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Coronal heating by phase-mixed shear Alfvén waves

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Summary. We consider the physical processes which occur when a shear Alfvén wave propagates in a structure with a large gradient of the Alfvén velocity. Although these waves do not possess local resonances (unlike magneto acoustic modes) they nevertheless suffer intense *phase mixing* during which the oscillations of neighbouring field lines become rapidly out of phase. We study this effect and show that the resulting large growth of gradients dramatically enhances the viscous and ohmic dissipation. The cases of propagating and standing waves are considered, and a detailed calculation is given of the rate of dissipation achieved in a finite length structure like a loop, in the presence of a random excitation at its ends. We prove that, after a long enough time, phase mixing can actually ensure the dissipation of all the wave mechanical energy that a loop can pick up from the excitation, in agreement with a previous claim by Ionson.

MHD instabilities developed in the phase-mixed flow play a decisive role in hastening the ultimate dissipation by promoting momentum exchange between neighbouring layers which vibrate out of phase. The stability to Kelvin-Helmholtz perturbations is investigated in some detail, and it is shown that propagating waves are stable. However, standing waves are hihgly unstable in the vicinity of velocity antinodes, and so they decay in a few periods. Similarly, the stability to tearing perturbations is examined, and it is shown that standing waves suffer tearing near velocity nodes, while propagating waves appear to be stable.

The general conclusion of the study is that phase mixing is the process most able to ensure the dissipation of shear Alfvén waves in loops and in open regions of strong reflectivity, and that loops, in particular, must be in a permanent state of Kelvin-Helmholtz and tearing turbulence.

Key words: coronal heating – Alfvén waves – MHD waves – MHD instabilities – solar corona

I. Introduction

The problem of coronal heating has been reconsidered in recent years from a new point of view after it was realized that this heating is probably related to the magnetic field structure of the corona (Vaiana et al., 1973) and that the measured acoustic flux is too small (Athay and White, 1978; Mein et al., 1981). A number of recent reviews have stressed the renewed interest in electric heating by electric currents or magnetic waves (Kuperus et al., 1981; Heyvaerts and Schatzman, 1980; Chiuderi, 1981; Priest, 1982a, b). Up to now, the possibility of pure Joule heating of the corona has not been studied in depth. Simple Joule dissipation in the presence of anomalous resistivity seems to pose more problems than it solves (Rosner et al., 1978; Heyvaerts, 1982) and the theory of heating turbulence remains to be developed further (Galeev et al., 1981).

Concerning the dissipation of electro-mechanical energy, the difficulty of damping Alfvén waves by simple friction processes was first recognized by Osterbrok (1961). Subsequent studies have been concerned with both the accessibility of Alfvén waves to the corona (Hollweg, 1979, 1981a, b; Leroy, 1980, 1981) and also the effect of horizontal stratification. The latter effect is often very pronounced and gives rise to special phenomena, such as surface magnetohydrodynamic waves (Wentzel, 1979; Roberts, 1981), or oscillations having a continuum of frequencies for a given wave vector (Kadomtsev, 1976) which in turn produces so called "dissipationless" damping (Sedlacek, 1971; Ionson, 1978; Rae and Roberts, 1972; Hasegawa and Chen, 1974). The phenomena associated with the gradient of the Alfvén velocity have recently received much attention, especially those associated with that particular polarization which has a component of the displacement in the direction of the inhomogeneity. Standard shear Alfvén waves, however, have a displacement perpendicular to the direction of inhomogeneity and have received little attention, probably because the simplicity of their propagation properties was not expected to lead to rapid damping.

The aim of this paper is to show that, on the contrary, the frequency (or wavelength) detuning between neighbouring oscillating magnetic surfaces leads, both in the case of propagating or standing shear Alfvén waves, to several interesting phenomena, which are all produced by the phase mixing that these oscillations exhibit as they propagate (or evolve in time). Phase mixing naturally generates small scale motions in the flow and may in some cases enhance the dissipation by a rather large factor. This takes different aspects according to whether we consider vibrations produced in open or closed magnetic field lines. Also we shall demonstrate that phase mixing in the oscillatory flow may drive fast MHD instabilities. These, in turn, may create small scale structure, thereby increasing the effective dissipation coefficients and making the above mentioned damping mechanism even more efficient.

Inhomogeneous structures in the corona have various geometries, but, for the sake of simplicity, we shall consider here a planar inhomogeneity. We denote by $\hat{x}$ the direction of this inhomogeneity and by $\hat{z}$ the direction of the unperturbed magnetic field, assumed

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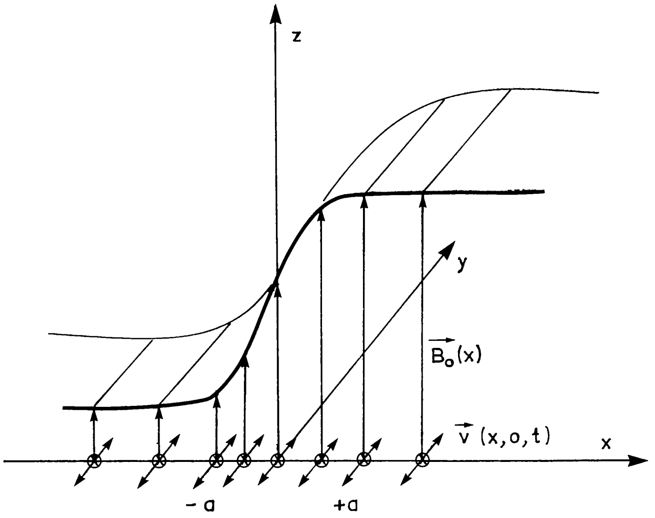

Fig. 1. The geometry of the basic magnetic field $B(x)\hat{z}$ excited by footpoint motions (double sense arrows) in the $\hat{y}$-direction, that generate shear Alfvén waves. The variation of the Alfvén speed with x has a similar profile

to be in planes perpendicular to $\hat{x}$ (see Fig. 1). In the Sun, the feet of these magnetic field lines are anchored in a dense medium, the photosphere, which is subject to motions induced possibly by an even more dense region, the convection zone. It is usual to consider that these foot-point motions are unaffected by processes going on in the chromosphere and corona. They appear as boundary conditions for motions of the fluid in the upper layers and represent the driver of such motions.

The study is presented in several parts. Part II presents the basic equations and Part III considers the laminar motion and its dissipation in response to imposed boundary conditions at the base of open field lines. The damping length associated with friction induced by phase mixing is estimated analytically.

In Part IV, the same problem is considered merely in the context of closed loop structures, which develop standing waves, rather than propagating ones (e. g., Hollweg, 1979, 1981). A damping time for the dissipation due to phase mixing of an initially coherent perturbation is calculated. The resulting rate of energy dissipation in a structure permanently excited by random foot-point motions is calculated in some detail with mathematical completeness. The result, anticipated by Ionson (1982), that this structure reacts as a high quality resonator excited by white noise is completely supported by this detailed calculation. It shows moreover that one given structure can in fact absorb frequencies not only in the immediate vicinity of one "resonance frequency", but also over a finite bandwidth, because of the finite range of Alfvén velocities that the structure can support. It is moreover shown that this resonant excitation and damping process is just due to the effect of phase mixing of shear Alfvén waves, particularly effective in this case because they are standing.

In the next sections, we discuss stability of the phase-mixed flows to the tearing and, more important, to the Kelvin-Helmholtz instabilities. An estimate of whether these might enhance the dissipation is made by means of a local approximation (both in space and time). Section V describes the structures which have been analysed for stability. The Kelvin-Helmholtz instability of propagating phase-mixed Alfvén waves is considered in Sect. VI. In the limit of fast growing modes, they are demonstrated to be stable. More difficult is the question of instability in standing waves which is considered in Sect. VII. We show how the problem may be reduced to a rather general type of Rayleigh stability equation which in this case is of Hill type, with periodic coefficients. Many previous results on this instability in bounded media are generalized for our problem. These are sometimes trivial, but the proof that there actually exists unstable modes is not. The growth time is found to be of the order of the transit time through the fine scale dimension produced by phase mixing. This is supported by an analytical result which gives explicitly the growth-rate versus wave numbers for a simplified model. Section VIII considers the possibility that due to phase mixing, the magnetic structure produced by the wave might filament by the tearing mode in a time less than one period. The quasi-periodicity of the structure enhances slightly the growth rate as compared to the case of a plane sheet pinch. Section IX discusses how effective these instabilities are in producing wave damping; it is concluded that they actually play an important role.

Our overall conclusion is that there are good reasons to think that phase-mixed shear Alfvén waves are an excellent candidate for heating inhomogeneous structures that should be explored in much more detail in future, including hitherto ignored effects such as vertical stratification, anomalous processes and non-linear developments.

II. Equations relevant to the problem

In the case of shear Alfvén waves in a planar inhomogeneity, the equations of motion are linear, even though the perturbation itself may be of finite amplitude. This is a well-known result which leads to much simplification. The basic magnetic structure on which shear Alfvén waves will be considered to develop is given by

$$\boldsymbol{B}_0 = B_0(x)\hat{z}; \quad \varrho = \varrho_0(x); \quad p = p_0(x); \tag{1}$$

and the perturbations are assumed independent of y, with a polarization in the $\hat{y}$ direction. Hence, in the perturbed state, we write:

$$\begin{aligned} \boldsymbol{B} &= B_0(x)\hat{z} + \mathrm{b}(\mathrm{x,z,t})\,\hat{y} \\ \boldsymbol{v} &= v(x,z,t)\,\hat{y}. \end{aligned} \tag{2}$$

It is a simple matter to show that when there is no dissipation the induction equation, Ohm's law, and the (non-linear) equation of motion lead to the well known propagation equation, for $v(x,z,t)$, say:

$$\frac{\partial^2 v}{\partial t} - v_A^2(x)\,\frac{\partial^2 v}{\partial z^2} = 0, \tag{3}$$

where

$$v_A^2(x) = \frac{B_0^2(x)}{\mu_0 \varrho_0(x)}. \tag{4}$$

Inclusion of resistivity and viscosity, assumed small (as long as phase mixing is not too large at least) leads instead to the following set of linearized MHD equations:

$$\frac{\partial v}{\partial t} = \frac{B_0(x)}{\mu_0 \varrho_0(x)}\,\frac{\partial b}{\partial z} + v_v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{5}$$

$$\frac{\partial b}{\partial t} = B_0(x)\,\frac{\partial v}{\partial z} + v_m\left(\frac{\partial^2 b}{\partial x^2} + \frac{\partial^2 b}{\partial z^2}\right), \tag{6}$$

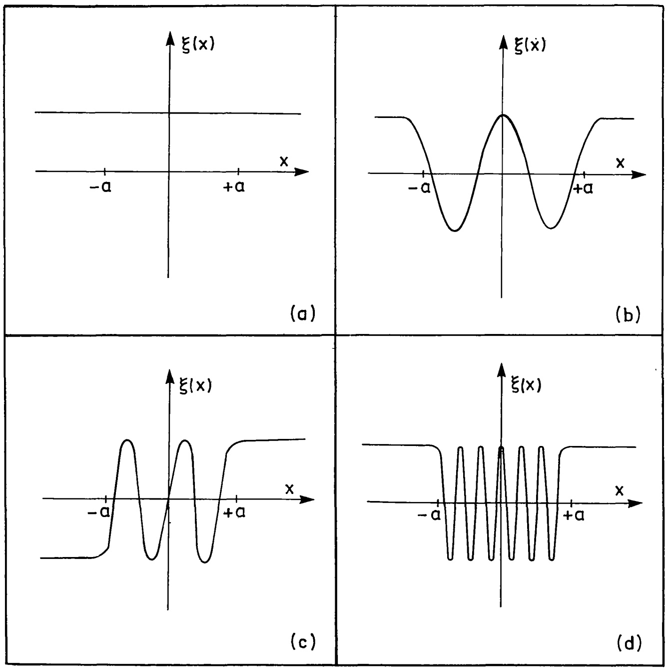

Fig. 2a–d. The profile of the plasma displacement $\xi(x)\hat{y}$ in a propagating wave of fixed frequency at several heights $z_0 < z_1 < z_2 \ll z_3$. The profiles also apply to a standing wave of fixed wavelength at several times $t_0 < t_1 < t_2 \ll t_3$. z_0 (or t_0) is assumed to be such that all displacements are in phase **a**. In **b** and **d**, the displacements for $x < -a$ or $x > +a$ are assumed to be in phase, while they are out of phase in **c**. Phase mixing increases as height increases (or time elapses) and strong x gradients develop in the region with an Alfvén velocity gradient

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where v_v is the kinematic viscosity coefficient and v_m is the magnetic diffusivity, both assumed uniform. These equations become extremely simple in the cases when either v_v or v_m is zero or v_v and v_m are so small that their squares and products can be neglected. Then b may be eliminated to give:

$$\frac{\partial^2 v}{\partial t^2} = v_A^2(x)\frac{\partial^2 v}{\partial z^2} + (v_m + v_v)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial v}{\partial t}. \tag{7}$$

If dissipation were ignored completely, all the magnetic surfaces with $x = \text{const}$ would oscillate independently of their neighbours. For open field lines excited at their base at a fixed frequency ω, the wavelength, $\lambda_{\parallel}(x)$ is given by:

$$\lambda_{\parallel}(x) = \frac{2\pi}{k_{\parallel}(x)} = 2\pi\frac{v_A(x)}{\omega} \quad \text{(Case 1).} \tag{8}$$

If on the other hand the wavelength ($\lambda_{\parallel}$) is fixed by boundary conditions (which may correspond to the situation of closed field lines or of reflected waves in an open field) each magnetic surface has its own oscillation frequency ($\omega(x)$) given by

$$\omega(x) = k_{\parallel} v_A(x) \quad \text{(Case 2).} \tag{9}$$

In both cases, after having propagated a number of wavelengths (Case 1) or after having evolved for a number of periods (Case 2), the oscillations of neighbouring sheets with different values of x become more and more out of phase. The profile of fluid displacement $\xi(x)$ at progressively larger heights (Case 1) or times (Case 2) therefore looks qualitatively as shown in Fig. 2. This leads us to expect that, as we proceed to high enough altitudes, z, (Case 1) or large enough times (Case 2), the dominant effect in dissipative process will be the x-inhomogeneity. This can be proved exactly, provided:

$$k_{\parallel} a \ll 1 \tag{10}$$

so that the wavelength along the field is much larger than the width ($2a$) of the inhomogeneity. In that case one can neglect second derivatives in z in Eqs. (5) and (6), which is correct up to first order in ($k_{\parallel} a$) included.

Equation (7) then reduces to our basic equation:

$$\frac{\partial^2 v}{\partial t^2} = v_A^2(x)\frac{\partial^2 v}{\partial z^2} + (v_m + v_v)\frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial t}. \tag{11}$$

III. Damping by phase mixing in propagating waves (laminar)

In order to estimate the possible damping produced by phase mixing, we consider a propagating wave, for which the frequency is given and

$$v(x,z,t) = \hat{v}(x,z)\exp i(\omega t - k_{\parallel}(x)z). \tag{12}$$

The x-dependence of the amplitude $\hat{v}(x,z)$ is given by boundary conditions, while its z-dependence is determined by the rate of damping. Substituting in Eq. (7) the form (12) yields the following equation for $\hat{v}(x,z)$:

$$\underbrace{(\omega^2 - k_{\parallel}^2(x)v_A^2(x))\hat{v}}_{o} = v_A^2(x)\left[\frac{\partial^2 v}{\partial z^2} - 2ik_{\parallel}(x)\frac{\partial v}{\partial z}\right]$$
$$+ i\omega(v_m + v_v)\left[\frac{\partial^2 \hat{v}}{\partial x^2} - iz\frac{d^2 k_{\parallel}}{dx^2}v - 2iz\frac{dk_{\parallel}}{dx}\frac{\partial \hat{v}}{\partial x} - z^2\left(\frac{dk_{\parallel}}{dx}\right)^2\hat{v}\right], \tag{13}$$

where $k_{\parallel}(x)$ is defined by (8) so that the left hand side vanishes. Equation (13) can be further simplified in the *weak damping approximation* (if v_m and v_v vanished, $\hat{v}$ would be constant with z)

$$\frac{1}{k}\frac{\partial}{\partial z} \ll 1 \tag{14}$$

and in the *strong phase mixing limit*, expressed by

$$\frac{z}{k_{\parallel}}\frac{dk_{\parallel}}{dx} \gg 1. \tag{15}$$

Remembering that $\hat{v}$ is the (complex) amplitude of the wave, and hence a rather regular function of x, it then can be seen that Eq. (13) reduces to:

$$\frac{\partial \hat{v}}{\partial z} = -\frac{1}{2}\frac{k_{\parallel}}{\omega}z^2\left(\frac{dk_{\parallel}}{dx}\right)^2(v_m + v_v)\hat{v} \tag{16}$$

which can be readily integrated in z to give:

$$\hat{v}(x,z) = \hat{v}(x,0)\exp -\frac{1}{6}(k_{\parallel}(x)z)^3\left(\frac{v_v + v_m}{\omega}\right)\left(\frac{d\,\mathrm{Log}\,k_{\parallel}(x)}{dx}\right)^2. \tag{17}$$

The length $(d\,\mathrm{Log}\,k_{\parallel}/dx)^{-1} = a(x)$ is of the order of the size of the inhomogeneity, and a total Reynolds number may be defined as:

$$R_{\mathrm{Tot}}(x) = \frac{\omega a^2(x)}{(v_v + v_m)}. \tag{18}$$

Then, the solution (17) may be conveniently rewritten:

$$\hat{v}(x,z)=\hat{v}(x,0)\ \exp\ -\frac{1}{6}\left(\frac{k_{\|}z}{R_{\mathrm{Tot}}^{1/3}}\right)^3 \tag{19}$$

which exhibits a characteristic damping length in the laminar situation of:

$$\Lambda_{\mathrm{lam}}=\frac{(6\,R_{\mathrm{Tot}})^{1/3}}{k_{\|}(x)}=\frac{\lambda_{\|}}{2\pi}\,(6\,R_{\mathrm{Tot}})^{1/3}. \tag{20}$$

This shows that damping could occur after several wavelengths, $\lambda_{\|}$. To appreciate numerically the effect, note that

$$v_m=(\mu_0\sigma)^{-1}=\left(\frac{T}{10^6\ \mathrm{K}}\right)^{-3/2}\ \mathrm{m}^2\ \mathrm{s}^{-1}, \tag{21}$$

whereas the ion viscosity in a plasma (in the absence of magnetic field) is given by

$$v_{v0}=\left(\frac{T}{10^6\ \mathrm{K}}\right)^{5/2}\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)^{-1}\ 7.02\ 10^8\ \mathrm{m}^2\ \mathrm{s}^{-1}. \tag{22}$$

This transport coefficient is strongly reduced by the magnetic field by a factor $(\varrho_{Gi}/l_{Ti})^2$, where

$$\left(\frac{\varrho_{Gi}}{l_{Ti}}\right)=\left(\frac{1}{B_{\mathrm{Gauss}}}\right)\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)\left(\frac{T}{10^6}\right)^{3/2}\ 3.5\ 10^{-3} \tag{23}$$

and ϱ_{Gi} and l_{Ti} are the thermal ion gyroradius and mean free path, respectively. Usually this ratio is much smaller than 1, and we should then take for v_v in the corona the value:

$$v_v\simeq\left(\frac{T}{10^6\ \mathrm{K}}\right)^{1/2}\frac{1}{B_{\mathrm{Gauss}}^2}\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)\ 3.6\ 10^3\ \mathrm{m}^2\ \mathrm{s}^{-1}. \tag{24}$$

With the figures taken for normalization, viscous dissipation dominates ohmic dissipation but this would not be the case in somewhat larger magnetic fields, and we should also keep in mind the possibility that the molecular coefficients of transport may not be representative of the state of the corona.

Denoting by P the wave period, the viscous, laminar damping length by phase mixing is obtained as:

$$\Lambda_{\mathrm{lam\ visc}}=9.7\ 10^4\left(\frac{v_{v\ \mathrm{classical}}}{v_{v\ \mathrm{real}}}\right)^{1/3}B_{\mathrm{Gauss}}^{5/3}\left(\frac{P}{100\ \mathrm{s}}\right)^{2/3}$$
$$\cdot\left(\frac{10^{10}\ \mathrm{cm}^{-3}}{n}\right)^{5/6}\left(\frac{10^6\ \mathrm{K}}{T}\right)^{1/6}\left(\frac{a}{100\ \mathrm{km}}\right)^{2/3}\ \mathrm{km}, \tag{25}$$

whereas the Joule laminar damping length is given by:

$$\Lambda_{\mathrm{lam\ Joule}}=\left(\frac{v_{m\ \mathrm{Spitzer}}}{v_{m\ \mathrm{real}}}\right)^{1/3}B_{\mathrm{Gaus}}\left(\frac{10^{10}\ \mathrm{cm}^{-3}}{n}\right)^{1/2}$$
$$\cdot\left(\frac{T}{10^6\ \mathrm{K}}\right)^{1/2}\left(\frac{a}{100\ \mathrm{km}}\right)^{2/3}\left(\frac{P}{100\ \mathrm{s}}\right)^{2/3}\ 6.93\ 10^5\ \mathrm{km}. \tag{26}$$

The relevant value is the shortest, which is often the viscous one. These damping lengths are in practice quite long, because a may be larger than 100 km in the corona and B can be somewhat more than 1 Gauss. We note however that this is due to the rather high value of the classical Reynolds numbers:

$$R_{\mathrm{visc}}=\frac{\omega a^2}{v_v}=1.74\ 10^5\ B_{\mathrm{Gauss}}^2\left(\frac{10^6\ \mathrm{K}}{T}\right)^{1/2}\left(\frac{100\ \mathrm{s}}{P}\right)$$
$$\cdot\left(\frac{10^{10}\ \mathrm{cm}^{-3}}{n}\right)\left(\frac{a}{100\ \mathrm{km}}\right)^2 \tag{27}$$

$$R_{\mathrm{Joule}}=\frac{\omega a^2}{v_m}=6.28\ 10^8\left(\frac{T}{10^6}\right)^{3/2}\left(\frac{100\ \mathrm{s}}{P}\right)\left(\frac{a}{100\ \mathrm{km}}\right)^2.$$

It is then to be suspected that anything which lowers this number may have a drastic effect on wave dissipation. According to (20) dissipation could then take place on a length of the order of the parallel wavelength:

$$\lambda_{\|}=2.8\ 10^3\left(\frac{P}{100\ \mathrm{s}}\right)\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)^{-1/2}\left(\frac{B}{1\ \mathrm{Gauss}}\right)\ \mathrm{km}. \tag{28}$$

Note however that phase mixing needs several wavelengths to develop. The waves most likely to dissipate by this mechanism are the short period waves propagating in fields with a modest value. For example, considering waves of 10 s periods instead of 100 s in an environment somewhat denser than 10^{10} cm^{-3} (10^{11}, say) reduces $\Lambda_{\mathrm{lam\ visc}}$ to something of the order of 2000 km. The same value results for wave periods of 100 s if diffusion were enhanced by a factor 10^3 due to turbulence. However, this damping length can be much larger for longer period in stronger fields. A major enhancement in the efficiency of the process is found when the waves are for some reason trapped and phase-mix in time without propagating away. Such standing waves are encountered in loops and also wherever reflection from the transition region occurs (Hollweg, 1981; Leroy, 1981) even if the fields are open. The following section details the calculation and circumstances under which such damping occurs.

We finish this examination of damping by phase mixing in open structures by considering the problem when the motion of the (single) foot-point of an open field line is given as a stationary random process. The dissipation rate is given by

$$W=\frac{1}{2}\ v\varrho(x)\left(\frac{\partial v}{\partial x}\right)^2. \tag{29}$$

If $v(x,0,t)$ is given, and has a Fourier transform $v_0(x,\omega)$, for a situation in which no wave reflection occurs higher in the corona, the velocity field at altitude z and time t will be given by

$$v(x,z,t)=\int d\omega e^{i\omega t}v_0(x,\omega)\ \exp\ (-ik_{\|}(x)z)$$
$$\cdot\ \exp\ -\frac{1}{6}\,(k_{\|}(x)z)^3\,\frac{v}{\omega}\left(\frac{d\ \mathrm{Log}\ k_{\|}}{dx}\right)^2 \tag{30}$$

and the main term of this x-gradient comes from the variations of the phase $k_{\|}(x)z$. Then

$$\frac{\partial v}{\partial x}=\int d\omega e^{i\omega t}v_0(x,\omega)\left(-iz\,\frac{dk_{\|}}{dx}\right)$$
$$\cdot\ \exp\ (-ik_{\|}z)\ \exp\ -\frac{1}{6}\,(k_{\|}z)^3\,\frac{v}{\omega}\left(\frac{d\ \mathrm{Log}\ k_{\|}}{dx}\right)^2. \tag{31}$$

This expression gives the Fourier transform of the response $(\partial v/\partial x)$ in terms of the excitation $v(x,0,t)$ in the form:

$$\left(\frac{\partial v}{\partial x}\right)_{\omega}=\mathscr{Z}_1(\omega)\,v_0(\omega). \tag{32}$$

In the case when $v(x,0,t)$ does not admit a Fourier transfrom but is a stationary random process, the average power dissipated can be expressed in terms of its power spectrum $(\mathscr{P}_{v_0}(\omega))$ (Champeney, 1973) as:

$$\langle W\rangle=\frac{1}{4\pi}\ v\varrho(x)\int_{-\infty}^{+\infty}\mathscr{P}_{\frac{\partial v}{\partial x}}(\omega)=\frac{1}{2\pi}\int_{0}^{+\infty}\gamma\varrho(x)\,|\mathscr{Z}_1(\omega)|^2\,\mathscr{P}_{v_0}(\omega). \tag{33}$$

Note that the power spectrum $\mathscr{P}_{v_0}$ integrated over ω gives the time average of v_0^2. Using Eqs. (31) and (32), Eq. (33) can be converted to:

$$\langle W\rangle=\frac{1}{2\pi}\int_0^{+\infty}\omega\varrho(x)\mathscr{P}_{v_0}(\omega)\left\{(k_{\|}(x)z)^2\frac{\nu}{\omega}\left(\frac{d\,\mathrm{Log}\,k_{\|}}{dx}\right)^2\right.$$
$$\left.\cdot\exp-\left[\frac{1}{3}(k_{\|}(x)z)^3\frac{\nu}{\omega}\left(\frac{d\log k_{\|}}{dx}\right)^2\right]\right\}d\omega. \quad (34)$$

As expected, the response function in curly brackets reaches a maximum near an altitude of the order of the damping length for the relevant frequency. We could then speak of an altitude of dissipation for waves of frequency ω by phase mixing.

IV. Damping by phase mixing in standing waves (laminar)

Some structures do not allow propagation up to infinity, such as coronal loops or open field lines with an important enough stratification that reflection is produced (Leroy, 1981). Then phase mixing occurs in time rather than in space, which makes it much more efficient as far as coronal heating is concerned. This case has been discussed in a penetrating paper by Ionson (1982), who gave a description of what is going on in a loop by an averaging procedure which implies some analogy with the behaviour of a high quality resonator excited by white noise. It was not clear in this paper, however, what exactly is the nature of the resonant damping considered. Also the relation to phase mixing was not stressed, and a more detailed consideration, avoiding averaging procedures, may be felt desirable. Moreover, it may have become clear from the preceding section that we should not regard an inhomogeneous structure as a single oscillator, but rather as a collection of oscillators weakly coupled by friction.

In this section we return to this question with some more detail, and a bit more mathematical sophistication, in such a way that the variation of the local heating function with the transverse coordinate may be obtained.

We consider again the configuration described in Fig. 1, with basic state given by Eq. (1), but we now suppose that this structure is bounded from above and below by boundaries at altitude $z=0$ and $z=l$, with a velocity field (still in the y-direction) prescribed at these planes, namely:

$$\left.\begin{array}{l}v(x,0,t)=v_1(x,t)\,\hat{\mathbf{y}}\\ v(x,1,t)=v_2(x,t)\,\hat{\mathbf{y}}.\end{array}\right\} \quad (35)$$

The equation to be solved is again the basic Eq. (11). It is a trivial matter to show that boundary motions can be accounted for in the form of a driving term by performing a very simple change of variable, namely:

$$v(x,z,t)=v_1(x,t)+(z/l)\;[v_2(x,t)-v_1(x,t)]+\xi(x,z,t). \quad (36)$$

Then (11) with boundary conditions (35) reduces to the problem:

$$\frac{\partial^2\xi}{\partial t^2}-v_A^2(x)\frac{\partial^2\xi}{\partial z^2}-\nu\frac{\partial^2}{\partial x^2}\frac{\partial\xi}{\partial t}=-\left\{\frac{\partial^2 v_1}{\partial t^2}+\frac{z}{l}\frac{\partial^2 v_2}{\partial t^2}-\frac{z}{l}\frac{\partial^2 v_1}{\partial t^2}\right\}$$
$$=f(x,z,t) \quad (37)$$
$$\xi(x,0,t)=0;\qquad \xi(x,l,t)=0.$$

As ξ vanishes at both boundaries, we may further reduce the problem to a one dimensional-collection of oscillators, coupled by friction. First rescale z by defining:

$$z=\frac{l}{2}+\frac{\zeta}{\pi}\frac{l}{2};\qquad -\pi\leqq\zeta\leqq+\pi \quad (38)$$

and express ζ by the trigonometric expansion:

$$\xi(x,z,t)=\tfrac{1}{2}a_0(x,t)+\sum_1^\infty a_n(x,t)\cos n\zeta+b_n(x,t)\sin n\zeta. \quad (39)$$

The transformation of the right hand side of (37) involves a piecewise linear function, namely $\zeta/2$, whose Fourier coefficients a_n vanish, while $b_n=(-1)^{n+1}/n$. Using this and (39) in (37), we find that all the a_n's are zero except for a_0, which obeys the equation

$$\frac{\partial^2 a_0}{\partial t^2}-\nu\frac{\partial^2}{\partial x^2}\frac{\partial a_0}{\partial t}=-\left(\frac{\partial^2 v_1}{\partial t^2}+\frac{\partial^2 v_2}{\partial t^2}\right). \quad (40)$$

The b_n's obey independent equations as well:

$$\frac{\partial^2 b_n}{\partial t^2}+n^2\frac{4\pi^2 v_A^2(x)}{l^2}b_n-\nu\frac{\partial^2}{\partial x^2}\frac{\partial b_n}{\partial t}=\frac{(-1)^{n+1}}{n\pi}\left(\frac{\partial^2 v_1}{\partial t^2}-\frac{\partial^2 v_2}{\partial t^2}\right). \quad (41)$$

In the limit of small viscosity (and small resistivity as well), the damping term can play a significant role only when phase mixing occurs to enhance the gradients. No propagating term able to play that role appears in (40), which is actually a diffusion equation with source for $(\partial a_0/\partial t)$. Neglecting diffusion, its solution can be conveniently written as:

$$a_0(x,t)=-(v_1(x,t)+v_2(x,t)), \quad (42)$$

which is seen not to develop very large gradients (assuming that v_1 and v_2 are not so pathological as to have extremely small-scales by themselves). This term a_0 will therefore contribute a negligible part to the total damping and so we ignore it and concentrate on the Eq. (41). We introduce the resonant frequency for harmonic n

$$\Omega_n(x)=n\frac{2\pi}{l}v_A(x) \quad (43)$$

and put

$$f_n(x,t)=\frac{(-1)^{n+1}}{n\pi}\left(\frac{\partial^2 v_1}{\partial t^2}-\frac{\partial^2 v_2}{\partial t^2}\right). \quad (44)$$

We also provisionally drop the indices n, and denote b_n by y below, so that now we have to solve, instead of (37), a series of equations that are similar to those of a one-dimensional array of coupled randomly excited oscillators:

$$\frac{\partial^2 y}{\partial t^2}+\Omega^2(x)y-\nu\frac{\partial^2}{\partial x^2}\frac{\partial y}{\partial t}=f(x,t). \quad (45)$$

One could consider tackling (45) by taking a time Fourier transform and solving the resulting singular differential equation in x for $y_\omega(x)$, namely:

$$-i\nu\omega y''_\omega+(\Omega^2(x)-\omega^2)y_\omega=f_\omega(x). \quad (46)$$

This equation has a singularity near the point x_ω where $\Omega(x_\omega)=\omega$. This feature reminds us of the similar problem found in the study of vibrations with the other polarization (Rae and Roberts, 1981, 1982) in inhomogeneous structures. Although a solution by proper matching is perhaps accessible, it is actually much easier to find the Green's function of (45) in the small damping limit by two time scaling. Let us first solve (45) without forcing terms as we did for the altitude dependent problem before. In the absence of damping the relevant solution of (45) would be

$$y=A(x,t)\cos(\Omega(x)t)+B(x,t)\sin(\Omega(x)t). \quad (47)$$

We now substitute (47) in (45), and calculate the x-derivative in the limit of large times, when important phase mixing has occurred, so that $\Omega(x)t \gg 1$, and most x-variations are due to the variations of this phase. Damping being small, the successive time derivatives of A and B are treated as being of increasing order in ν. This gives equations for A and B:

$$\left.\begin{aligned}\frac{\partial \operatorname{Log} A}{\partial t} &= -\frac{\nu}{2}\left(\frac{d\Omega}{dx}\right)^2 t^2 \\ \frac{\partial \operatorname{Log} B}{\partial t} &= -\frac{\nu}{2}\left(\frac{d\Omega}{dx}\right)^2 t^2\end{aligned}\right\} \tag{48}$$

and the solution is found to be

$$y(x,t) = \{A_0(x)\cos\Omega(x)t + B_0(x)\sin\Omega(x)t\}\exp -\frac{\nu}{6}t^3\left(\frac{d\Omega}{dx}\right)^2. \tag{49}$$

We recognize in the exponential a damping time for waves of frequency $\omega = \Omega(x)$ (which are the only ones excitable near point x) of:

$$\tau_{\text{lam}} = \frac{(6R_{\text{Tot}})^{1/3}}{\omega} = \frac{P}{2\pi}(6R_{\text{Tot}})^{1/3}. \tag{50}$$

These phase mixing damping times, for figures quoted in the preceding paragraph, would be of the order of $20P$, which is short enough to be of interest.

Now, Eq. (49) tells us something else also, which is very important. It shows that $A(x,t)$ on field line x is determined only by those initial conditions $A(x',0)$ which actually refer to the same field line, namely $A(x,0)$; the coupling to the neighbouring field lines might have produced a certain delocalization in x of the response, but in the present weak damping approximation, only the closely neighbouring field lines interact dissipatively, as manifested by the presence of the term $(d\Omega/dx)^2$ in the damping factor in (49).

Hence, at this approximation, the response is still local in x. This very important property gives us now in a straight forward way Green's function for the problem (45). Actually, this Green's function, for positive times, is one of the possible solutions to the initial value problem without forcing term. It is easy to know which one to choose for, in the limit of times much smaller than the damping time, τ_{lam}, we should recover the undamped percussional response, namely:

$$G_0(\tau) = \frac{\sin\Omega(x)\tau}{\Omega(x)}. \tag{51}$$

Hence the Green function we need is

$$G(\tau) = \frac{\sin\Omega(x)\tau}{\Omega(x)}\exp -\left\{\frac{\nu}{6}\tau^3\left(\frac{d\Omega}{dx}\right)^2\right\} \tag{52}$$

and the solution of (45) is, in the same approximation,

$$y(x,t) = \int_{-\infty}^{t} dt' f(x,t')\frac{\sin\Omega(x)(t-t')}{\Omega(x)}\exp -\frac{\nu}{6}\left(\frac{d\Omega}{dx}\right)^2(t-t')^3. \tag{53}$$

This can be rearranged to give the Fourier transform $(y(x,\omega))$ of the response, y, in terms of that of the excitation f. Inserting:

$$f(x,t) = \int_{-\infty}^{+\infty} d\omega e^{i\omega t} f(x,\omega) \tag{54}$$

in (53) we obtain:

$$y(x,t) = \int_{-\infty}^{+\infty} d\omega e^{i\omega t} f(x,\omega)\int_0^\infty e^{-i\omega\tau}\frac{\sin(\Omega(x)\tau)}{\Omega(x)}\exp -\frac{\nu}{6}\left(\frac{d\Omega}{dx}\right)^2\tau^3 d\tau, \tag{55}$$

and so:

$$y(x,\omega) = f(x,\omega)\,\mathscr{Z}_2(x,\omega), \tag{56}$$

where the admittance, $(\mathscr{Z}_2)$, can be found by comparing (54), (55), and (56). We actually need the admittance which relates $\partial y/\partial x$ to f. This is obtained by differentiating (55) with respect to x. In the strong phase mixing approximation $(\Omega(x)\tau \gg 1)$ we get in terms of the original notations:

$$\frac{\partial b_n}{\partial x}(x,\omega) = f_n(x,\omega)\,\mathscr{A}_n(x,\omega)$$

$$\mathscr{A}_n(x,\omega) = \int_0^\infty e^{-i\omega\tau}\left(\frac{d\operatorname{Log}\Omega_n(x)}{dx}\right)\tau\cos(\Omega_n(x)\tau) \cdot \exp\left[-\frac{\nu}{6}\left(\frac{d\Omega_n}{dx}\right)^2\tau^3\right]d\tau. \tag{57}$$

The rate of energy dissipation is still given by (29), in which we insert the expansion (39) and retain only the b_n's coefficients. Taking into account the orthogonality of the base functions, we obtain a simple result for the rate averaged in z, namely:

$$\bar{W}(x,t) = \frac{1}{2}\varrho(x)\nu\int_0^l \frac{dz}{l}\left(\frac{\partial v}{\partial x}\right)^2 = \sum_1^\infty \frac{1}{4}\varrho(x)\,\nu\left(\frac{\partial b_n}{\partial x}\right)^2. \tag{58}$$

As explained in Sect. III, when f_n, instead of being Fourier-transformable is a stationary random process, the time averaged dissipation rate may be obtained in terms of the power spectrum of b_n as shown in Eqs. (33) and (32) (Champeney, 1973); then

$$\langle\bar{W}(x)\rangle = \sum_{n=1}^\infty \frac{1}{4\pi}\nu\varrho(x)\int_0^\infty |\mathscr{A}_n(x,\omega)|^2\frac{1}{n^2\pi^2}\mathscr{Q}(\omega,x)\,d\omega, \tag{59}$$

where $\mathscr{Q}(\omega,x)$ is the power spectrum of the process $(\ddot{v}_1(x,t) - \ddot{v}_2(x,t))$. We may for simplicity assume v_1 and v_2 to have identical statistical properties, and to be independent, in which case:

$$\mathscr{Q}(\omega,x) = 2\omega^4\mathscr{P}_v(\omega,x), \tag{60}$$

where now $\mathscr{P}_v$ is the spectral power of the process $v_1(x,t)$. The admittance $(\mathscr{A}_n)$, as given by Eq. (57), has obviously resonances around the frequencies $\pm\Omega(x)$. Only the resonance with positive frequency is of interest. Putting:

$$\varepsilon = \frac{\nu}{6}\left(\frac{d\Omega}{dx}\right)^2 \quad\text{and}\quad (3\varepsilon)^{1/3}\tau = \theta, \tag{61}$$

and retaining only this resonance, $\mathscr{A}_n$ can be transformed to:

$$\mathscr{A}_n(x,\omega) = \frac{1}{(3\varepsilon)^{2/3}}\left(\frac{d\operatorname{Log}\Omega_n}{dx}\right)\int_0^\infty \theta\exp -\left\{i\theta\left(\frac{\omega-\Omega_n(x)}{(3\varepsilon)^{1/3}}\right)+\frac{\theta^3}{3}\right\}d\theta. \tag{62}$$

A rescaling of frequencies can be used to evaluate (59) by putting

$$\xi = (\omega - \Omega_n(x))/(3\varepsilon)^{1/3}.$$

Ignoring the variation of all functions except the impedance with frequency around Ω_n, we obtain finally

$$\langle\bar{W}(x)\rangle = \frac{1}{\pi^3}\sum_{n=1}^\infty \varrho(x)\Omega_n^2(x)\mathscr{P}_v(\Omega_n(x),x)\left\{\int_0^\infty d\xi\left|\int_0^\infty \theta d\theta e^{-i\xi\theta}e^{-\theta^3/3}\right|^2\right\} \tag{63}$$

The numerical constant in curly bracket can be expressed, if needed, in terms of Airy functions. Actually

$$\left|\int_0^\infty d\theta\theta e^{-i\xi\theta}e^{-\theta^3/3}\right| = \left|\frac{d}{d\xi}\int_0^\infty d\theta e^{-\theta^3/3+i\xi\theta}\right| = \pi\left|\frac{d}{d\xi}Hi(i\xi)\right|, \tag{64}$$

where the function of complex argument $Hi(z)$ is defined in Abramowicz and Stegun (1965) and can be expressed in terms of Airy functions and their integrals. Finally, we can quote the result as:

$$\langle \bar{W}(x)\rangle = \frac{\mu}{\pi}\sum_{1}^{\infty} \varrho(x)\Omega_n^2(x)\mathscr{P}_v(\Omega_n(x),x)$$
$$\mu = \int_0^{\infty} d\xi \left|\frac{d}{d\xi} Hi(i\xi)\right|^2. \tag{65}$$

The important result of this calculation, which was stressed by Ionson (1982) is that the rate of heating, in this limit, appears to be independent of the dissipation coefficients. This is a property of a high quality resonator. In the present case, damping proceeds by friction on the neighbouring layers, and phase mixing can always proceed to a sufficiently advanced stage that the gradients match any dissipative efficiency no matter how small it is. What depends on the dissipation coefficients is the time it takes to reach the sufficiently phase mixed state [τ_{lam} in Eq. (50)]. But, if the structure has been excited for ever (by random stationary noise), then it finds itself, so to speak, always in a phase-mixed state.

We may then conclude that phase mixing of shear Alfvén waves in a bounded region which supports standing waves produces a state in which the rate of dissipation balances exactly the rate at which they are excited, no matter how small the dissipative coefficients may be. This could be compared with turbulent flows which develop finer and finer scales until dissipation at the proper rate becomes possible (although our problem does not exhibit the nonlinearities of a turbulent situation!). Finally, note that the total heating rate of the structure (i.e. in a slab of length L_y in the y direction) is

$$W = \frac{\mu}{\pi} L_y l \sum_{1}^{+\infty}\int_{-\infty}^{+\infty} dx \varrho(x) n^2\Omega_1^2(x)\mathscr{P}_v(n\Omega_1(x),x), \tag{66}$$

which can tap at a finite size bandwidth in the oscillation spectrum. However, each particular layer picks up only a small part of power spectrum available, unlike open flux tubes, but the heat deposition is more concentrated in closed structures than in open ones.

V. Instantaneous and local configurations of interest for stability

As realized in the preceding paragraph, some of the physics in phase mixing dissipation depends on the actual value of the transport coefficients. These may in turn be irrelevant if some type of instability disrupts the sheared structure, in which case they should be replaced by effective transport coefficients. It is therefore very important to look at the stability of the flows produced, especially for those instabilities which might grow on a time-scale shorter than one wave period. This represents a difficult problem of stability in an inhomogeneous, time dependent structure. To keep it at a reasonable level of simplicity, we assume again that the phase mixing is complete enough that the length d characteristic of the phase variations in the cross field x-direction has become much smaller than the inhomogeneity scale a. We then perform a local stability analysis considering for simplicity $B_0(x)$ and $\varrho_0(x)$ as uniform, and we consider several types of perturbation.

The phase mixing scale (d) achieved depends on the distance from the photospheric plane, $z=0$, in open structures excited from below by, say, a velocity field of given frequency ω. This is the situation achieved if the waves are propagating or if they are reflected very far away and form a semi-infinite system of standing waves of given uniform frequency. In the case of propagating waves with dissipative effects neglected, the motion achieved may be written in the form:

$$\left.\begin{aligned} \boldsymbol{B} &= B_0(x)\,\{\hat{\mathbf{z}} - A(x)\cos[\omega t - k_\parallel(x)z]\,\hat{\mathbf{y}}\},\\ \boldsymbol{v} &= v_{A_0}(x)\,\{A(x)\cos[\omega t - k_\parallel(x)z]\,\hat{\mathbf{y}}\}, \end{aligned}\quad \text{(propagating)}\right\} \tag{67}$$

where $k_\parallel$ is defined in Eq. (8). In the case of a system of standing waves with a given frequency ω, we can write on the other hand:

$$\left.\begin{aligned} \boldsymbol{B} &= B_0(x)\,\{\hat{\mathbf{z}} - A(x)\sin\omega t\sin k_\parallel(x)z\hat{\mathbf{y}}\}\\ \boldsymbol{v} &= v_{A_0}(x)\,\{A(x)\cos\omega t\cos k_\parallel(x)z\hat{\mathbf{y}}\}. \end{aligned}\quad \text{(standing)}\right\} \tag{68}$$

For large enough z, the major inhomogeneities in these oscillations are those associated with the fast variation of the phase with x. We can make a local analysis by taking $A(x)$, $\varrho_0(x)$, $B_0(x)$ constant while the phase variations are expanded around those at a certain location, $(x,z)=(0,z_0)$, say. We then write:

$$\left.\begin{aligned} z &= z_0 + \zeta\\ k_\parallel(x)z &= k_\parallel(0)z + xk'_\parallel(0)z\\ &= k_\parallel(0)z_0 + k_\parallel(0)\zeta + xk'_\parallel(0)z_0 + xk'_\parallel(0)\zeta. \end{aligned}\right\} \tag{69}$$

This can be simplified even more in the long-wavelength limit if we examine scales with $k_{\parallel 0}^{-1} \gg \zeta$. Then we can write approximately:

$$k_\parallel(x)z = k_\parallel(0)z_0 + (k'_\parallel(0)z_0)x = \Phi_0 + qx, \tag{70}$$

where q is the shear wave vector in the x-direction and is defined as:

$$q = k'_\parallel(0)z_0. \tag{71}$$

By a suitable choice of initial time, the local approximation to (67) and (68) will then be, in the propagating case

$$\left.\begin{aligned} \boldsymbol{B} &= B_0(\hat{\mathbf{z}} - A\cos(\omega t - qx)\,\hat{\mathbf{y}})\\ \boldsymbol{v} &= v_A(A\cos(\omega t - qx)\,\hat{\mathbf{y}}), \end{aligned}\right\} \tag{72}$$

and in the standing case

$$\left.\begin{aligned} \boldsymbol{B} &= B_0(\hat{\mathbf{z}} - A\sin\omega t(\sin\Phi_0\cos qx + \cos\Phi_0\sin qx)\,\hat{\mathbf{y}})\\ \boldsymbol{v} &= v_A A\cos\omega t(\cos\Phi_0\cos qx - \sin\Phi_0\sin qx)\,\hat{\mathbf{y}}. \end{aligned}\right\} \tag{73}$$

Note that both are, as expected, perfectly valid solutions of the MHD equations in a uniform medium. They represent Alfvénic perturbations with a large $(k_\perp/k_\parallel)$.

If we are interested in physical instability processes which occur in structures with a finite dimension in the direction of z, we can consider, for example, fundamental oscillations, for which the exact expression is given by

$$\left.\begin{aligned} \boldsymbol{B} &= B_0(x)\,\{\hat{\mathbf{z}} - A(x)\sin[\omega(x)t]\sin[k_\parallel z]\,\hat{\mathbf{y}}\}\\ \boldsymbol{v} &= v_A(x)A(x)\cos[\omega(x)t]\cos[k_\parallel z]\,\hat{\mathbf{y}}, \end{aligned}\quad \text{(standing)}\right\} \tag{74}$$

where $\omega(x)$ and $k_\parallel$ (assumed fixed) are related by:

$$\omega(x) = k_\parallel v_A(x). \tag{75}$$

A local expansion of the frequency may be made around the reference point $(x,z)=(0,z_0)$ and reference time t_0. After neglecting again the z variations and putting $t=t_0+\tau$, we find:

$$\left.\begin{aligned} \boldsymbol{B} &= B_0\{\hat{\mathbf{z}} - A\sin(k_\parallel z_0)\sin(\omega\tau + qx)\,\hat{\mathbf{y}}\}\\ \boldsymbol{v} &= v_A A\cos(k_\parallel z_0)\cos(\omega\tau + qx)\,\hat{\mathbf{y}}, \end{aligned}\right\} \tag{76}$$

where now

$$q = +\frac{d\omega}{dx_0}\,t_0. \tag{77}$$

The stability of the structures described by (72), (73) or (76) may be considered at various reference positions (z_0) and various reference times (t_0) either including or neglecting the full time-dependence. In this preliminary report we shall look only for instabilities that develop rapidly, i. e. in less than one period, and we shall fix then the reference time. In that limit the problem of MHD stability of the propagating wave reduces to that of considering the stationary flow described by

$$\left.\begin{aligned} &\boldsymbol{B}=B_0(\hat{z}-A\cos qx\,\hat{y})\\ &\boldsymbol{v}=v_A A\cos qx\,\hat{y}, \end{aligned}\right\} \tag{78}$$

where q is given in this case by (71).

Similarly the problems for standing waves with a fixed frequency or fixed wavelength reduce to that of considering stability of the stationary structure:

$$\left.\begin{aligned} &\boldsymbol{B}=B_0(\hat{z}-A\sin\varphi_0\sin qx\,\hat{y})\\ &\boldsymbol{v}=v_A A\cos\varphi_0\cos qx\,\hat{y}, \end{aligned}\right\} \tag{79}$$

where $\varphi_0=\omega t_0$ (or $k_\| z_0$) in the case of Eq. (73) [or Eq. (76) respectively] and the phases Φ_0 (or $\omega\tau$) have been transformed away. The standing wave problem finally gives rise to two extreme cases corresponding to the structure near nodes or antinodes of the magnetic field. Near nodes of the magnetic field:

$$\left.\begin{aligned} &\boldsymbol{B}=B_0\,\hat{z}\\ &\boldsymbol{v}=v_A A\cos qx\,\hat{y}, \end{aligned}\right\} \tag{80}$$

whereas near nodes of the velocity:

$$\left.\begin{aligned} &\boldsymbol{B}=B_0(\hat{z}-A\sin qx\,\hat{y})\\ &\boldsymbol{v}=0. \end{aligned}\right\} \tag{81}$$

The stationary configurations (78), (80), and (81) will be explored below for their stability, against tearing modes and Kelvin-Helmholtz modes. Both of these instabilities produce interactions between sheets of different x values and are thus candidates for increasing the effective viscosity. In this first paper, we have put more emphasis on the study of the Kelvin-Helmholtz problem, which is simpler, and, as we shall see, more likely to enhance the dissipation.

The possibility of microinstabilities arises because the development of large gradients in phase-mixed Alfvénic oscillations implies a build-up of strong electric current densities. However, micro-instabilities require extremely high current densities. Taking as a representative value the ion thermal velocity threshold, we get a critical current

$$J^*=nev_{Ti}\simeq 160\left(\frac{T}{10^6}\right)^{1/2}\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)\mathrm{Am}^{-2}. \tag{82}$$

The current density in the wave, at altitude z, say, is

$$J_0\simeq\frac{AB_0}{\mu_0}\,q\simeq k_\| z_0\,\frac{AB_0}{a\mu_0}. \tag{83}$$

Thus, numerically:

$$\frac{J_0}{J^*}=A\left(\frac{z_0}{\lambda_{\|0}}\right)3\ 10^{-4}\,B_{\mathrm{Gauss}}\left(\frac{a}{100\ \mathrm{km}}\right)^{-1}\left(\frac{T}{10^6\ \mathrm{K}}\right)^{-1}\cdot\left(\frac{n}{10^{10}\ \mathrm{cm}^{-3}}\right)^{-1} \tag{84}$$

which means that it needs 3000 wavelengths to reach a sufficiently phase mixed situation in open structures and 3000 periods in a closed one, i.e. three days if $P=100$ s.

VI. Kelvin-Helmholtz stability of a propagating wave

Kelvin-Helmholtz instability develops in some shear flows; it is described at length in many textbooks (Chandrasekhar, 1981, for example) and we benefited greatly from the excellent presentation given in the book by Drazin and Reid (1981) and also from the work done in the thesis of Roberts (1971).

Kelvin-Helmholtz instability destroys the regularity of a shear flow pattern. It needs inflexion points in the velocity profile to develop, but not all flows with an inflexion point are unstable. As limiting cases, an unmagnetized flow with a discontinuity in the velocity profile is always unstable, and the growth rate is $\frac{1}{2}k\Delta v$, where k is the wave-number parallel to the flow direction and Δv is the velocity jump. The same is true if a uniform magnetic field $B\hat{z}$ perpendicular to the flow is added and is therefore not perturbed by the unstable motions. By contrast, a flow profile linear in x is not unstable. A magnetic field (uniform) has a stabilizing effect when the perturbation distorts it. This is the case if the field has a component B_y along the direction of the flow. In that case, even the flow with a discontinuity may be stabilized if the Alfvén velocity $B_y/(\mu_0\varrho)^{1/2}$ is larger than Δv. In the case of the flow (78) we consider such a case that v in the Alfvén wave is of the order of Av_A (the Alfvén velocity based on B_y). The possibility that these waves are then actually stable to Kelvin-Helmholtz perturbations is justified later in this section. On the other hand the configuration (80), appropriate to the neighbourhood of a magnetic field node in a standing wave is most likely to be unstable since $\boldsymbol{v}$ is normal to $\boldsymbol{B}$. We demonstrate in Sect. VII that this is actually the case.

The general perturbation equations, in perfect MHD, for this problem can be written conveniently by introducing the vorticity field $\boldsymbol{\Omega}=\mathrm{rot}\ \boldsymbol{v}$. The equations of motion then become

$$\left.\begin{aligned} &\mathrm{div}\ \boldsymbol{B}=0\\ &\mathrm{div}\ \boldsymbol{v}=0\\ &\frac{\partial\boldsymbol{B}}{\partial t}=\mathrm{rot}\ (\boldsymbol{v}\times\boldsymbol{B})=(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{B}\\ &\varrho\frac{\partial\boldsymbol{\Omega}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{\Omega}-(\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{v}=(\boldsymbol{B}\cdot\nabla)\boldsymbol{J}-(\boldsymbol{J}\cdot\nabla)\boldsymbol{B}. \end{aligned}\right\} \tag{85}$$

We consider incompressible motions, and seek two dimensional rolls, periodic in y, with any perturbation of the form

$$\tilde{p}(x,y,z,t)=p(x,t)\,e^{iky}. \tag{86}$$

Note that these perturbations are taken to be independent of z. We also introduce the components of velocity and field perturbations

$$\left.\begin{aligned} &\boldsymbol{v}_1=(u\boldsymbol{e}_x+v\boldsymbol{e}_y)\,v_A\\ &\boldsymbol{B}_1=(\alpha\boldsymbol{e}_x+\beta\boldsymbol{e}_y)\,B_0. \end{aligned}\right\} \tag{87}$$

After linearizing as usual we arrive at a system of equations for α, β, u, v.

This finally gives for the case of a propagating wave represented by Eq. (78) the following set of equations for the perturbations:

$$\frac{\partial\alpha}{\partial x}+ik\beta=0, \tag{88a}$$

$$\frac{\partial u}{\partial x}+ikv=0, \tag{88b}$$

$$\frac{\partial \alpha}{\partial t}+ikAv_{A_0}\cos qx\ (\alpha+u)=0, \tag{88c}$$

$$\left(\frac{\partial}{\partial t}+ikAv_{A_0}\cos qx\right)\left(\frac{\partial v}{\partial x}-iku\right)$$
$$=q^2Av_{A_0}\cos qx\ (\alpha+u)-ikAv_{A0}\cos qx\left(\frac{\partial \beta}{\partial x}-ik\alpha\right). \tag{88d}$$

By elimination of β and v, this set can be reduced to two equations only. These are then Fourier analyzed in time, as:

$$u(x,t)=\hat{u}(x)e^{-i\gamma t};\quad \alpha(x,t)=\hat{\alpha}(x)e^{-i\gamma t} \tag{89}$$

and we introduce for convenience the dimensionless variables:

$$x'=qx;\quad f=\gamma/(kAv_{A_0});\quad K=\frac{k}{q}. \tag{90}$$

We then obtain the system for the x-components of the perturbed velocity ($\hat{u}$) and magnetic field ($\hat{\alpha}$)

$$\left.\begin{aligned} &f\hat{\alpha}=\cos x'(\hat{\alpha}+\hat{u})\\ &(f-\cos x')\ (\hat{u}''-K^2\hat{u})=\cos x'(\hat{\alpha}+\hat{u})-\cos x'(\hat{\alpha}''-K^2\hat{\alpha}).\end{aligned}\right\} \tag{91}$$

Differentiation is with respect to x. Finally (91) reduces to a single equation for the combination:

$$\hat{w}=\hat{\alpha}+\hat{u} \tag{92}$$

namely

$$(2\cos x'-f)\ (\hat{w}''-K^2\hat{w})=2\sin x'\hat{w}'. \tag{93}$$

This can be rewritten in a form similar to a Sturm-Liouville equation, namely:

$$\frac{d}{dx'}(2\cos x'-f)\frac{d\hat{w}}{dx'}=K^2\ (2\cos x'-f)\,\hat{w}. \tag{94}$$

Since the problem is not posed in a finite domain, we cannot use known results for this case. As we show in the next paragraph this equation, which has periodic even coefficients, admits a theorem according to which the solutions are very nearly periodic if they are bounded at all. The boundedness of the solution is of course the condition which we have to impose. Multiply (94) by $\hat{w}^*$, the complex conjugate of $\hat{w}$, and integrate over a convenient interval I such that $\hat{w}$ and $\cos x'$ are very nearly periodic on it. Integrating by parts, we use the fact that the integrated term is small due to quasi-periodicity. Then, for any ε, we can find an interval $I(\varepsilon)$ such that:

$$\left|\int_{I(\varepsilon)}(2\cos x'-f)\left(\left|\frac{d\hat{w}}{dx'}\right|^2+K^2|\hat{w}|^2\right)dx'\right|\leqq\varepsilon.$$

In particular, the imaginary part of the integral should be as small as one wishes also. Then:

$$\left|\mathrm{Im}\,f\int_{I(\varepsilon)}\left(\left|\frac{d\hat{w}}{dx'}\right|^2+K^2|\hat{w}|^2\right)dx'\right|\leqq\varepsilon. \tag{95}$$

From Eq. (95), we deduce that $\mathrm{Im}\,f$ vanishes, and that the configuration considered is always stable. This is because of the stabilizing role of the magnetic field mentioned at the beginning of this section.

VII. Kelvin-Helmholtz instability at velocity antinodes of a standing wave

We now turn to consider the stability of standing waves. As far as the Kelvin-Helmholtz instability is concerned, the most likely places for it to occur, are where the velocity is largest, i.e. near antinodes of the velocity field. These turn out to be nodes of the magnetic field as well, which is also quite favourable for instability. We then consider the flow represented by Eq. (80), and perturb it as described in Eqs. (85)–(88). This gives in the present case the system:

$$\left.\begin{aligned} &\frac{\partial \alpha}{\partial x}+ik\beta=0\\ &\frac{\partial u}{\partial x}+ikv=0\\ &\frac{\partial \alpha}{\partial t}+ikAv_{A_0}\cos qx\ \alpha=0\\ &\left(\frac{\partial}{\partial t}+ikAv_{A_0}\cos qx\right)\left(\frac{\partial v}{\partial x}-iku\right)=q^2Av_{A_0}\cos qx\ u.\end{aligned}\right\} \tag{96}$$

The magnetic field appears to suffer no disturbance while the elimination of v gives the equation for the perturbation of the x-component (u) of the velocity. Using variables defined in Eqs. (89) and (90) we obtain:

$$(f-\cos x')\left(\frac{d^2\hat{u}}{dx'^2}-K^2\hat{u}\right)-\cos x'\hat{u}=0. \tag{97}$$

This is a particular case of the so-called Rayleigh stability equation. Actually, had we taken the velocity field in Eq. (80) to have a profile $a(x')$ instead of $\cos x'$, we would have obtained the equation for $\hat{u}$ as:

$$(f-a(x'))\left(\frac{d^2\hat{u}}{dx'^2}-K^2\hat{u}\right)+\frac{d^2a}{dx'^2}\,\hat{u}=0 \tag{98}$$

which is exactly the Rayleigh stability equation. Here f and $\hat{u}$ are complex, x' is real, and the equation should be solved subject to the condition that $\hat{u}$ remains finite when x' tends to infinity (both $+\infty$ and $-\infty$). Equation (97) is of Hill type, with periodic and even coefficients of period 2π. Solutions of such equations are governed by the Floquet theorem, which states that there exists a number λ such that one particular solution is of the form:

$$F_+(x)=e^{i\lambda x}P(x), \tag{99}$$

where $P(x)$ is a periodic function of period 2π. Similarly, by changing x to $-x$, the function

$$F_-(x)=e^{-i\lambda x}P(-x) \tag{100}$$

is also a solution. It can be shown that, if λ is not an integer, F_+ and F_- are linearly independent. Then if λ – called the Floquet exponent – is real, any solution is bounded, while if λ is complex, no solution can be bounded. Physically meaningful solutions (i.e. obeying our boundary conditions) should then be associated with a real Floquet exponent. This exponent plays, for this periodic basic state, much the same role as a wavelength in the x direction. Finding the "dispersion relation" for Eq. (44) means exhibiting the proper relation between the "wave-numbers" k and λ (real!) and the complex frequency f. This is not a straight-forward task. Nevertheless stability and, to some extent, the growth rate can be examined without a detailed knowledge of this relation by using a series of known theorems which we extend to the present situation.

First, consider Fjørtoft's theorem, valid for perturbations in bounded media, which states that a necessary condition for instability is that the velocity profile $a(x)$ has an inflexion point. Let u^* be the complex conjugate of u. Let us consider (97) in the case of an unstable mode, with complex f. Then we can divide by $(\cos x' - f)$, which, because $\operatorname{Im} f \neq 0$, never vanishes, multiply by u^* and integrate on a certain finite interval $I = [a, b]$. We get (dropping ^ from now on):

$$\int_I u^* u'' - K^2 \int_I u^* u + \int_I \frac{\cos x'}{\cos x' - f} u^* u = 0. \tag{101}$$

After integrating the first term by parts we arrive at:

$$-\int_I u' u'^* + u^* u'|_a^b - K^2 \int_I |u|^2 + \int_I \frac{\cos x' (\cos x' - f^*)}{|\cos x' - f|^2} |u|^2 = 0. \tag{102}$$

The integral term usually vanishes when the boundary conditions are imposed. This is not so here, but we can make it as small as we wish by a proper choice of bounds a and b. Actually we know from Floquet's theorem that:

$$u = \alpha_1 e^{i\lambda x} P(x) + \alpha_2 e^{-i\lambda x} P(-x).$$

If λ is a rational number, $(\lambda \in Q)$, u has a period, and we can take a and b distant by that period so that $u^* u'(b) = u^* u'(a)$. If $\lambda \notin Q$, it is always possible to find a number in Q very near it, r say. Let us take a and b such that $\alpha_1 e^{irx} P(x) + \alpha_2 e^{-irx} P(-x)$ has a period $(b-a)$. Then, after putting $\alpha = r + \eta$, with η very small, we see that the difference in values of u^* and u' between a and b can be made as small as we wish by a proper choice of r, and hence of a and b. Then, for any ε we can find an interval $I(\varepsilon)$ such that:

$$\left.\begin{aligned} -\varepsilon \leqq \int_{I(\varepsilon)} |u'|^2 + K^2 |u|^2 - \int_{I(\varepsilon)} \frac{\cos x' (\cos x' - \operatorname{Re} f)}{|\cos x' - f|^2} |u|^2 \leqq +\varepsilon \\ -\varepsilon \leqq \operatorname{Im} f \int_{I(\varepsilon)} \frac{|u|^2 \cos x'}{|\cos x' - f|^2} dx' \leqq +\varepsilon. \end{aligned}\right\} \tag{103}$$

The profile $\cos x'$ actually allows the integral in the second equation of (103) to vanish. Hence this necessary condition for instability is satisfied.

Let us now derive a theorem which puts bounds on the wavelengths of unstable modes. We combine the two inequalities of (103) by substracting $(\operatorname{Re} f / \operatorname{Im} f)$ times the second from the first to get

$$-\varepsilon' \leqq \int_{I(\varepsilon)} |u'|^2 + K^2 |u|^2 - \int_{I(\varepsilon)} \frac{(\cos x' - \operatorname{Re} f)^2 - \operatorname{Re} f^2}{|\cos x' - f|^2} |u|^2 \leqq +\varepsilon'. \tag{104}$$

But we have the inequality:

$$\frac{(\cos x' - \operatorname{Re} f)^2 - \operatorname{Re} f^2}{(\cos x' - \operatorname{Re} f)^2 + \operatorname{Im} f^2} \leqq 1, \tag{105}$$

and so

$$\begin{aligned} K^2 \int |u|^2 &\leqq \varepsilon' + \int \frac{(\cos x' - \operatorname{Re} f)^2 - \operatorname{Re} f^2}{(\cos x' - \operatorname{Re} f)^2 + \operatorname{Im} f^2} |u|^2 - \int |u'|^2 \\ &\leqq \varepsilon' + \int (|u|^2 - |u'|^2) \end{aligned} \tag{106}$$

or, for small enough ε'

$$K^2 \leqq 1 + \frac{\varepsilon' - \int |u'|^2}{\int |u|^2} \leqq 1 \tag{107}$$

This result (that for unstable modes $K^2 \leqq 1$) is a generalization of a well known theorem (Drazin and Reid, 1981). It shows that unstable modes have $|K| < 1$.

Finally we extend a theorem which gives useful bounds on the growth rate. Following the usual procedure (Drazin and Reid, 1981) we consider the Rayleigh equation (97) and the equation

$$\left(\frac{d^2}{dx'^2} - K^2\right)(\cos x' - f) u^+ + \cos x' u^+ = 0, \tag{108}$$

where $u^+ = \dfrac{u}{\cos x' - f}$. The latter equation can be rewritten as

$$\frac{d}{dx'} (\cos x' - f)^2 \frac{du^+}{dx'} - K^2 (\cos x' - f)^2 u^+ = 0. \tag{109}$$

Let us multiply by u^{+*} and integrate (by parts) over a finite interval. By the same type of argument as above it is always possible to choose this interval in such a way that boundary values are either exactly equal or almost equal. To simplify the reasoning consider the case when they are strictly equal. Then

$$\begin{aligned} \int_a^b \left[K^2 (\cos x' - f)^2 |u^+|^2 + (\cos x' - f)^2 \left|\frac{du^+}{dx}\right|^2 \right] \\ = u^{+*} (\cos x' - f)^2 u'^+ |_a^b = 0. \end{aligned} \tag{110}$$

Now separate real and imaginary parts of this relation and consider first the imaginary part which becomes

$$\operatorname{Im} f \int_a^b (\cos x - \operatorname{Re} f) \left(\left|\frac{du^+}{dx}\right|^2 + K^2 |u^+|^2 \right) = 0. \tag{111}$$

This is possible only for $|\operatorname{Re} f| \leqq 1$ (Rayleigh's theorem). The real part gives

$$\int_a^b \left((\cos x - \operatorname{Re} f)^2 - \operatorname{Im} f^2 \right) \left(\left|\frac{du^+}{dx}\right|^2 + K^2 |u^+|^2 \right) = 0. \tag{112}$$

Also it is true that:

$$\int_a^b (1 - \cos^2 x') \left(\left|\frac{du^+}{dx'}\right|^2 + K^2 |u^+|^2 \right) \geq 0. \tag{113}$$

In Eq. (113) the part of the integral involving $\cos^2 x$ can be obtained from Eq. (112) in terms of similar expressions involving $\cos x$ and $(\operatorname{Re} f)^2 - (\operatorname{Im} f)^2$, and finally Eq. (110) gives the expression of that involving $\cos x$. Equation (113) can then be reduced to:

$$(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2 \leqq 1$$

which is the well known Howard semi-circle theorem, rederived and specialized to our particular geometry.

We know at present that if unstable modes exist they must have $K < 1$ [Eq. (107)] and their growth rate, as well as their real frequency, is smaller than 1 in the present units [Eq. (90)]. It is now easy to find marginally stable modes to Eq. (97). These are obtained putting $f = 0$, and we can find bounded solutions for $K^2 < 1$, namely:

$$u_0 = b_+ e^{i\mu x'} + b_- e^{-i\mu x'}, \tag{114}$$

where

$$\mu = \sqrt{1 - K^2} \tag{115}$$

is recognized as being the Floquet exponent for this particular case, while b_+ and b_- play the role of $P(\pm x)$. This does not by itself prove that the state is unstable. To show this we have to show that truly unstable modes exist in the vicinity of the marginal one. The classical demonstration (Drazin and Reid, 1981) cannot be adapted in this case, so we shall derive a special proof of this result, which takes advantage of the periodicity of the coefficients of the equation. We recall that modes are labelled by K and by λ, and that the condition for λ to be real defines a dispersion relation between (f, K, λ). We know that the set of numbers $f=0, K, \lambda=\sqrt{1-K^2}$ satisfies this relation. We shall then take, at constant K, a neighbouring value of λ and see what the corresponding value of f is : or, conversely, we can vary f and seek which of those variations are related to changes in λ by a real value. This requires that we have some means to calculate the Floquet exponent. One such technique is as follows. Consider a particular solution $y(x')$ of (97) which can be expanded as:

$$\begin{aligned} y(x') &= \alpha F_+(x') + \beta F_-(x') \\ &= \alpha P(x') e^{i\lambda x'} + \beta P(-x') e^{-i\lambda x'}, \end{aligned} \tag{116}$$

where $F_+(x')$ and $F_-(x')$ are defined by Eqs. (99) and (100). Its derivative is:

$$\begin{aligned} y'(x') &= \alpha(i\lambda P(x') + P'(x')) e^{i\lambda x'} \\ &\quad - \beta(i\lambda P(-x') + P'(-x')) e^{-i\lambda x'} \end{aligned} \tag{117}$$

and we calculate:

$$\left.\begin{aligned} y(0) &= (\alpha+\beta) P(0) \\ y'(0) &= (\alpha-\beta)\ (i\lambda P(0) + P'(0)) \\ y(2\pi) &= \alpha e^{2i\pi\lambda} P(0) + \beta e^{-2i\pi\lambda} P(0) \\ y'(2\pi) &= (\alpha e^{2i\pi\lambda} - \beta e^{-2i\pi\lambda})\ (i\lambda P(0) + P'(0)), \end{aligned}\right\} \tag{118}$$

where 2π-periodicity of $P(x)$ has been used. Now consider that particular solution which has $y(0)=1$ and $y'(0)=0$; let us call it y_1. For that solution $\alpha=\beta=1/2$, and we have:

$$y_1(2\pi) = \cos 2\pi\lambda. \tag{119}$$

Equation (119) may therefore be regarded as an implicit form of the dispersion relation, because, being the solution of Eq. (97), $y_1(2\pi)$ implicitely depends on K and f.

Let us now consider growing modes, i.e. bounded solutions of Eq. (97) with f not real, but small. It is straightforward to convert Eq. (97) into the more convenient form:

$$u'' + \mu^2 u + f \frac{\cos x' - f^*}{|\cos x' - f|^2} u = 0. \tag{120}$$

The third term in this equation is to be regarded as a perturbation. Unfortunately it exhibits a rather singular behaviour when $\mathrm{Im}\, f \to 0$. Let us write:

$$f = f_1 + i f_2. \tag{121}$$

Then (120) can be written explicitely:

$$u'' + \mu^2 u = -f \left\{ \frac{\cos x' - f_1}{(\cos x' - f_1)^2 + f_2^2} + i \frac{f_2}{(\cos x' - f_1)^2 + f_2^2} \right\} u, \tag{122}$$

where the functions in the curly bracket approach $\mathscr{P}(1/\cos x')$ and $\delta(\cos x')$ respectively as f_1 and $f_2 \to 0$. It is then easier to deal with these terms by converting (122) into an integral equation. The general solution of

$$u'' + \mu^2 u = s(x') \tag{123}$$

is

$$\begin{aligned} u &= b_+ e^{i\mu x'} + b_- e^{-i\mu x'} \\ &\quad + \int_0^{x'} \exp(i\mu(x'-x''))\, dx'' \int_0^{x''} dx''' \exp(-i\mu(x''-x''')) \, s(x''') \end{aligned} \tag{124}$$

and so (122) can be converted to:

$$\begin{aligned} u(x') &= b_+ e^{i\mu x'} + b_- e^{-i\mu x'} - f \int_0^{x'} \exp(i\mu(x'-x''))\, dx'' \\ &\quad \cdot \int_0^{x''} dx''' \exp(-i\mu(x''-x''')) u(x''') \frac{\cos x''' - f_1 + i f_2}{(\cos x''' - f_1)^2 + f_2^2}. \end{aligned} \tag{125}$$

This relation, which is essentially exact, may be used to calculate λ from the dispersion relation Eq. (109). Actually, the function y_1 which appears in this relation must have $u(0)=1$ and $u'(0)=0$. It is a trivial matter to show that this corresponds to $b_+ = b_- = 1/2$ in (124). Then we obtain the integral relation:

$$\begin{aligned} y_1(2\pi) &= \cos 2\pi\mu - f \int_0^{2\pi} \exp(i\mu(2\pi - x'))\, dx' \\ &\quad \cdot \int_0^{x'} dx'' \exp(-i\mu(x'-x'')) y_1(x'') \frac{(\cos x'' - f_1) + i f_2}{(\cos x'' - f_1)^2 + f_2^2}. \end{aligned} \tag{126}$$

When f is small, we can iterate to first order in f using Eq. (125) (with $b_\pm = 1/2$) to obtain the approximate dispersion relation Eq. (119) in the form:

$$\begin{aligned} \cos 2\pi\lambda &= \cos 2\pi\mu - f e^{2i\pi\mu} \int_0^{2\pi} dx' e^{-i\mu x'} \\ &\quad \cdot \int_0^{x'} dx'' \exp(-i\mu(x'-x'')) \cos \mu x'' \frac{(\cos x'' - f_1) + i f_2}{(\cos x'' - f_1)^2 + f_2^2}. \end{aligned} \tag{127}$$

The factor of f in the second term is, in the limit $f_1, f_2 \to 0$, a perfectly well defined complex number, Z_0, which can be calculated as:

$$Z_0 = \frac{1}{2\mu} \mathscr{P} \oint dx'' \frac{\sin 2\mu(\pi - x'')}{\cos x''} + \frac{i\pi}{\mu} \operatorname{sgn}(f_2) \sin 2\mu\pi, \tag{128}$$

where $\mathscr{P}$ denotes the principal part of the integral. In the present case, parity considerations show that it vanishes. Therefore,

$$Z_0 = \frac{i\pi}{\mu} \operatorname{sgn}(f_2) \sin 2\mu\pi \tag{129}$$

and for small f, the dispersion relation Eq. (119), approximated by Eq. (127), gives after separating real and imaginary parts:

$$\begin{aligned} & f_1 \simeq 0 \\ & |f_2| = \frac{(\cos 2\pi\lambda - \cos 2\pi\mu)\mu}{\pi \sin 2\pi\mu} \simeq \frac{\mu(\mu-\lambda)}{\pi}. \end{aligned} \tag{130}$$

This shows that growing modes exist. They have $\lambda < \mu$, and the real part of the frequency vanishes near $\lambda = \mu$, presumably as $(\lambda - \mu)^2$.

To complete this study, we should calculate the actual growth rate of the instability and its maximum value in particular. This requires some numerical work and is left for a future paper. Actually, we can guess from the Howard semi-circle theorem that $\mathrm{Im}\, f$ should be a number ξ_1 smaller than one, but probably not

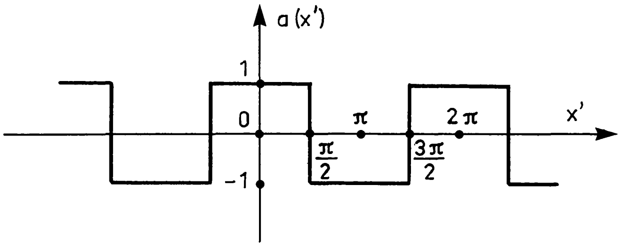

Fig. 3. A square wave velocity profile

much smaller. Hence, in dimensional form, we can write:

$$\operatorname{Im} \gamma = \xi_1 A k v_{A_0}. \tag{131}$$

In order to support this argument, and to complement this study further, we here present a calculation of the growth rate of the instability in the much easier case when the velocity profile, instead of being a cosine, is taken as a succession of positive and negative squares of period 2π as shown in Fig. 3. This simple model has the extreme advantage of lending itself to analytical calculation. The result, as will now be shown, is that the limit put by Howard semi-circle theorem is actually reached in that case : roots of the dispersion relation actually lie on that circle.

The procedure adopted is as follows. We want to solve the dispersion relation (119) for real λ and K and complex f. Therefore we need to solve the (periodic) Rayleigh equation (198) for y, with the profile $a(x)$ shown in Fig. 3 with the initial conditions appropriate to the function y_1 [i.e. $\alpha=\beta=1/2$ in Eq. (116)]:

$$\left.\begin{aligned} y_1(0)&=1 \\ y_1'(0)&=0. \end{aligned}\right\} \tag{132}$$

As the velocity profile $a(x)$ is discontinuous, jump conditions must be derived, which are just the ones used in the study of stability of vortex sheet (see Drazin and Reid, 1981, for example, for details). These are:

$$\left.\begin{aligned} &(f-a(x))\frac{du}{dx}+u\frac{da}{dx} && \text{continuous} \\ &\frac{u}{f-a(x)} && \text{continuous.} \end{aligned}\right. \tag{133}$$

In each part of the square profile, the solution of Eq. (98) is trivial and can be written:

$$u=A_i e^{Kx}+B_i e^{-Kx}, \tag{134}$$

where A_i and B_i are constants which differ on each interval and communicate through the jump conditions (133). Therefore, we can solve piecewise (98) on the three successive intervals $[0, \pi/2]$, $[\pi/2, 3\pi/2]$, $[3\pi/2, 2\pi]$, starting with $A_1=B_1=1/2$ to satisfy the conditions (132). We find the values (A_2, B_2) appropriate to $[\pi/2, 3\pi/2]$, and (A_3, B_3), which refer to $[3\pi/2, 2\pi]$. Once A_3 and B_3 are known $y_1(2\pi)$ can be calculated, and the dispersion relation (119) can be written down explicitly. A little algebra gives:

$$\cos 2\pi\lambda = \mathrm{ch}^2 K\pi + \mathrm{sh}^2 K\pi \frac{f^4+6f^2+1}{f^4-2f^2+1}, \tag{135}$$

which readily shows that no solution is to be found with real f at all. This biquadratic equation can be solved for f, and gives the result:

$$\left.\begin{aligned} f&=\pm i \exp\left(\pm i\frac{\Psi}{2}\right) \\ \Psi&=\arccos\frac{2\,\mathrm{sh}^2 K\pi-(1-\cos 2\pi\lambda)}{2\,\mathrm{sh}^2 K\pi+(1-\cos 2\pi\lambda)}. \end{aligned}\right. \tag{136}$$

For a given value of K, Im f is between 1 (the value appropriate to a vortex sheet) and th $(K\pi)$. As is seen from (136), the roots are on the Howard circle (which in this case is given by $|f|=1$). Another noteworthy feature is that all λ's differing by an integer give modes with the same growth rate, but not the same spatial structure. This corrponds to the well-known band structure found in solid state physics, except that here the bands are not separated by gaps in f values, but join smoothly. There is a difference between this case and that of the cosine profile, which results from the fact that the square profile is unstable near $\pi/2$ and $3\pi/2$, where it locally reduces to a step function while the cosine profile is locally stable. Therefore, the square profile is most unstable for large K's, whereas the cosine profile is unstable only for $K<1$.

VIII. Tearing mode instability at velocity nodes of a standing wave

One possible way to enhance the dissipation would be if neighbouring magnetic surfaces were to reconnect in a time shorter, or of the order of, the wave period. In situations when phase mixing is highly developed the time scale ω^{-1} (or P) is much larger than the Alfvén crossing time through the smallest dimension in x. It is therefore conceivable that tearing could proceed on a time scale faster than P.

The study of this equation should take full account of the periodic structure of the magnetic configuration subject to the instability. This may vary the tearing growth rate by quite important factors (Cross and Van Hoven, 1971; Bobrova and Syrovatsky, 1979). Also, the flow field of the basic state should be considered in its full generality. Among the three particular geometries that we decided to consider [Eqs. (78), (80), and (81)] the one studied in Sect. VII which was found to be Kelvin-Helmholtz unstable [Eq. (81)] contains no current and so is not subject to tearing instability.

The magnetic configuration in a propagating wave has been schematized by Eq. (78). We have seen in Sect. VI that it is stable to perfect MHD perturbations. This would imply that no marginally stable state exists at all, and hence, that unstable tearing mode cannot be found. This is because the magnetic configuration exterior to the resonant layer of a tearing mode instability, is quasistatic since the instability is a slow process, and hence should correspond to an ideally marginally stable perturbation. This argument should certainly be examined in more detail, but it does at least indicate that the appearance of tearing in propagating waves is unlikely.

The configuration (81) describes schematically the vicinity of velocity nodes (and field antinodes) in the pattern of a standing Alfvén wave. A detailed investigation of the instability in such a state has already been published by Bobrova and Syrovatsky (1979). Their work actually concerns plane force free fields. But, as the value of $\boldsymbol{k}\cdot\boldsymbol{B}$ is the main quantity which matters in this problem it is applicable here as it stands. The procedure is quite similar to the one described in the study of the Kelvin-Helmholtz instability of the square periodic profile (Sect. VII). It turns out that the equation for the so-called exterior solution, i.e. between resonant surfaces, turns out to be solvable in simple terms. On the other hand, the logarithmic derivative jump can just be taken from the well-known paper by Fürth et al. (1963), because it results from a local analysis. These jump conditions allow us to communicate between successive perfect MHD layers through the resonant surfaces. As this problem is also periodic, the dispersion relation can again be written in the form (119); simply the Fürth et al. (1963) jump relations have to be used when crossing a resonant layer.

This gives, with our notation, the following value for the growth rate:

$$(\tau_R \operatorname{Im} \gamma) = \left[\frac{(\cos 2\pi\lambda - \cos 2\pi\sqrt{1-K^2})\sqrt{K(1-K^2)}\,\Gamma(1/4)}{\pi \sin(2\pi\sqrt{1-K^2})\,\Gamma(3/4)}\right]^{4/5} S^{2/5}, \tag{137}$$

where

$$\tau_R = \mu_0 \sigma q^{-2} \tag{138}$$

is the resistive time scale and

$$S = \frac{\tau_R(q^{-1})}{\tau_A(q^{-1})} = \frac{\mu_0 \sigma q^{-1}}{A v_{A_0}} \tag{139}$$

is the usual parameter calculated here for a characteristic length q^{-1} and the maximum shear field amplitude. Bobrova and Syrovatsky also found that the maximum growth rate occurs for $K \cong S^{-1/7}$ and is of the order of $S^{4/7}$. For smaller K's the "constant ψ" approximation is no longer valid. For our purpose we note that the configuration (81) is tearing unstable, with a growth time of the order of:

$$(\operatorname{Im} \gamma)^{-1} \simeq \tau_A^{4/7} \tau_R^{3/7}. \tag{140}$$

IX. Conditions for instabilities to develop fast enough, and their effect

We have now approximately delineated the conditions under which a shear Alfvén motion turns unstable. Now, it is necessary that this occur on a time scale shorter than one period. In the case of the Kelvin-Helmholtz instability, we can see from Eq. (131) that this would happen if:

$$\xi_1 q A v_{A_0} > \omega = k_\| v_A, \tag{141}$$

where q changes in time or space according to Eqs. (71) or (77). In the case of a propagating wave of fixed frequency, we can then restate the inequality (141) as a condition stating that the distance from the excitation surface should be larger than a critical distance, Λ_{KH}, which from (141) and (71) is found to be:

$$\Lambda_{KH} = \frac{1}{\xi_1 A}\left(\frac{d \operatorname{Log} v_A}{dx}\right)^{-1}_{x=0} \simeq \frac{a}{\xi_1 A}. \tag{142}$$

Note that we assume here that the level $z=0$ is actually an antinode of Alfvénic perturbations so that the phase evolves in such a way that (86) applies. Λ_{KH} can be rather short for A large enough. Remember that A is not restricted to small values only. Note, however, that Λ_{KH} cannot be less than one wavelength, if we assume that photospheric motions do not create by themselves the very fine scales involved, but rather that these result from phase mixing. This shows that the laminar picture loses meaning rather quickly, and momentum transfer could become much more efficient, so reducing drastically the damping length Λ_{lam} estimated in Eq. (20), or the damping time τ_{lam} given in Eq. (50).

It is difficult to give reliable estimates of Λ_{KH}, which depends on essentially two unknown parameters: the inhomogeneity thickness a, and the dimensionless amplitude A. For the sake of estimating a value let us consider the chromosphere, where velocities $A v_{A_0}$ may be of the order of 10 km s^{-1}, judging from spectroscopic microturbulence, and a field of say, 50 Gauss. Then we find $v_A = 350$ km s^{-1} and $A^{-1} = 35$, so that for $a \simeq 100$ km we would have $\Lambda_{KH} \simeq 3500$ km.

In the case of standing waves existing in structures with a definite length, the condition (141) translate into a condition on the phase mixing time, which must become larger than a certain minimum value τ_{KH}

$$\tau_{KH} = \frac{a}{\xi_1 A v_A} = \frac{\Lambda_{KH}}{v_A}.$$

With the figures just quoted, τ_{KH} may be of the order of 10 s. We then reach the interesting conclusion that coronal loops might just find themselves in a state of permanent "Kelvin-Helmholtz turbulence", at least where the gradient of Alfvén velocity is large enough.

The effect of a fully-developed Kelvin-Helmholtz instability will be difficult to analyse in detail. We expected rolls to form in the vicinity of inflexion points in the flow (Fig. 4a). These rolls will have a size (l_E) of the order of q^{-1}, and the velocity will be of the order of $A v_{A_0}$. Many rolls will interact because of the closeness of neighbouring unstable layers, and we may expect turbulent eddies of this size to be set up. We may judge their effect qualitatively by putting an effective viscosity in Eqs. (18) and (25) with a value:

$$\nu_{eff} = l_E v_E \sim A v_A q^{-1}. \tag{144}$$

When condition (141) is met, the corresponding effective Reynolds number [Eq. (18)] becomes:

$$R_{eff} \sim \xi_1 q^2 a^2 \sim \frac{1}{\xi_1 A^2}(k_\|^2 a^2) \tag{145}$$

which involves the ratio of the inhomogeneity thickness to the wavelength squared. The effective Reynolds number may then not be so large, and from Eq. (25) we expect the wave to be damped in a wavelength or so, or if we consider a structure of given size, in a few periods.

Let us now consider the effect of tearing instability in causing reconnection between neighbouring sheets (Fig. 4b). According to the estimate (140) of the growth-rate, the condition for sufficiently rapid development is:

$$\tau_A^{4/7} \tau_R^{3/7} \ll P, \tag{146}$$

where τ_A, τ_R are defined in (137), (139), and P is the period of the wave. The scale length q^{-1} should be again estimated from Eqs. (71) or (77). In the case of a standing wave of given frequency, the condition translates again into a lower bound, Λ_{TM} say, on the altitude z which we estimate using (71) and (146) as:

$$\Lambda_{TM} = \left(\frac{\lambda_\|}{2\pi}\right)\frac{1}{A^{4/10}}\left(\frac{\mu_0 \sigma a^2}{P}\right)^{7/10} S(a)^{-2/5}, \tag{147}$$

where $S(a)$ is the expression (139) with q^{-1} replaced by the inhomogeneity scale a. Numerically we obtain:

$$\Lambda_{TM} = 2.5\ 10^4 A^{-2/5}\left(\frac{\eta_{\rm spitzer}}{\eta_{\rm real}}\right)^{3/10}\left(\frac{P}{100\ {\rm s}}\right)^{3/10}(B_{\rm Gauss})^{3/5} \cdot \left(\frac{n}{10^{10}\ {\rm cm}^{-3}}\right)^{-3/10}\left(\frac{T}{10^6\ {\rm K}}\right)^{9/20}\left(\frac{a}{100\ {\rm km}}\right) {\rm km}. \tag{148}$$

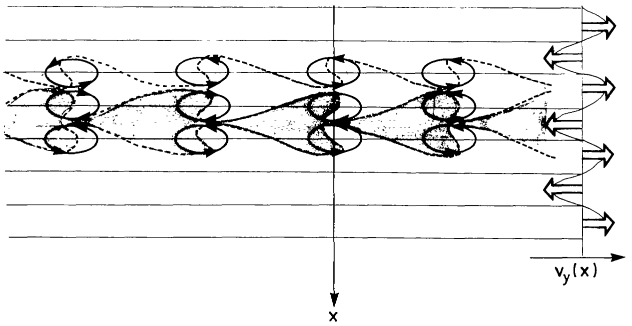

Fig. 4a. A schematic view for the possible geometry of fluid displacement (circular arrows), and interface perturbation (dotted lines) in a Kelvin-Helmholtz instability taking place in a phase-mixed Alfvén wave. The velocity profile in the unperturbed wave is shown

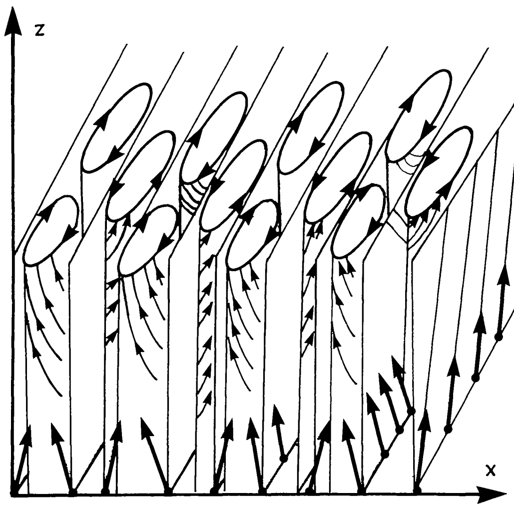

Fig. 4b. A schematic view for the possible geometry of the magnetic field in a tearing instability taking place on the phase-mixed Alfvén wave. The field in the unperturbed wave is shown by heavy arrows on plane $z=0$

It is interesting to note that this length is not extremely large and reduces to even smaller values in the chromosphere ($T=10^4$ K, $n=10^{12}\,\mathrm{cm}^{-3}$, $\Lambda_{\mathrm{TM}}\simeq 800$ km). Even in the corona, it may decrease to small values by the effect of some state of anomalous resistivity. The corresponding time scale, by which a shear Alfvén wave in a finite structure suffers this instability is correspondingly very short:

$$\tau_{\mathrm{TM}}=\frac{\Lambda_{\mathrm{TM}}}{v_A}. \tag{149}$$

X. Conclusions

We here call attention to the importance of phase mixing effects in the dissipation of shear Alfvén waves which are propagating in structures with a transverse gradient of the Alfvén velocity. Unlike magneto-acoustic modes, shear Alfvén waves do not obey a singular propagation equation. They simply propagate independently on each magnetic surface. But precisely because of this independence, the vibrations become more and more out of phase as the propagation proceeds, and friction grows. We have shown that this process is particularly efficient in structures where multiple reflections set up standing waves, for example (but not only) in loops. The idea that a loop picks up the energy in a narrow frequency band, out of the rather broad spectrum of excitation provided by the foot point motions is not new. Nevertheless, we have presented in this paper the first full MHD study of this process, including the aspects brought about by the dissipation and the subtle effect that phase mixing has in promoting the efficient dissipation which we find at the end. As a side product of this study, it has become clear that what is actually important is the standing character of the waves rather than the closed or open configuration in which they propagate. Standing waves can also be found in open structures, and so may represent an efficient heating mechanism for coronal holes. Nevertheless, even propagating waves may suffer damping by phase mixing during their propagation. Though less effective, the damping of reasonably high frequency waves in a magnetic environment with low average Alfvén velocity could be completed in a few thousand kilometers especially if the dissipation coefficients are weakly turbulent rather than classical. Whether this might meet the requirements put by coronal hole heating remains to be debated.

A consistent discussion of dissipation processes cannot by-pass the problem of stability. In a highly phase mixed shear flow, the most likely instabilities are those associated with field and velocity gradients, especially the Kelvin-Helmholtz and the tearing modes. Current driven micro-instabilities seem very unlikely, though it may prove useful to consider more exhaustively all the possibilities. The stability study presented here has been somewhat simplified by treating the problem in a local approximation both in space and time. We nevertheless believe that this is a useful first step. The propagating wave appears to be strictly stable to both kinds of perturbation. For the Kelvin-Helmholtz instability this was expected, and the tearing stability is just a consequence of the non-existence of marginally stable ideal perturbations. By contrast, standing waves are unstable to both modes but at different locations in the wave pattern. Though the posed stability problem may be oversimplified in some respects, we have tried to tackle it with a reasonable degree of precision, by proving theorems of enough generality that the conclusion can be regarded as only weakly dependent on the model. For example, we have discussed the stability of two different velocity profiles (cosine and square). Some of the results presented in their relevant sections appear to be new to the field of fluid dynamics, where flow instability in periodic media has been only occasionally considered, and instabilities in unbounded media have been considered for vanishing perturbations at infinity (Drazin and Reid, 1981). The treatment has required special methods similar to those developed for electron wave mechanics in solids. This is also true of our discussion of tearing instability, a case where an analytical result is available.

On the whole, the stability study shows that phase mixed standing shear Alfvén waves, which should exist in coronal loop structures, are highly subject to instability. We also draw the rather interesting conclusion that any normal loop pervaded by Alfvénic noise has its inhomogeneous parts in a state of permanent tearing and Kelvin-Helmholtz turbulence. No doubt this should be an essential element of the transport properties of these loops as well as representing a source of their heating.

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