half the projected mass).

Justification for this choice can be found, for example, in de Vaucouleurs (1962), Freeman (1970), de Vaucouleurs and Freeman (1972), Kormendy (1977*a*, *b*), Burstein (1979*a*, *b*) and Whitmore and Kirshner (1981). One perhaps has also to consider exponential disks cut off below a certain radius (Freeman 1970; Kormendy 1977*b*) or cut off at large radii (e.g., van der Kruit and Searle 1981*a*, *b*). I assume that the mass distribution is the same as the observed light distribution in each component separately.

One can write for a sphere + disk galaxy

$$\gamma(s) = \alpha_d \gamma_d(s) + (1 - \alpha_d)\gamma_d(s), \tag{8}$$

where $\alpha_d = M_d/M$, and the spheroidal contribution $\gamma_s(s)$ is simply the fractional mass in the sphere enclosed in s (we have $\gamma_d, \gamma_s \xrightarrow{s \to \infty} 1$).

For an exponential disk, an analytic expression for $\gamma_d(s)$ can be obtained from Freeman (1970)

$$\gamma_d(s) = (s^3/2)\left[I_0(s/2)K_0(s/2) - I_1(s/2)K_1(s/2)\right], \tag{9}$$

with h being the exponential scale of the disk $h = h_d$ and where I and K are the modified Bessel functions. The sphere contribution $\gamma_s(s)$ will then depend on the parameter $s_e = r_e/h$. For the cases with a sphere only, I take $h = r_e$. To proceed I now need to specify the function $\mu(x)$. In calculating velocity curves I have arbitrarily used

$$\mu(x) = x(1 + x^2)^{-1/2}. \tag{10}$$

I have also used other forms for $\mu(x)$ such as $\mu(x) = 1 - e^{-x}$ or $\mu = x$ for $x < 1$ and $\mu = 1$ for $x \geq 1$. Although the details of the results depend on the choice of μ, the major features are insensitive to the choice of μ from the above three. Hopefully, detailed comparisons between calculated and observed velocity curves will help constrain μ in the future. I shall give the results for μ given by equation (10) and in a few comparative cases for $\mu = 1 - e^{-x}$.

Consider first pure disks. The velocity curves are given by solutions of equation (5) with $\gamma(s, t_i) = \gamma_d(s)$ from equation (9). The curves $v(s, \xi)$ form a one parameter family.

When using the conventional dynamics, the dimensionless velocity curve, for objects with one scale length (h), is unique. However, by introducing a new constant a_0 we can form a quantity of the dimensions of length in addition to h, i.e., $r_t = (MG/a_0)^{1/2}$, so the dimensionless curve depends on the ratio $r_t/h = \xi$.

I give in Figure 1 the results for a few values of ξ (for $\xi = 2$, I also give the curve for $\mu = 1 - e^{-x}$). We note the following: (*a*) For $\xi \leq 1$ the rotation curve is already very nearly the limiting curve for $\xi = 0$; i.e., $v(s) = [\gamma_d(s)]^{1/4}$. (*b*) For $\xi = 2$ the curve stays rather flat down to about two disk scale lengths. (*c*) For $\xi \gtrsim 3$ the velocity curve acquires a considerable hump and deviates more and more from flatness for increasing values of ξ.

Point (*c*) is an important general result. If most of the mass in the galaxy is contained well within the radius r_t (i.e., if $\xi \gg 1$), there is a region between h and r_t which is outside most of the mass, but in which $\mu \approx 1$. In this region the dynamics is Newtonian, and $v(s)$ has to decrease like $s^{-1/2}$ and to deviate from a flat curve.

Consider now galaxies with no disk component. We now define $\xi = (MG/a_0 r_e^2)^{1/2}$ and $s = r/r_e$. Some curves are shown in Figure 2. As before, we get a one-parameter family. For $\xi = 2$ we get a very flat curve ($\pm 5\%$ down to $r = 0.1 r_e$).

Compound model galaxies constitute a three-parameter family. In addition to ξ one has to specify $\alpha_d(\equiv M_d/M)$ and $s_e(\equiv r_e/h)$. In Figure 2, I show the velocity curves for $\alpha_d = 0.25$, 0.5, and 0.75, each with $s_e =$ 0.25, 0.5, and 1. (Kormendy 1977*a* and Burstein 1979*b* find for all the galaxies they studied and all fitting procedures $s_e \leq 1$.) For each pair (α_d, s_e), I have plotted the curve for a value of ξ, which gives a nearly flat curve, and for two values of ξ above and below that.

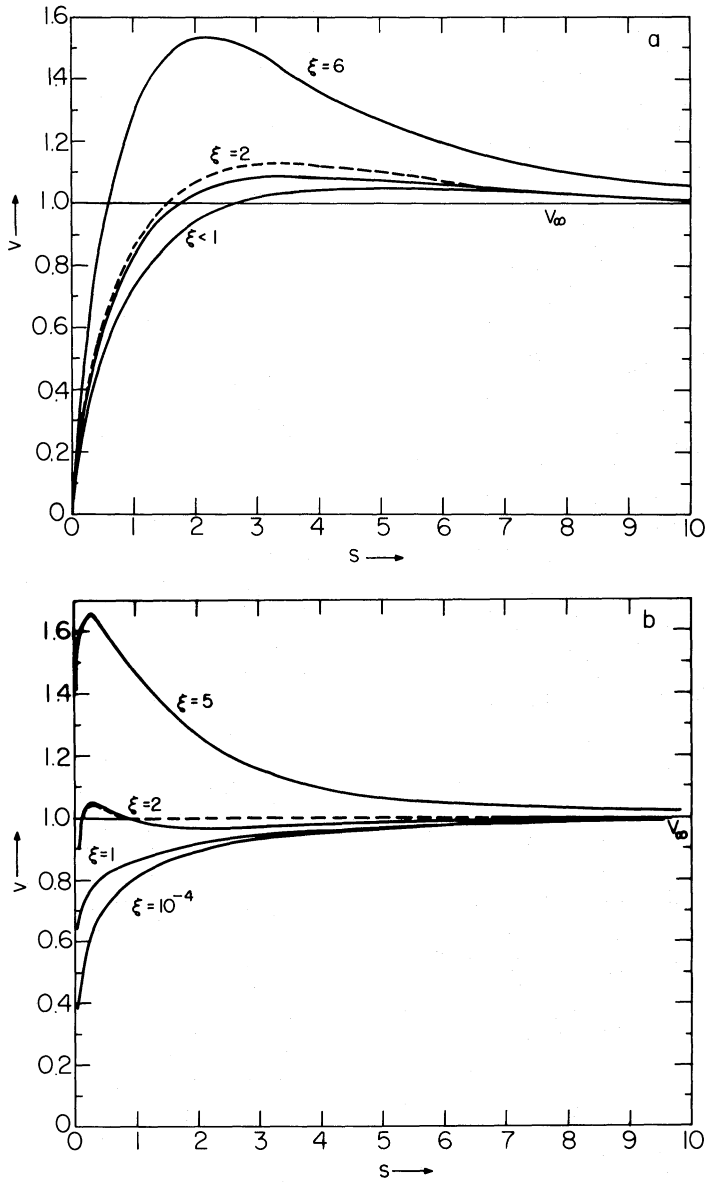

FIG. 1.—Calculated velocity curves for a pure exponential disk (Fig. 1*a*) and pure de Vaucouleurs sphere (Fig. 1*b*). Velocity is in units of V_∞ and radius in units of the exponential scale length and effective radius, respectively. The values of ξ are marked on the curves. Continuous line is for $\mu(x) = x(1 + x^2)^{-1/2}$ and dashed line for $\mu(x) = 1 - e^{-x}$.

Note that velocity curves which are flat down to small radii are obtained only for certain combinations of (α_d, s_e, ξ). In particular they occur only near $\xi = 1$. As I shall show in § VI, ideal spheres or disks which have an exactly flat rotation curve also have a unique average surface mass density of order $a_0 G^{-1}$ (or a unique value of ξ of order 1).

We are now in a position to give an estimate of a_o, the acceleration constant. Since many galaxies have a rather flat rotation curve down to small radii, we can assume that they have ξ, say, between 1 and 2. The typical value of the extrapolated central surface brightness for exponential disk galaxies is $B = 21.65$ mag arsec^{-2} (e.g., Freeman 1970), which corresponds to

$$L_{B,\text{disk}}/2\pi h^2 = 145 L_{B,\odot} \text{ pc}^{-2}. \qquad (11)$$

Assuming that this value corresponds to $1 < \xi < 2$ we take $M_d G/a_0 h^2 = \alpha_d \xi^2 \sim 0.5$–2, to get $a_0 \sim (0.7$–$3) \times 10^{-8} P$ cm s^{-2}, where $P = M_d/L_{B,\text{disk}}$ in solar units and is between 1 and 4 depending on galaxy type. This is of course only a rough estimate of a_0. The proper de-

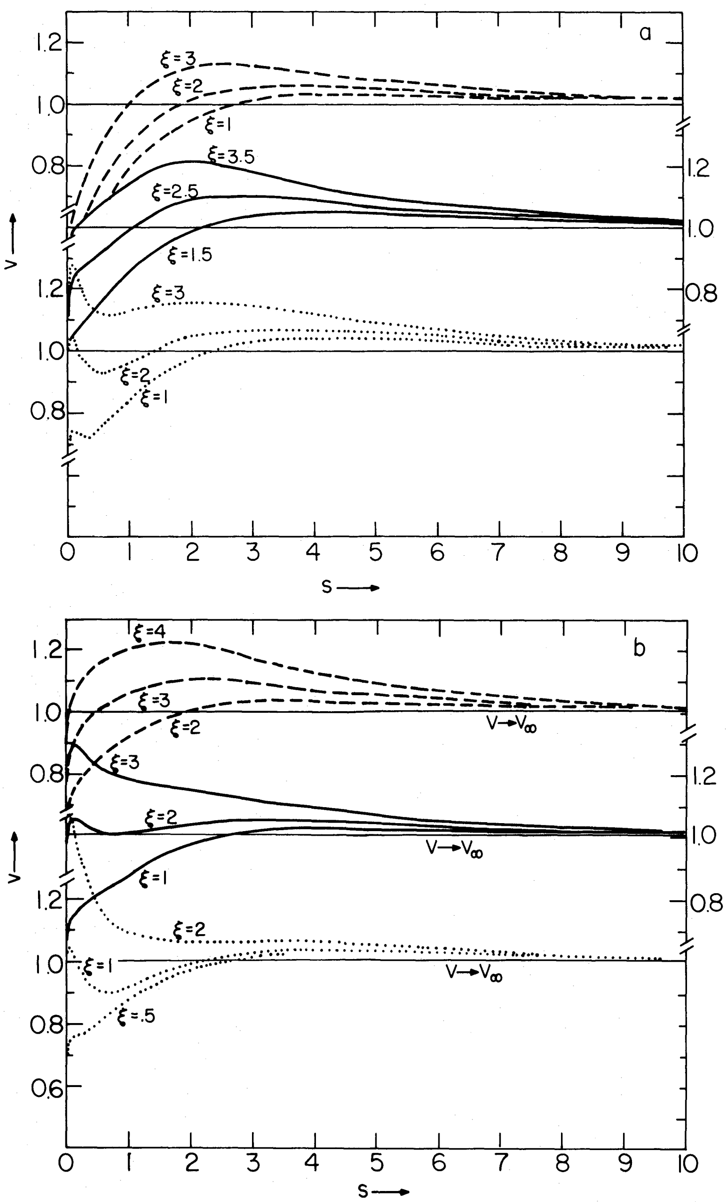

FIG. 2.—Calculated velocity curves for sphere + disk galaxies, with disk to total mass ratios of $\alpha_d = 0.75$ (Fig. 2*a*), $\alpha_d = 0.5$ (Fig. 2*b*), $\alpha_d = 0.25$ (Fig. 2*c*). In each case the results are given for three values of the sphere-to-disk scale ratio $s_e = 1$(- - -), $s_e = 0.5$(——), $s_e = 0.25$(. . .). The values of ξ are marked on the curves. The ordinate scale for $s_e = 1$, 0.25 are given on the left. That for $s_e = 0.5$ is given on the right.

termination of a_0 in this way should involve a fit to the observed velocity curves of specific galaxies using the observed mass distribution.

The value of $a/a_0 = v^2 \xi s^{-1}$, in galaxies, is in general much smaller than 1 at $s \gg 1$, it increases beyond 1 as s becomes smaller than 1 where v is still near 1 and then goes down again as v decreases rapidly inward. The velocity curve thus does not sample $\mu(x)$ at very large values of x, and it may not be possible to determine the asymptotic behavior of $\mu(x)$ from the velocity curves. If, however, specific forms of $\mu(x)$ have to be tested, the velocity curves provide a potentially very useful test.

Galaxies are, in general, accelerated in the gravitational field of a neighboring galaxy or of a group of galaxies. As a result, the acceleration of objects within the galaxy with respect to the fundamental frame differs somewhat from its acceleration with respect to the center of the galaxy.

If the strong equivalence principle still holds in the modified dynamics, the external acceleration does not

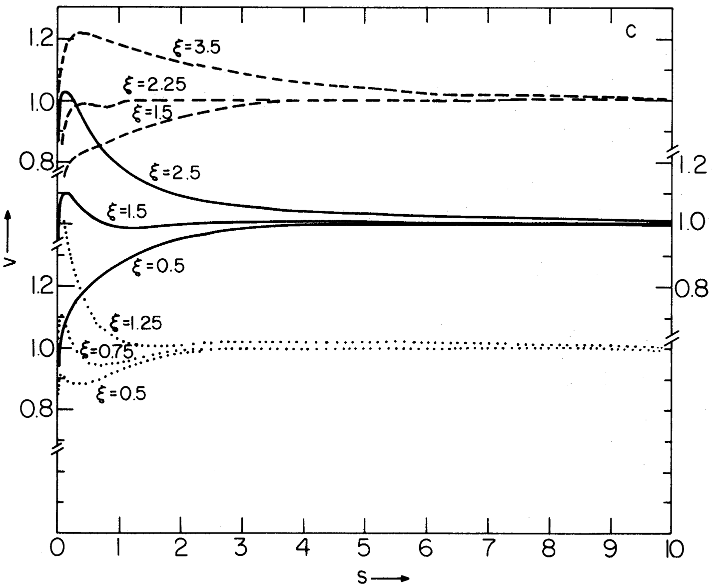

FIG. 2c

affect the internal dynamics. It may be, however (see Paper I) that, in the modified dynamics, the acceleration which determines μ in Equation (1) is the net acceleration with respect to the distant matter. In this case, the effects of an external mass on the internal dynamics of a galaxy may be (when the external acceleration is larger than the internal one and the latter is much smaller than a_0) much stronger than its tidal effects from which they differ in the following ways: (a) The new effects do not result from nonuniformity of the external field across the galaxy. They exist in full strength in a uniform field. (b) The effects of an external, spherically symmetric field (e.g., that of a point mass) are not symmetric with respect to the center of the galaxy. Thus, asymmetries in the structure of the galaxy may be produced. (c) Due to (a), the effective "internal" force produced by the external field may be much larger than its tidal force. For example, a point mass M at a distance R from an object of size r produces tidal forces of the order of $MGrR^{-3}$ but may produce effective internal forces in that object of order MGR^{-2} (i.e., R/r times larger), due to the corrections to the internal accelerations. In this case, at large distances from the center of the galaxy, where the external acceleration dominates, μ becomes a constant, and the orbital velocity in the galaxy will start to decrease with radius.

To summarize this section: (a) The asymptotic flatness of galaxy rotation curve is ensured by assumption (b) of the modified dynamics (linearity of μ for small values of its argument). (b) Rotation curves for realistic mass distribution behave like observed curves (i.e., stay nearly flat to small radii) for values of ξ (or equivalently the average surface brightness) in a certain narrow range. (c) For a given mass distribution the deduced rotation curve is rather insensitive to the choice of $\mu(x)$ as long as μ has the required behavior at small and large values of x.

III. THE $M - V_\infty$ RELATION AND THE TULLY-FISHER CORRELATION

Tully and Fisher (1977, hereafter TF) were the first to find a relation (hereafter the TF relation) between the luminosity (L) of galaxies and the width of their 21 cm line (which is a measure of the galaxy's rotational velocity). Since then a few analyses of this relation have been published (e.g., Sandage and Tammann 1976; Aaronson, Huchra, and Mould 1979; Bottinelli *et al.* 1980; Rubin, Burstein, and Thonnard 1980; Visvanathan 1981; de Vaucouleurs *et al.* 1982).

Different groups chose their galaxy samples in different ways, use luminosities defined in different bands, apply the various corrections necessary in different ways, etc. All suggest a relation of the form $L \propto V^\delta$, where V is either the velocity at some prespecified point taken from the velocity curve, or the velocity inferred from the 21 cm line width. The values of δ found are between 2.5 and 5.

The existence of such a relation and its exact form seem to be of little theoretical use if one assumes the existence of hidden mass to explain the rotation curves. V then measures the total mass (luminous + hidden) and the $L(V)$ relation can, at best, provide a relation between the amount of *total* mass within a certain radius and the luminosity. As in the case of the velocity curves, the TF type relations become an extremely important constraint on theories which assume no hidden mass.

As we saw in § II, I predict (from assumption [*b*]):

$$V_\infty^4 = a_0 GM. \tag{12}$$

This is a major prediction and an absolute relation independent of galaxy type or any other property of the galaxy.

In order to test relation (12) one has to obtain a TF type relation with the following points in mind.

1. The velocity to be used should be V_∞, not the maximum velocity, nor the velocity at some isophotal radius, etc. (although all of these velocities are in many cases close to each other).

2. The luminosity can be used only as a measure of M. One should use a sample of galaxies for which there are good reasons to believe that the mass-to-luminosity ratio (M/L) is constant (for the standard matter) in the band for which L is measured.

Requirement 2 can perhaps be satisfied if one uses galaxies of one type (Rubin, Burstein, and Thonnard 1980, but see de Vaucouleurs *et al.* 1982), or if one chooses a photometric band for which one can safely assume that M(luminous)$/L$ is independent of galaxy type. The choice of the luminosity in the IR seems particularly appealing, in this respect, for the reasons discussed by Aaronson, Huchra, and Mould (1979). It is thus encouraging that their derived relation agrees so well with equation (12) (see also Visvanathan 1981 and Weekes 1981).

Alternatively, one can use a mixed type sample at a band for which M/L is not constant but obtains the luminous mass for each galaxy from its L and a theoretical value of M/L proper for its type.

I analyzed the galaxy data given in Table 1 of FG in this way. Given are 51 galaxies with their luminosities and rotational velocities $V(R_{\rm H})$ at the Holmberg radius. I have not considered the Sdm and Sm type galaxies for which the inclination determination is very uncertain (see, e.g., Tully *et al.* 1978). I have taken out M81, N4736, M51, M83, and M101 for which the velocity curve is still sharply rising or decreasing at the last measured point. One is then left with 40 galaxies. I have divided them into four type groups as did FG (SO–Sa, Sab–Sb, Sbc–Sc, Scd–Sd). For each group I have calculated the average value of V^4/L_B and found (10.5, 8.3, 3.5, 1.8)$\times 10^{-2}$ (km s^{-1})$^4 L_{B,\odot}^{-1}$. The respective standard deviations are (5.9, 5.5, 1.9, 1.2)$\times 10^{-2}$ (km s^{-1})$^4 L_{B,\odot}^{-1}$ and the number of galaxies in each group (6, 11, 15, 8). If the $M \propto V_\infty^4$ relation holds, the values of V^4/L_B should be proportional to M/L_B for the stars and gas.

I use the values of M/L_B in solar units derived by Larson and Tinsley (1978) for the stellar population from their models as given by FG, corrected for the mass of the gas using Roberts's (1975) result (again as quoted by TF), to obtain the following model values of M/L_B for the above groups (3.56, 2.89, 1.71, 0.89). The ratios of the values of V^4/L_B to these values of M/L_B are, respectively, (2.95, 2.87, 2.05, 2.02)$\times 10^{-2}$ (km s^{-1})$^4 M_\odot^{-1}$ which are very close to each other considering the uncertainties involved and the fact that the values of M/L themselves vary by a factor of 4.

I use the average value of $V^4/L_B/(M/L_B)$ over the four groups (2.47×10^{-2} (km s^{-1})$^4 M_\odot^{-1}$) to obtain the value of a_0:

$$a_0 = 1.9\times 10^{-8} h_{50}^2 \text{ cm s}^{-2}, \tag{13}$$

with $h_{50} = H_0/(50$ km s^{-1} Mpc^{-1}). The uncertainty in a_0 is of about a factor of 2.

To test the $M - V_\infty$ relation for this mixed sample, in more detail, I obtained the theoretical mass of each galaxy in the sample from its quoted L_B and the theoretical M/L values as discussed above. I obtain the best fit slope of 3.1 and 4.2 for the $\log M$ versus $\log V$ and $\log V$ versus $\log M$. Constraining the slope to be 4, I obtain the best fit intercept which yields $a_0 = 1.5\times 10^{-8} h_{50}^2$ cm s^{-2} again with a factor of 2 uncertainty (which correspond to a scattering of 0.3 in $\log M$ about the $\log M$-$\log V$ regression line).

Constraining the slope for the data, on Sc galaxies, of Rubin, Burstein, and Thonnard (1980) to be 4, I obtain the best fit value of the intercept which gives $a_0 = 3.7\times 10^{-8} P^{-1} h_{50}^2$ cm s^{-2}, where $P = (M/L_B)$ in solar units, assumed constant for the sample. I use the value of P adequate for Sc galaxies from Larson and Tinsley (1978) as quoted by FG: $P = 1.71$, to obtain $a_0 = 2.1\times 10^{-8} h_{50}^2$ cm s^{-2}, with an uncertainty similar to the one above.

The determination of a_0 from the proportionality factor in the TF relation is independent of that which is described in § II. The second is only based on the *shape* of the rotation curve and makes use of surface mass densities (or surface brightness); hence, it does not depend on knowledge of H_0 and, for that matter, is insensitive to errors in determining inclinations or errors in determining L due to deviations from the Hubble flow, absorption in our Galaxy, or even internal absorption. The determination of a_0 as described in this section makes use of total luminosities and asymptotic velocities, so it is sensitive to the above factors; on the other hand, it is not affected at all by uncertainties in the mass distribution within the galaxy or the exact form of $\mu(x)$ to which the other method is sensitive. Note also that an overall error in the assumed values of M/L affects the value of a_0 in opposite ways in the two determinations.

Our present knowledge of the TF relation is sufficient to rule out a modification of only the distance dependence of gravity as the sole explanation of the mass discrepancy, as such a modification implies a M-V relation of the form $M \propto V^2$.

IV. THE DYNAMICS OF THE z-MOTIONS

There is an important aspect of galactic stellar dynamics for which the major implications of the modified dynamics are quite straightforward. This has to do with the structure of the galactic disk and motion of stars, in the direction, z, perpendicular to the plane of the disk.

I shall adopt the standard assumptions which are made for studying this problem (e.g., Oort 1965; Mihalas 1968), i.e., that the particle trajectories are nearly circles around the galactic center and in the disk plane. The z-motions (as well as other deviations) are small perturbations. Namely, the z-excursions of the particles are small compared with the orbital radius and the deviations of velocities and accelerations from those of circular motion are small. One can then decouple the dynamics of the z-motion from that in the other directions and find relations between the z-component of the galaxy's gravitational acceleration (g_z^N), the z-velocity dispersion, and z-dependence of the density of some group of test particles. One can then determine g_z^N and via Poisson's equation determine the total gravitational mass density in the central plane or the surface mass density in the disk (Oort 1960, 1965).

On the basis of the above assumptions, all the stars involved in the analysis have approximately the same acceleration $xa_0 = V^2/r$ in the galaxy's field, where V is the circular velocity and r is the distance from the galactic center. As can be shown explicitly from equation (1), keeping terms to the lowest order in the acceleration deviations from the average galactic rotation, the effect of the modification on the z-dynamics is to make the effective inertial mass of all particles involved a factor $\mu(x)$ smaller than their gravitational mass, otherwise leaving the dynamics Newtonian. The true (modified) effective gravitational acceleration, g_z, which is derived from the stellar dynamics, in the conventional manner, is larger, by a factor approximately equal to $[\mu(x)]^{-1}$, than the Newtonian acceleration, g_z^N, which is related to ρ via the Poisson equation. This factor increases with r (at large r and with a flat rotation curve it is proportional to r).

For example, Spitzer's (1942) solution for the z-structure for a plane symmetric isothermal, self-gravitating disk, when applied to galaxy disks, will now give

$$\rho(z) = \rho(0)\,\mathrm{sech}^2(z/z_0), \tag{14}$$

with

$$z_0 = \left[\mu(x)\langle V_z^2\rangle / 2\pi G\rho(0)\right]^{1/2}, \tag{15}$$

where $\rho(0)$ is the density in the mid-plane and $\langle V_z^2\rangle$ is the (z-independent) z-dispersion.

Another modification takes place when one employs the Poisson equation to obtain the mass density ρ from g_z^N. For an axisymmetric galaxy, we have for the radial terms in the Laplacian, in the midplane of the galaxy:

$$\partial g_r^N/\partial r + g_r^N/r = \mu(x)(A-B)^2 \times \{f(x) + 2[1+f(x)] \times (A+B)/(A-B)\},$$
$$f(x) \equiv d\ln[\mu(x)]/d\ln(x). \tag{16}$$

Here A and B are the Oort rotation parameters.

Thus the relation between ρ and $\partial g_z^N/\partial z$ depends differently on the galactic rotation parameters then in the Newtonian case ($\mu = 1, f = 0$).

We shall see below that near the Sun, $\partial g_r^N/\partial r + g_r^N/r$ is much smaller than $\partial g_z^N/\partial z$.

V. THE MILKY WAY

I first obtain the value of a_0 from estimates of the Galaxy's mass (M_G) and its asymptotic rotational velocity. I take the estimated mass of the standard matter in the Galaxy given by Bahcall and Soneira (1980) (without the mass which they add to account for the Oort discrepancy) $M_G \approx 3\times 10^{10} M_\odot$. Assuming, as in Gunn, Knapp, and Tremaine (1979), that the velocity curve of the Galaxy is flat beyond the Sun and normalizing to their value of the rotational velocity at the Sun's position, I get:

$$a_0 = V_\infty^4 G^{-1} M_G^{-1} = (5.8\pm 1.0)\times 10^{-8}\left(V_\infty/220\ \mathrm{km\ s^{-1}}\right)^4 \times \left(M_G/3\times 10^{10} M_\odot\right)^{-1}, \tag{17}$$

where the error quoted reflects only that in $V_\odot$.

That this value of a_0 is consistent with that obtained in § III is not really surprising. After all we know that the Galaxy does not deviate much from the TF relation. However, in terms of the astrophysical assumptions made, this is an independent estimate of a_0 (as can be seen from the fact that it is independent of h_{50} and that it does not make use of a model value for M/L).

The value of ξ I get for the Galaxy normalized to the above values of M_G and V_∞ is (using a_0 from eq. [17])

$$\xi = V_\infty^2/a_0 h = 0.8(h/3.5\ \mathrm{kpc})^{-1}\left(V_\infty/220\ \mathrm{km\ s^{-1}}\right)^{-2} \times \left(M_G/3\times 10^{10} M_\odot\right). \tag{18}$$

Next consider the question of the mass density near the Sun. The observational situation is summarized by FG. The dynamical mass density (Oort 1965) (deduced with the conventional dynamics) is $\rho_d \approx 0.15\ M_\odot\ \mathrm{pc^{-3}}$. Estimates of the mass density in conventional forms give $(0.08\text{–}0.12) M_\odot\ \mathrm{pc^{-3}}$ (FG and references therein). One finds that it is practically impossible to account for the remaining mass density of, say, $0.06\ M_\odot\ \mathrm{pc^{-3}}$ by the contribution from an unseen halo which one assumes to explain the velocity curve, unless the halo is very flat (see, for example, Bahcall and Soneira 1980).

With the modified dynamics one deduces a dynamical mass smaller by a factor approximately equal to $\mu(V_\odot^2/r_\odot a_0)$ (assuming that the radial terms in the Poisson equation can be neglected; see below). If the Oort discrepancy results from the use of Newtonian dynamics when in fact the modified dynamics should be used, we can write $\mu(V_\odot^2/r_\odot a_0) \approx 0.7$. This then implies that the argument of μ is of order 1 (from assumption [c] of the modified dynamics). This in turn gives $a_0 \sim V_\odot^2/r_\odot = 1.9\times10^{-8}(V_\odot/220\ \text{km s}^{-1})^2(r_\odot/8.5\ \text{kpc})^{-1}$ cm s^{-2}.

I now estimate the contribution of the radial terms in Poisson's equation. I take the values of A and B from Gunn, Knapp, and Tremaine (1979): $A = (13\pm2)$ km s^{-1} kpc^{-1}, $B = -A$, to get (from eq. [16])

$$\partial g_r^N/\partial r + g_r^N/r = x_\odot \mu'(x_\odot)(A-B)^2$$
$$= x_\odot \mu'(x_\odot) 7.5\times10^{-31}\ \text{s}^{-2}. \quad (19)$$

This value is to be compared with that of $-\partial g_z^N/\partial z = \mu(x_\odot)8.8\times10^{-30}$ s^{-2} as given by Oort (1965) [and corrected by a factor $\mu(x_\odot)$].

It is hard to estimate $x_\odot\mu'(x_\odot)$. Note, however, that if $\mu(x)$ is monotonic ($\mu' > 0$) and concave ($\mu'' < 0$) we have $x\mu'(x) < \mu(x)$ (see § VI after eq. [22] for an important implication of this inequality). In this case, the radial term given by equation (19) is at least a factor of 10 smaller than the z-term.

VI. GALAXY STRUCTURE

Two of the major questions concerning galaxy structure have to do with the following observations.

1. Galaxies have particular density distributions, i.e. they involve disks which are approximately described by a surface brightness $\Sigma(r) = \Sigma^{\exp} e^{-r/h}$ (perhaps cut off at small and/or large radii) and spheres or ellipsoids which are approximately described by the de Vaucouleurs law $\Sigma = \Sigma^{\text{sph}} e^{-(r/r_d)^{1/4}}$ (e.g., de Vaucouleurs 1962; Freeman 1970, 1978; Kormendy 1977*a*, *b*). Other forms have also been suggested for describing spherical components in galaxies and they work better in some cases (Hubble 1930 and the King 1966 models).

2. There are values of the central surface density Σ^i, different for disks and spheres, which appear to play a special role. They are $\Sigma_0^{\exp} \approx 145 L_{B,\odot}$ pc^{-2} ($S_0 = 21.65$ mag arcsec^{-2}) for the disks, and $\Sigma_0^{\text{sph}} \approx 2.3\times10^5 L_{B,\odot}$ pc^{-2} ($S_0 = 13.65$ mag arcsec^{-2}) for spheres. Here Σ_0^i is the extrapolated central surface brightness in the B band using an exponential law for the disks and de Vaucouleurs law for the spheres. These were first pointed out by Fish (1964) and by Freeman (1970).

Allen and Shu (1979) discuss the evidences and conclude that the above values constitute upper cutoffs. Galaxies with higher surface brightnesses are very rare.

The modified dynamics involve a constant, a_0, which, with the proper transformation from mass to luminosity, defines a special value of surface brightness, i.e., $[a_0 G^{-1}(M/L)^{-1}]$. As I now show it is indeed this value (within a numerical factor of order one) which corresponds to Σ_0 for both disks and spheres.

The observed values of $\Sigma_0^{\exp}$ and Σ_0^{sph} define certain values of the parameter ξ defined in § II.

For an exponential disk

$$(\xi^{\exp})^2 \equiv GM^{\exp}/a_0 h^2 = P_B (M/L_B)_\odot (G/a_0) 2\pi \Sigma_0^{\exp}, \quad (20a)$$

For a de Vaucouleurs sphere:

$$(\xi^{\text{sph}})^2 \equiv GM^{\text{sph}}/a_0 r_e^2$$
$$= 1.1\times10^{-2} P_B (M/L_B)_\odot (G/a_0)\Sigma_0^{\text{sph}}. \quad (20b)$$

Here, $M^{\exp}$ and M^{sph} are the masses of the exponential disk and sphere, respectively, P_B is M/L_B in solar units, r_e is the effective radius of the sphere. The numerical factors are introduced through the relations between Σ_0^i and the quantities L/h^2 and L/r_e^2, respectively. With the above values of $\Sigma_0^{\exp}$ and Σ_0^{sph} we get:

$$\xi^{\exp} = 0.85(P_B)^{1/2} h_{50}^{-1}; \qquad \xi^{\text{sph}} = 1.4(P_B)^{1/2} h_{50}^{-1}, \quad (21)$$

with $a_0 = 2\times10^{-8} h_{50}^2$ cm s^{-2}.

Thus, the preferred values of Σ_0^i correspond in both disks and spheres to values of ξ near 1.

As we saw in § II, a value of ξ which is not much larger than 1 is a necessary condition for the velocity curve not to deviate much from flatness. I thus find that galaxies possess, in general, the two properties between which I found a causal link, i.e., rotation curves which stay flat well within the luminous mass on the one hand, and on the other, values of ξ near (or at least below) 1. As I show below, ideal spheres or disks which have an exactly flat rotation curve, indeed have a unique mass distribution and a unique average surface density. It is not clear, at the moment, which of the two properties is the cause and which is the consequence. It is tempting to suggest that the flatness of the rotation curve is the cause, i.e., that many galaxies are built in such a way so as to make their rotation curve as flat as possible. This then requires that they have a certain mass distribution and size to match their mass, so as to produce a value of

ξ close to or smaller than 1. This must remain a conjecture for the time being.

I now consider idealized galaxies which have an exactly flat rotation curve (down to $r = 0$). They can be used as reference models for real galaxies. Just as a spherical mass with $M(r) \propto r$ produces a constant rotational velocity if the conventional dynamics applies, there is a spherical density distribution which has a constant rotational velocity with the modified dynamics. This distribution is

$$M^*(r) = M\mu(r_0/r)(r/r_0), \qquad r_0 \equiv (MG/a_0)^{1/2}. \tag{22}$$

I shall call such spheres flat rotation curve (FRC) spheres.

Three important properties of the FRC sphere are (a) It is finite: for $r \gg r_0$, $M^*(r) \approx M$; (b) It has a unique mass distribution up to a scale length; and (c) It has a unique surface mass density $M/r_0^2 = a_0 G^{-1}$. I shall hereafter use the name FRC-like (FRCL) body for a spherical object which has the density law of an FRC but does not necessarily have the unique surface density (and is thus not an FRC). Expression (22) for $M^*(r)$ can be a mass law only if it nowhere decreases with r. This is equivalent to the condition $d\ln[\mu(x)]/d\ln(x) \leqslant 1$ which I shall assume. (See end of § V.) I find that if the de Vaucouleurs sphere is required to be a FRC-like object $\mu(x)$ is determined to be very nearly of the form $1 - e^{-x}$ for $x \lesssim 10$.

There is also a unique surface density distribution $\Sigma^*(r)$ in a thin disk which produces a flat rotation curve. Using formulae from Mestel (1963) (modified to be compatible with the modified dynamics) I find for this distribution as a function of the distance from the center, r

$$\Sigma^*(r) = Mh^{-2}\pi^{-2}\eta^{-1} \times \int_0^1 \int_0^1 \frac{u^2\,du\,t\,dt}{(1-u^2)^{1/2}(1-t^2)^{1/2}} \frac{d}{du}\left[-u^{-1}\mu(u\eta^{-1}t^{-1})\right], \tag{23}$$

where $\eta = r/h$ and $h = (MGa_0^{-1})^{1/2}$. As for the spherical case, $\Sigma^*(r)$ has a finite mass M, it is unique up to one scale parameter (h), and it has a unique average surface density.

A compound galaxy with a total mass M, and a FRCL spheroid of mass M_s, has an exactly flat rotation curve if its disk surface density $\bar{\Sigma}^*(r)$ is given by

$$\bar{\Sigma}^*(r) = \Sigma^*(\eta) - \alpha_s s_s^{-2}\Sigma^*(\eta/s_s). \tag{24}$$

Here Σ^* is that given by equation (23), $\alpha_s = M_s/M$, $s_s = r_s/h$, r_s is the scale radius of the sphere (replacing r_0 in the first of eqs. [22]), and η and h are defined as before.

It should be stressed that the existence of a preferred value for the surface mass density is not a necessary result of the modified dynamics as the asymptotic flatness of the rotation curve, and the $M \propto V_\infty^4$ relation are. It is only necessary if the rotation curve is to stay flat down to small radii. When the average surface density much exceeds $a_0 G^{-1}$, the rotation curve must have a large bump. Such bumps are not observed for actual galaxies. When the average surface density is smaller than $a_0 G^{-1}$, the velocity rises more slowly from the center out.

Burstein (1982) recently found that there exists a strong correlation between the infrared average surface brightness of galaxies and their asymptotic rotational velocity. Rubin, Burstein, and Thonnard (1980) find, in their sample of Sc's, correlation between the luminosity (or rotational velocity) of galaxies and the steepness with which the velocity rises near the origin. There should then exist a correlation between surface brightness and steepness of rise of the velocity. The modified dynamics implies such a correlation.

VII. ELLIPTICAL GALAXIES

Ellipticals form a rather heterogeneous group. They have different degrees of ellipticity, perhaps different M/L, and although many are well approximated by de Vaucouleurs's law, many depart from this law in different ways (see, for example, Oemler 1976, Kormendy 1977*b*, Strom and Strom 1978). Also, the orbits in ellipticals may vary within the galaxies and from galaxy to galaxy. It is thus difficult to make general predictions as was possible for disk galaxies.

Elliptical galaxies exhibit, however, some properties which very much resemble analogous properties of disk galaxies.

1. In the one case I know of (Knapp, Kerr, and Williams 1978), where a 21 cm rotation curve is available for an elliptical (NGC 4278), it stays flat to large radii, [instead of $M/L_B = 19.8(M/L_B)_\odot$ which FG give for this case, I get $(M/L_B) = 7.5(M/L_B)_\odot$ with the modified dynamics].

2. Various groups find correlations between the luminosity (L) and the central velocity dispersions, σ, in ellipticals (Faber and Jackson 1976; Sargent *et al.* 1977; Schechter and Gunn 1979; Tonry and Davis 1981; Terlevich *et al.* 1981; Tonry 1981). All fit a relation of the form $L \propto \sigma^\delta$. All except Tonry (who finds $\delta = 3.2 \pm .2$), find δ to be nearly or at least consistent with 4.

3. I have already mentioned the evidence that ellipticals tend to have values of L/r_e^2 which are close to, or at least not larger than, a certain critical value which is nearly the same as that for disks.

A major step in understanding ellipticals can be made if we can identify them, at least approximately, with idealized structures such as the FRCL spheres discussed above. I have also studied isotropic and nonisotropic isothermal spheres, in the modified dynamics, as such possible structures. I found that they have properties which very much resemble those of ellipticals and galactic bulges. I describe these in Milgrom (1983*c*).

VIII. PREDICTIONS

The main predictions concerning galaxies are as follows.

1. Velocity curves calculated with the modified dynamics on the basis of the observed mass in galaxies should agree with the observed curves. Elliptical and S0 galaxies may be the best for this purpose since (*a*) practically no uncertainty due to obscuration is involved and (*b*) there is not much uncertainty due to the possible presence of molecular hydrogen.

2. The relation between the asymptotic velocity (V_∞) and the mass of the galaxy (M) ($V_\infty^4 = MGa_0$) is an absolute one.

3. Analysis of the z-dynamics in disk galaxies using the modified dynamics should yield surface densities which agree with the observed ones. Accordingly, the same analysis using the conventional dynamics should yield a discrepancy which increases with radius in a predictable manner.

4. Effects of the modified dynamics are predicted to be particularly strong in dwarf elliptical galaxies (for review of properties see, e.g., Hodge 1971 and Zinn 1980). For example, those dwarfs believed to be bound to our Galaxy would have internal accelerations typically of order $a_{\text{in}} \sim a_0/30$. Their (modified) acceleration, g, in the field of the Galaxy is larger than the internal ones but still much smaller than a_0, $g \approx (8\ \text{kpc}/d)a_0$, based on a value of $V_\infty = 220\ \text{km s}^{-1}$ for the Galaxy, and where d is the distance from the dwarf galaxy to the center of the Milky Way ($d \sim 70$–220 kpc). Whichever way the external acceleration turns out to affect the internal dynamics (see the discussion at the end of § II, the section on small groups in Paper III, and Paper I), we predict that when velocity dispersion data is available for the dwarfs, a large mass discrepancy will result when the conventional dynamics is used to determine the masses. The dynamically determined mass is predicted to be larger by a factor of order 10 or more than that which can be accounted for by stars. In case the internal dynamics is determined by the external acceleration, we predict this factor to increase with d and be of order (d/8 kpc) (as long as $a_{\text{in}} \ll g$, $h_{50} = 1$).

Prediction 1 is a very general one. It is worthwhile listing some of its consequences as separate predictions, numbered 5–7 below (note that, in fact, even prediction 2 is already contained in prediction 1).

5. Measuring *local* M/L values in disk galaxies (assuming conventional dynamics) should give the following results: In regions of the galaxy where $V^2/r \gg a_0$ the local M/L values should show no indication of hidden mass. At a certain transition radius, local M/L should start to increase rapidly. The transition radius should occur where $V^2/r \approx a_0$. This test has the following advantages: (*a*) It does not require an absolute calibration of M/L as we are concerned only with variations of this quantity; (*b*) Effects of the modified dynamics manifest themselves more clearly in local mass determination than in the integrated masses; and (*c*) In many cases this test requires information on local behavior in the disk only while the spheroid can be neglected. This makes the determination of mass from velocity more certain.

6. Disk galaxies with low surface brightness provide particularly strong tests (a study of a sample of such galaxies is described by Strom 1982 and by Romanishin *et al.* 1982). As low surface brightness means small accelerations, the effects of the modification should be more noticeable in such galaxies. We predict, for example, that the proportionality factor in the $M \propto V_\infty^4$ relation for these galaxies is the same as for the high surface density galaxies. In contrast, if one wants to obtain a correlation $M \propto V_\infty^4$ in the conventional dynamics (with additional assumptions), one is led to the relation $M \propto \Sigma^{-1}V_\infty^4$ (see, for example, Aaronson, Huchra, and Mould 1979), where Σ is the average surface brightness. This implies that low surface density galaxies, of a given velocity, have a mass higher than predicted by the M-V relation derived for normal surface density galaxies.

We also predict that the lower the average surface density of a galaxy is, the smaller is the transition radius, defined in prediction 5, in units of the galaxy's scale length. In fact, if the average surface density is very small we may have a galaxy in which $V^2/r < a_0$ everywhere, and analysis with conventional dynamics should yield local M/L values starting to increase from very small radii.

7. As the study of model rotation curves shows, we predict a correlation between the value of the average surface density (or brightness) of a galaxy and the steepness with which the rotational velocity rises to its asymptotic value (as measured, for example, by the radius at which $V = V_\infty/2$ in units of the scale length of the disk). Small surface densities imply slow rise of V.

IX. DISCUSSION

The main results of this paper can be summarized by the statement that the modified dynamics eliminates the need to assume hidden mass in galaxies. The effects in galaxies which I have considered, and which are commonly attributed to such hidden mass, are readily explained by the modification. More specifically:

1. The velocity of a particle in a circular orbit around a finite galaxy becomes independent of the radius of the orbit at large radii. This results in asymptotically flat rotation curves of galaxies.

2. The asymptotic rotational velocity V_∞ depends *only on the mass of the galaxy*, M, $V_\infty^4 = a_0 GM$. This relation, I find, is in agreement with the observational relations (of the Tully-Fisher type) between the velocity and the luminosity of galaxies, when the properly defined luminosity is used.

3. The mass density in the solar neighborhood deduced from the z-dynamics with the *modified* dynamics is smaller than that deduced with the conventional dynamics and is consistent with that which is observed directly.

4. I have calculated velocity curves for model galaxies. I find that the resulting rotation curves can be flat down to small radii, deep within the mass distribution (in accord with the observed behavior of many galaxies) only when the surface mass density is near a certain critical value (of order $a_0 G^{-1}$). This again is in accord with the observational fact that galaxies tend to prefer a certain value for their surface brightness. For galaxies with lower surface density, the velocity is expected to rise more slowly toward the asymptotic value.

I wish to stress again that the main results of this paper only depend on the bare-bones assumptions of the modified dynamics. They do not depend on the particular formulation which one builds on these assumptions, nor on the exact form of $\mu(x)$ in the formulation I have used. The asymptotic flatness of the rotation curves and the relation $MGa_0 = V_\infty^4$ result from assumption (b). The relation between average surface density and shape of the velocity curve result in the following way from the basic assumptions. If the average surface density is much larger than $a_0 G^{-1}$, most of the mass of the galaxy is deep within the Newtonian regime. In this case there should be a wide range of radii within which V decreases as $r^{-1/2}$. There must then be, in this case, a large bump on the velocity curve. The statement that the galactic acceleration at the Sun's position is of order a_0 also follows from assumption (c) of the modified dynamics and the fact that the Oort discrepancy is of order unity.

It is of great importance to determine the value of a_0 and the form of $\mu(x)$. As I discuss in Paper I, there is, hopefully, a theory in which these are calculable (together with G) from the distribution of matter in the universe and its manner of expansion.

I have determined a_0 (with uncertainty of about a factor of 2 on each side) in a few ways: (a) From the proportionality constant in the $M \propto V_\infty^4$ relation: $a_0 \approx 2\times10^{-8} h_{50}^2 (P/P_0)^{-1}$ cm s^{-2}, where P is the M/L ratio for standard matter in galaxies, in solar units, and P_0 is a model value I used for it. (b) From the $M \propto V_\infty^4$ relation applied to the Milky Way: $a_0 \approx 6\times 10^{-8}(V_G/220 \text{ km s}^{-1})^4 (M_G/3\times10^{10} M_\odot)^{-1}$ cm s^{-2}, where M_G is the Galaxy's mass and V_G its asymptotic velocity. (c) From the requirement that the preferred surface brightness, observed in galaxies, equals the one I find is necessary to obtain rotation curves which stay flat to small radii: $a_0 \approx 3\times10^{-8}(P/2)$ cm s^{-2}. (d) Assuming that the discrepancy factor between the Oort density and the observed one is $\mu(V_\odot^2/r_\odot a_0) = 0.7$, where $V_\odot$ and $r_\odot$ are the Sun's orbital velocity and radius respectively, I get $a_0 \approx 2\times10^{-8}(V_\odot/220 \text{ km s}^{-1})^2 (r_\odot/8.5 \text{ kpc})^{-1}$ cm s^{-2}.

Determination (a) is insensitive to the form of $\mu(x)$ or to details of the mass distribution in galaxies. It does require knowledge of H_0 and M/L and involves uncertainties in the inclinations and obscuration corrections as reflected in uncertainties in V_∞ and L. Determination (b) does not require knowledge of M/L or of H_0. It is susceptible to uncertainties in the galaxy's mass and its asymptotic velocity. Determination (c) does not require knowledge of H_0 or the inclination of galaxies but (for an accurate result) requires a detailed analysis of individual galaxies, and unlike (a) and (b) may require knowledge of $\mu(x)$. Determination (d) again requires more information on $\mu(x)$ and involves uncertainties in the velocity and orbital radius of the Sun.

It should be noted that the range of values I get for a_0 contain the value of $CH_0 = 5\times10^{-8} h_{50}$ cm s^{-2}, where C is the speed of light. This fact may be most significant as is discussed in Paper I.

As to the form of $\mu(x)$. As already said, most of the results of this paper are not sensitive to the exact form of μ. As shown in Paper I, solar system experiments put some constraint on the asymptotic form of μ (for $x \to \infty$).

It appears to me that the best way to determine $\mu(x)$ observationally will be through detailed comparison of calculated and observed velocity curves. In galaxies, though, $x = a/a_0$ does not much exceed 10, and so $\mu(x)$ can be obtained only up to $x \sim 10$.

Another potential method of determining $\mu(x)$ is through identifying ellipticals with specific structures such as FRCL objects or isothermal spheres. If such identification can be made, $\mu(x)$ can be deduced directly from the observed surface brightness distribution of ellipticals. For example, I have demonstrated in § VI, that if isolated ellipticals are FRCL bodies (except perhaps at very small radii), $\mu(x)$ is fixed and at least for $x < 10$ is well approximated by $\mu(x) = 1 - e^{-x}$.

Some additional major aspects of galaxy dynamics which are bound to be greatly affected if the ideas presented here and in Paper I are basically correct, are:

1. Galaxy formation. With smaller effective inertial mass than thought before, galaxies collapse easier and at lower masses of the perturbation.

2. Galaxy evolution. I mention in Paper I the possibility that a_0 vary with cosmic time. This will lead to evolution of galaxies due to changes in the inertia field with which they interact.

3. The question of stability of self-gravitating disk

galaxies (Ostriker and Peebles 1973) has to be rechecked.

4. The dynamics within globular clusters should be greatly affected. The typical gravitational acceleration in them is $g_N = 1.5 \times 10^{-8}(M/10^5 M_\odot)(r/10 \text{ pc})^{-2}$ cm s^{-2}, compared with $a_0 \sim 2 \times 10^{-8}$ cm s^{-2} we find in previous sections.

5. The formation and maintenance of warps in the outer parts of discs (e.g., Hunter and Toomre 1969) will have to be reconsidered.

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REFERENCES

Aaronson, M., Huchra, J., and Mould, J. 1979, *Ap. J.*, **229**, 1.
Allen, R. J., and Shu, F. H. 1979, *Ap. J.*, **227**, 67.
Bahcall, J. N., and Soneira, R. M. 1980, *Ap. J. Suppl.*, **44**, 73.
Bosma, A. 1978, Ph.D. thesis, University of Groningen.
______. 1981*a*, *A.J.*, **86**, 1791.
______. 1981*b*, *A.J.*, **86**, 1825.
Bottinelli, L., Gouguenheim, L., Paturel, G., de Vaucouleurs, G. 1980, *Ap. J. (Letters)*, **242**, L153.
Burstein, D. 1979*a*, *Ap. J. Suppl.*, **41**, 435.
______. 1979*b*, *Ap. J.*, **234**, 435.
______. 1982, *Ap. J.*, **253**, 539.
Davies, R. L. 1981, *M.N.R.A.S.*, **194**, 879.
de Vaucouleurs, G. 1962, in *IAU Symposium 15, Problems of Extragalactic Research*, ed. G. C. McVittie (New York: Macmillan), p. 3.
de Vaucouleurs, G., Buta, R., Bottinelli, L., Gouguenheim, L., and Paturel, G. 1982, *Ap. J.*, **254**, 8.
de Vaucouleurs, G., and Freeman, K. C. 1972, in *Vistas Astr.*, **14**, 163.
Faber, S. M., and Gallagher, J. S. 1979, *Ann. Rev. Astr. Ap.*, **17**, 135 (FG).
Faber, S. M., and Jackson, R. E. 1976, *Ap. J.*, **204**, 668.
Fish, R. A. 1964, *Ap. J.*, **139**, 284.
Freeman, K. C. 1970, *Ap. J.*, **160**, 811.
______. 1978, in *IAU Symposium 77, Structure and Properties of Nearby Galaxies*, ed. E. M. Berkhuijsen and R. Wielebinski (Dordrecht: Reidel), p. 3.
Gunn, J. E., Knapp, G. R., and Tremaine, S. D. 1979, *A.J.*, **84**, 1181.
Hodge, P. W. 1971, *Ann. Rev. Astr. Ap.*, **9**, 35.
Hubble, E. 1930, *Ap. J.*, **71**, 231.
Hunter, C., and Toomre, A. 1969, *Ap. J.*, **155**, 747.
King, I. 1966, *A.J.*, **71**, 64.
Knapp, G. R., Kerr, F. J., and Williams, B. A. 1978, *Ap. J.*, **222**, 800.
Kormendy, J. 1977*a*, *Ap. J.*, **217**, 406.
______. 1977*b*, *Ap. J.*, **218**, 333.
Krumm, N., and Salpeter, E. E. 1977, *Astr. Ap.*, **56**, 465.
Larson, R. B., and Tinsley, B. M., 1978, *Ap. J.*, **219**, 46.
Mestel, L. 1963, *M.N.R.A.S.*, **126**, 553.
Mihalas, D. 1968, *Galactic Astronomy* (San Francisco; W. H. Freeman).
Milgrom, M. 1983*a*, *Ap. J.*, **270**, 365 (Paper I).
______. 1983*b*, *Ap. J.*, **270**, 384 (Paper III).
______. 1983*c*, preprint.
Oemler, A. Jr. 1976, *Ap. J.*, **209**, 693.
Oort, J. H. 1960, *Bull. Astr. Inst. Netherlands*, **15**, 45.
______. 1965, in *Stars and Stellar Systems*, Vol. **5**, *Galactic Structure*, ed. A. Blaauw and M. Schmidt (Chicago: University of Chicago Press), p. 455.
Ostriker, J. P., and Peebles, P. J. E. 1973, *Ap. J.*, **186**, 467.
Roberts, M. S. 1975, in *Stars and Stellar Systems*, Vol. **9**, in *Galaxies and the Universe*, ed. A. Sandage, M. Sandage, and J. Kristian (Chicago: University of Chicago Press), p. 309.
Romanishin, W., Krumm, N., Salpeter, E. E., Knapp, G., Strom, K. M., Strom, S. E. 1982, *Ap. J.*, **263**, 94.
Rubin, V. C., Burstein, D., and Thonnard, N. 1980, *Ap. J. (Letters)*, **242**, L149.
Rubin, V. C., Ford, W. K., and Thonnard, N. 1978, *Ap. J. (Letters)*, **225**, L107.
______. 1980, *Ap. J.*, **238**, 471.
______. 1982, *Ap. J.*, **261**, 439.
Salpeter, E. E. 1978, in *IAU Symposium 77, Structure and Properties of Nearby Galaxies*, ed. E. M. Berkhuijsen and R. Wielebinski (Dordrecht: Reidel), p. 23.
Sandage, A., and Tammann, G. A. 1976, *Ap. J.*, **210**, 7.
Sargent, W. L. W., Schechter, P. L., Boksenberg, A., and Shortridge, K. 1977, *Ap. J.*, **212**, 326.
Schechter, P. L., and Gunn, J. E. 1979, *Ap. J.*, **229**, 472.
Spitzer, L. 1942, *Ap. J.*, **95**, 329.
Strom, K. M, and Strom, S. E. 1978, *A.J.*, **83**, 73.
Strom, S. E. 1982, in *The Greenbank Conference on the H I Content of Normal Galaxies*, ed. R. Giovanelli and M. Haynes, NRAO Spec. Rept.
Terlevich, R., Davies, R. L., Faber, S. M., and Burstein, D. 1981, *M.N.R.A.S.*, **196**, 381.
Tonry, J. L. 1981, *Ap. J. (Letters)*, **251**, L1.
Tonry, J. L., and Davis, M. 1981, *Ap. J.*, **246**, 680.
Tully, R. B., Bottinelli, L., Fisher, J. R., Gougenheim, L., Sancisi, R., and van Woerden, H. 1978, *Astr. Ap.*, **63**, 37.
Tully, R. B., and Fisher, J. R. 1977, *Astr. Ap.*, **54**, 661 (TF).
van der Kruit, P. C., and Searle, L. 1981*a*, *Astr. Ap.*, **95**, 105.
______. 1981*b*, *Astr. Ap.*, **95**, 116.
Visvanathan, N. 1981, *J. Astr. Ap.*, **2**, 67.
Weekes, T. C. 1981, *A.J.*, **86**, 1415.
Whitmore, B. C., and Kirshner, R. P. 1981, *Ap. J.*, **250**, 43.
Young, P. J. 1976, *A.J.*, **81**, 807.
Zinn, R. 1980, in *Globular Clusters*, ed D. A. Hanes and B. F. Madore (Cambridge: Cambridge University Press), p. 191.

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