A Review of Blagg's Formula in the Light of Recently Discovered Planetary Moons and Rings

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Blagg's formula is a generalized form of Bode's law which, unlike Bode's law, can be equally well applied to the satellite systems.

This paper is a review of Blagg's formula in the light of the newly discovered satellites and rings of Jupiter, Saturn and Uranus. In the case of Uranus, new constants for Blagg's formula are suggested. This revised law allows incorporation of 12 of the 17 new objects discovered since Blagg's original formulation. Finally, the need for a better analytical fit to Blagg's empirical curve and the significance of the law are briefly discussed.

INTRODUCTION

In 1913 Blagg[1] proposed that the relative distances of the planets from the Sun could be represented by a geometric progression of ratio 1·7275 multiplied by a periodic function that represented the deviation from an exact progression. Blagg also found this to be true for the satellite systems of Jupiter, Saturn and Uranus. To be more explicit, she found that the distances of a system of satellites or planets from their primary could be represented by the formula

$$r_n = A(1{\cdot}7275)^n[B + f(\alpha + n\beta)] \qquad (1)$$

Here A and B are numerical constants, α and β are angular constants, f is a periodic function of 2π ranging from 0 to 1. The values of the constants are different for each system as shown in Table I. The parameter n is an integer (positive or negative) in a series representing the number of the object in the sequence outward from the primary. The αs have been increased from the original paper by 2β for the planets and Jupiter and by 4β for Saturn, to be consistent with the form of equation (1). The f function is shown in figure 1.

Table I

System	A	B	α	β
Planets	0·4162	2·025	112°·4	56°·6
Jupiter	0·4523	1·852	113°·0	36°·0
Saturn	3·074	0·0071	118°·0	10°·0
Uranus	2·98	0·0805	125°·7	12°·5

Since Blagg first formulated her 'law' there have been 16, or possibly 17, new objects discovered. The distances of these new objects from their primary should, if the 'law' is valid, agree with the distances predicted by Blagg's formula for some n [equation (1)]. This fact was first pointed out by Roy[2], who, in 1953, successfully compared the distances of all the (six) new bodies discovered up to that point with the distances predicted by the formula. This paper is thus primarily concerned with the 11 new objects discovered since the publication of Roy's paper.

Each of the four systems, planets, Jupiter, Saturn and Uranus, are dealt with in turn and, although no new planets have been discovered since Roy's paper was published, the planetary system has been included for completeness.

PLANETS

Table II shows the good agreement obtained by Blagg's 'law' with the observed distances of the planets. Pluto, the only new planet discovered since the law was first formulated, was dealt with by Roy.

Table II

Planet	*n*	*Distance*	*Blagg's law*
Mercury	−2	0·387	0·387
Venus	−1	0·723	0·723
Earth	0	1·000	1·000
Mars	1	1·524	1·524
Vesta	2	2·361	2·67
Juno		2·670	
Pallas		2·767	
Ceres		2·767	
Jupiter	3	5·203	5·200
Saturn	4	9·546	9·550
Uranus	5	19·20	19·23
Neptune	6	30·07	30·13
Pluto*	7	39·5	41·8

*New object, dealt with by Roy[2].

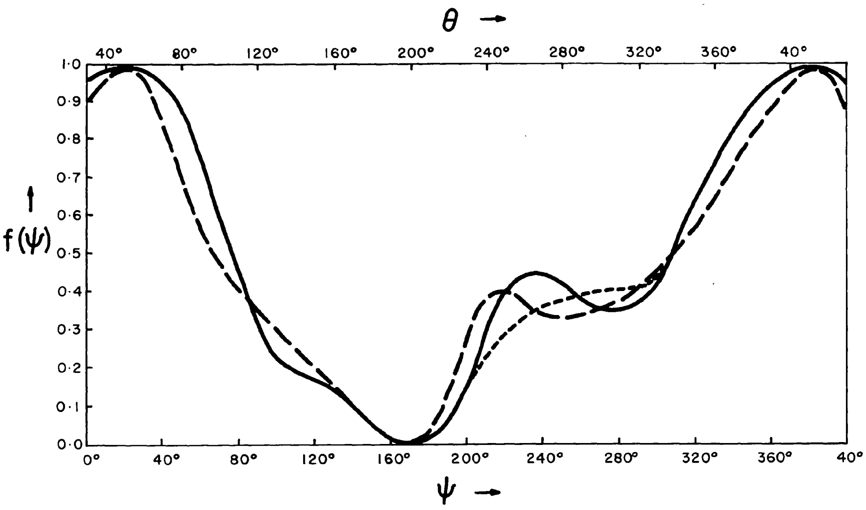

Figure 1. The full line represents Blagg's empirical function $f(\theta)$, where $\theta = \alpha + n\beta$ in equation (1). The dashed line is a representation of the analytic fit $f_N(\psi)$ as defined by equations (2) and (3). Nieto's suggested change to the shape of the empirical curve is shown 'dotted'.

JUPITER

The new satellites of Jupiter dealt with here are J13, 1979 J1[3], 1979 J2[4] and 1979 J3[4]. The rings of Jupiter are also discussed in view of the prediction made by Blagg that such collections of small bodies should occupy positions of relatively steep slope on the f function (figure 1).

As can be seen from Table III, J13, at a distance of 26·3 (Io = 1·0), fits into the $n = 6$ position along with J6, J7 and J10. The remaining three satellites are not so easily dealt with; 1979 J1 and 1979 J3 can be considered to occupy the same orbit at a distance of 0·306. The closest orbit to this, using Blagg's formula, is a rather unsatisfactory one of 0·233 obtained by taking $n = -3$. 1979 J2, at a distance of 0·528, lies between the orbits of J5 and Io. Blagg's formula does produce an unoccupied orbit between these two satellites, but at a distance of 0·718 and not 0·528 as one would expect if the law were valid.

Table III

Satellite	*n*	*Distance*	*Blagg's law*
1979 J1*	−3	0·306	0·233
1979 J3*		0·306	
J5	−2	0·429	0·429
1979 J2*	−1	0·528	0·718
Io	0	1·000	1·000
Europa	1	1·592	1·592
Ganymede	2	2·539	2·541
Callisto	3	4·467	4·467
	4		
	5		
J13*	6	26·30	27·54
J6		27·25	
J7		27·85	
J10†		27·85	
J12†	7	49·8	55·46
J11†		53·3	
J8		55·7	
J9†		56·2	

*New object, dealt with in this paper.
†New object, dealt with by Roy.

Preliminary analysis suggests, however, that by using different constants from those shown in Table I, a more satisfactory fit could be obtained. The new constants used for our preliminary analysis were A = 0·82, B = 1·03, $\alpha = 150°$ and $\beta = 25°$. However, further work is required in this direction before anything like a final result could be claimed, as discussed later. The need to alter the constants in this way casts doubt on the statistical significance of the law, though Blagg herself pointed out that this might be necessary with the discovery of new satellites and also that the details of the f function may have to be altered.

The rings of Jupiter, at a distance of 0·3, lie between the $n = -3$ and $n = -2$ orbits. This corresponds to a value of ~20° on the curve and, as Blagg predicted, the curve is relatively steep here. This is also the case for the revised values $\alpha = 150°$ and $\beta = 25°$ used in our preliminary analysis. In fact, these constants place the rings of Jupiter, Saturn and Uranus on approximately the same part of the curve. It does not seem satisfactory, however, that these do not correspond to an integer n.

SATURN

Six new satellites, all confirmed by *Voyager 1,* are dealt with here. The nomenclature used for these new satellites is that given in the *NASA News Release*[5]. This point is made in view of the fact that many people prefer the nomenclature used for the Jovian satellites in this paper (*e.g.* 1979 J1). Janus is not dealt with here due to the fact that a search, carried out by *Pioneer 11,* found no trace of it.

The six new satellites can be considered, for the purposes of this paper, to occupy three distinct orbits. S12, at a distance of 1·283 (Tethys = 1·0), is in the same orbit as Dione and is, in fact, also known as Dione B. As Dione occupies the $n = 1$ orbit, no work was required here. Of the remaining five satellites, S13, S14 and S15, at distances of 0·478, 0·468 and 0·460 respectively, can be placed around a previously unoccupied orbit corresponding to $n = -3$. Using Blagg's formula in its original form the value obtained for this orbit is $r_{-3} = 0{\cdot}463$ which compares favourably with the observed values.

This value, of 0·463, differs significantly from that of 0·54 obtained by Nieto[6]. However, when Nieto arrived at this value he was comparing it with Janus, which was reported to have a mean orbital radius of 0·538.

Table IV

Satellite	*n*	*Distance*	*Blagg's law*
S15*	−3	0·460	0·463
S14*		0·468	
S13*		0·478	
S10*		0·511	
S11*		0·511	
Mimas	−2	0·630	0·626
Enceladus	−1	0·808	0·807
Tethys	0	1·000	1·000
Dione	1	1·281	1·279
S12*		1·281	
Rhea	2	1·789	1·786
	3		
Titan	4	4·149	4·140
Hyperion	5	5·034	5·023
	6		
	7		
	8		
Iapetus	9	12·09	12·11
Phoebe	10	43·92	43·85

*New object, dealt with in this paper.

The remaining two satellites, S10 and S11, lie at a distance of 0·511 from Saturn. These could be placed in the $n = -3$ orbit, but this produces a 10% error in the distance. Alternatively, new constants may be necessary, to create a new orbit between the present $n = -3$ and $n = -2$ orbits. Again, as in the case of Jupiter, preliminary results show this to be possible.

One final point of interest is the position of the $n = -4$ orbit. Some authors[7,8] have suggested that satellites should form at the outer boundaries of the major rings. The value obtained for $n = -4$ is $r_{-4} = 0{\cdot}309$ which turned out to be on the outside edge of the C ring ($r = 0{\cdot}312$). If a new satellite should be discovered at this distance, it would perhaps confirm Blagg's choice of constants. An alternative solution to the deviations of S10 and S11 from the 'law', would then be required. This possibility is discussed later.

URANUS

Miranda is the only new, confirmed, satellite since the publication of Blagg's formula. There was another new satellite[9] reported in 1977, but this has never been confirmed.

The newly discovered rings[9,10], with a mean radius of 0·239 (Ariel = 1·0), lie on the curve at a position corresponding to $\theta \simeq 67°$. This, as previously mentioned, is on the same region of the curve as Saturn and Jupiter's rings which have corresponding values of ~83° and ~70° (using the preliminary revised constants) respectively.

Miranda has a mean orbital radius of 0·646 and was placed by Roy in the $n = -2$ slot. Nieto[11], however, quotes the distance of Miranda as being 0·678. This produces an error of 6% from the value of 0·64 obtained using the formula. To overcome this error Nieto suggests that with a slight alteration to the curve, as shown dotted in figure 1, and by using the constants $(B, \alpha, \beta) = (1{\cdot}04, 40°, -50°)$, Miranda can be more suitably placed in the $n = -1$ position.

Using the constants suggested by Nieto, the following values of r_n are obtained:

$$\left.\begin{aligned} r_{-1} &= A(1{\cdot}019) = 0{\cdot}502 \\ r_0 &= A(2{\cdot}030) = 1{\cdot}000 \\ r_1 &= A(2{\cdot}937) = 1{\cdot}447 \\ r_2 &= A(4{\cdot}297) = 2{\cdot}117 \\ r_3 &= A(6{\cdot}960) = 3{\cdot}428 \end{aligned}\right\} \quad \text{assuming } A = \frac{1}{2{\cdot}03}$$

Table V

Satellite	*Distance*	*n*	*Blagg's law*	*n*	*This paper*
U6*	0·277	−4	0·33	−2	0·273
		−3			
Miranda†	0·646	−2	0·64	−1	0·648
		−1			
Ariel	1·000	0	1·00	0	1·000
Umbriel	1·391	1	1·393	1	1·387
Titania	2·281	2	2·286	2	2·283
Oberon	3·057	3	3·055	3	3·059

*New object, dealt with in this paper.
†New object, dealt with by Roy.

This total lack of agreement with the observed values leads to the conclusion that either Nieto has been misquoted or he got it wrong.

By constructing a semi-log plot of distance against n, as in figure 2, it would appear that Miranda should indeed be placed in the $n = -1$ position. Note, however, that this semi-log plot also suggests a progression ratio of 1·5 and not 1·7275. This strong suggestion that Miranda should be placed in the $n = -1$ position, led to new constants being calculated. After much trial and error, the best values for these new constants were found to be

$$A = 0{\cdot}537 \qquad \alpha = 0°$$
$$B = 1{\cdot}105 \qquad \beta = -55°.$$

This new value of B is relatively large in comparison with the previous value of B = 0·0805 which, being small, makes the fitting of the satellite positions to Blagg's empirical curve very critical. Thus, by increasing the value of B, we have made the relative distances of the satellites less dependent on the empirical curve.

Note that the value of B for the Saturn satellite system is also very small and therefore makes the distances of the satellites very dependent on the shape of the curve.

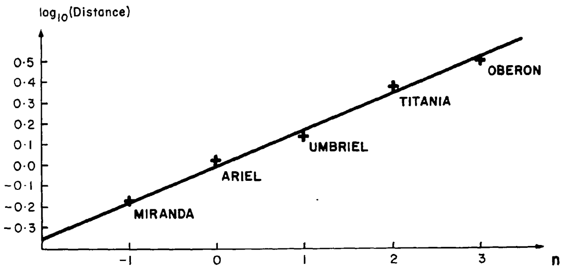

Figure 2. Semi-log plot of distance against n, for the Uranian satellite system, showing that Miranda is more suitably placed in the $n = -1$ position.

Nieto's suggested change to the shape of the curve was retained but not any of his constants. Note, however, that in both cases the value of β is negative and $\sim 50°$. When calculating the values of these constants, the distance of the new satellite, which shall be referred to as U6, was not considered owing to the uncertainty of its existence. It was, nevertheless, encouraging to find that its supposed position could then be satisfactorily placed in the new $n = -2$ position.

Table V shows the good agreement between the calculated and observed distances using these new constants. The new constants also eliminate the unoccupied orbits produced by Blagg's choice of constants.

CONCLUSION

As previously mentioned, the method used to find the new constants for Uranus was essentially one of trial and error. This was also the method adopted by Blagg. However, in these modern days it would perhaps be more appropriate to employ the use of a computer to obtain an optimized fit. To do this it would be necessary to obtain an analytic equation representing the Blagg f function. [Derivation of such an analytic form will also make it much easier to test the statistical significance of the law since it defines the number of parameters (degrees of freedom) incorporated in it.] Blagg did attempt this and arrived at the following equation:

$$f(\psi) = \frac{\cos\psi}{(3-\cos 2\psi)} + \frac{1}{[6-4\cos 2(\psi-30°)]} \quad (2)$$

where $\psi = \theta - 27°{\cdot}5$. This curve is unnormalized and needs to be written as

$$f_N(\psi) = 0{\cdot}249 + 0{\cdot}86 f(\psi) \quad (3)$$

to obtain values of $f_N(\psi)$ in the range 0·0 to 1·0. As can be seen from figure 1 this does not quite agree with the empirical form of the curve and, in view of the suggested change to the curve, further work is required in this direction before computer methods could be employed. The need for such computer methods becomes apparent in the cases of Saturn and Jupiter, both of which have large numbers of satellites. This, together with the fact that altering the shape of the curve for any one system may change the values of r_n in another system, makes the trial-and-error method too laborious. When dealing with Uranus, only five satellites were considered and the alteration to the curve did not affect any other system. This made the trial-and-error method feasible.

Of the 17 bodies discovered since Blagg first suggested her 'law', only five (1979 J1–2–3, S10 and S11) do not at present fit the law at all satisfactorily. Three conclusions are then possible:

(*a*) The constants and curve shape require further revision.

(*b*) If such revision allows acceptable fitting of these five wayward objects, without loss of fit for any of the other objects, then the law stands as a convenient description of *distances* of planets, moons and rings. Whether or not the law has any *theoretical* significance can only be tested by a proper statistical test comparing the number of data with the number of degrees of freedom in the law. In doing this one must also consider the existence of any unfilled integer values.

(*c*) If revisions of the law do not allow a fit for these five objects, the law might still be theoretically significant if the wayward objects above are affected by some additional factor. In particular, it may be important that the five objects involved here all lie close to or inside Roche's limit.

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