On the gravitational interaction of two planetesimals

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The dependence of the evolution of the orbits of two gravitationally interacting planetesimals (particles) moving about a massive central body (the sun) as a function of the initial data is investigated using numerical integration of the equations of motion of the plane three-body problem on a computer. Cases when the masses of the planetesimals are equal to each other and cases when the mass of one planetesimal is considerably greater than the other are considered. Special attention is paid to an investigation of cases when the heliocentric orbits of the planetesimals cross in the course of evolution. It is shown that with strong changes in the orbital elements the mean motions of the planetesimals can become commensurable, and these commensurabilities are retained for a rather long time. The results obtained are applied to an investigation of the supply zones of separate planetary embryos. Equations characterizing the motion of particles about triangular libration points are obtained through an analytical investigation of the plane, restricted, circular, three-body problem.

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INTRODUCTION

The problem of the formation of the planets from the circumsolar gas–dust cloud poses the task of investigating the evolution of a ring of gravitationally interacting planetesimals moving about a massive central body (the sun). If the mass of the cloud is small compared with the mass of the central body, then the interaction of two bodies in the field of a third massive body can be treated as an elementary process in a protoplanetary cloud consisting of a large number of bodies. The mutual gravitational influence of two planetesimals, particles, is investigated in the present report. To complete the picture the investigation is carried out over a rather long time interval, although in reality a "pure" interaction between two bodies cannot occur in a protoplanetary cloud and the influence of other bodies must also be taken into account. Cases of relatively strong changes in the orbital elements in the course of evolution, of the emergence of particles into resonance orbitals, and the motion of particles about triangular libration points are discussed. Equations are derived which allow one to estimate the limits of variations in the semimajor axes of the orbits from the initial data without intergrating the equations of motion. The evolution of the orbits of two gravitationally interacting planetesimals, particles, was investigated by numerical integration of the equations of motion of the plane three-body problem, and in certain cases analytically or by approximate methods.

We used the results of the present work to construct an algorithm for modeling the evolution of a ring of gravitating bodies. A description of this algorithm and the results of an investigation of the evolution of rings of bodies corresponding to the supply zones of the actual planets will be given in a later publication. We analyzed the evolution of rings of bodies not corresponding to the supply zones of the actual planets by the method of spheres of action in Ref. 1.

The results of the present work are presented in more detail in Refs. 2–5.

1. THREE TYPES OF EVOLUTION OF THE ORBITS OF TWO GRAVITATIONALLY INTERACTING PLANETESIMALS OR PARTICLES

Let us discuss the results of the investigation of the mutual gravitational influence of two particles moving in the same direction about a central body (the sun), the mass of which considerably exceeds the masses of these particles. The evolution of the orbits was investigated by numerical integration of the equations of motion of the plane three-body problem on a computer. We considered not only the restricted three-body problem but also cases when the masses of the planetesimals or particles were equal to each other ($m_1 = m_2$). The mass of the larger planetesimal was varied from 10^{-9} M⊙ to 10^{-3} M⊙. As these investigations show, for the same values of m_1 ($m_1 \geq m_2$) and initial values of the orbital elements the character of the variations in the orbital elements of particles of equal masses ($m_1 = m_2$) is about the same as that of the corresponding orbital elements of a particle of mass m_2 when $m_2 \ll m_1$, but the limits of the variations of the semimajor axes and eccentricities of the orbits when $m_1 = m_2$ are essentially smaller than when $m_2 \ll m_1$. The masses of the bodies are given in masses of the central body (the sun), the semimajor axes of the orbits in astronomical units, and the angles in radians. The initial value a_1^0 of the semimajor axis of the orbit of the first ($m_1 \geq m_2$) particle was taken as equal to 1 AU. By virtue of similarity laws, the dependence of a_i/a_i^0, e_i, and π_i (a_i, e_i, and π_i are the semimajor axis, eccentricity, and longitude of the pericenter of the orbit of the i-th particle, respectively) on the number N_{rev} of revolutions of the first particle about the central body is not changed if the initial values of the semimajor axes of the orbits are simultaneously increased by k_a times. In the process the time of one revolution increases by only $k_a\sqrt{k_a}$ times.[4] In the variants examined N_{rev} examined N_{rev} reached 25,000.[2,3] In the variants presented in Fig. 1a, b, c, d, N_{rev} equals 2500 and 5000.

The results obtained show[2,3] that the character of the evolution of the orbital elements of two gravitationally interacting particle-planetesimals can be divided arbitrarily into three types:

First type of evolution. Periodic (more precisely, almost periodic) variations in the semimajor axes and eccentricities of the orbits. The heliocentric orbits of the particles cross in the course of evolution. The limits of the variations of the semimajor axes ($\Delta a = \max\{\Delta a_1, \Delta a_2\}$, $\Delta a_i = \max a_i - \min a_i$, $i=1,2$) are relatively small and in the majority of cases are a little larger than the absolute value of the difference between the initial values of the semimajor axes of the orbits, $\Delta a_0 = |a_2^0 - a_1^0|$, in the case when the masses of the particles are equal to each other and a little greater than $2\Delta a_0$ in the case when the mass of one particle is considerably greater than the mass of the of other particle. With an increase in the masses of the particles (with $m_1 : m_2 = \text{const}$) or with a decrease in Δa_0 the ratio $\Delta a/\Delta a_0$ increases. For small initial eccentricities this ratio also grows with an increase in $|\Delta\varphi_0| - \pi/3$, if $\Delta\varphi_0 \in [-\pi, \pi]$, where $\Delta\varphi_0$ is the angle (in radians) at the initial time between the directions toward the particles from the apex at the central body, $\pi = 3.14...$.

In the case of initially circular orbits the limits Δe ($\Delta e = \max\{\Delta e_1, \Delta e_2\}$, $\Delta e_i = \max e_i - \min e_i$) of variation of the eccentricities are determined mainly by short-period (usually on the order of the synodic period of revolution) oscillations. With an increase in the initial eccentricities the values of Δe grow by hundreds of times in comparison with case of initially circular orbits owing to the appearance of long-period oscillations in the eccentricities. The values of Δe for two values of $e_1^0 = e_2^0 = e_0$ are presented in Table I for the case of $m_1 = 10^{-5}$, $m_2 = 10^{-7}$, and $a_2^0 = 1.01$.

The true anomaly of the i-th particle at the initial time is designated as ν_i^0 in Table I. In the variant corresponding to the second row of Table I the period of the long-period oscillations of the eccentricities equal 3000 revolutions of the first particle. In the case of $m_1 = m_2$ the graphs of the time dependence of e_1 and e_2 are close to each other for $e_1^0 = e_2^0 = 0$ (see Fig. 1a). For $e_1^0 = e_2^0 = e_0 \neq 0$, e_1 mainly decreased (in small steps) when e_2 grew (and vice versa) (see Fig. 1b).

For $e_1^0 = e_2^0 = 0$, during the entire evolution $\pi_1 - \pi_2 \approx \pi = 3.14...$ (radians), i.e., the apocenter of one orbit and the pericenter of the other orbit lay along the same ray with the apex at the central body. In the case of nonzero initial eccentricities the longitude of the pericenter π_2 of the second particle (as well as π_1 when $m_1 = m_2$) grew (i.e., moved about the central body in the same direction as the particles themselves) in small steps (see Fig. 1b). The time after which a new step appeared was equal to the half-period of the oscillations of a_2, while π_2 (and π_1 when $m_1 = m_2$) increased by 2π rad over the period of the long-period oscillations of e_2.

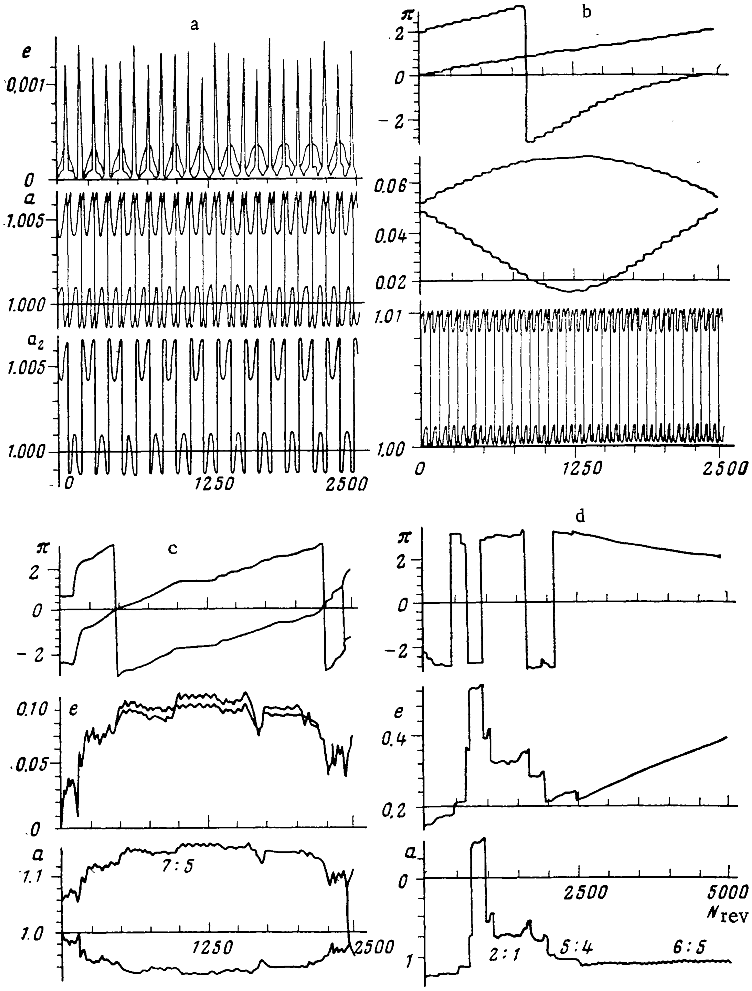

FIG. 1. Dependence of orbital elements of two gravitationally interacting planetesimals, particles, on the number of revolutions N_{rev} of the first planetsimal about the central body: a) $m_1 = m_2 = 10^{-5}$, $a_1^0 = 1$, $a_2^0 = 1.005$, $e_1^0 = e_2^0 = 0$, $\Delta\varphi_0 = 4$; b) $m_1 = m_2 = 10^{-5}$, $a_1^0 = 1$, $a_2^0 = 1.01$, $e_1^0 = e_2^0 = 0.05$, $\pi_1^0 = 0$, $\pi_2^0 = 2$, $\nu_1^0 = 1$, $\nu_2^0 = 0$; c) $m_1 = m_2 = 10^{-5}$, $a_1^0 = 1$, $a_2^0 = 1.05$, $e_1^0 = e_2^0 = 0$, $\Delta\varphi_0 = 4$; d) $m_1 = 10^{-5}$, $m_2 = 10^{-7}$, $a_1^0 = 1$, $a_2^0 = 1.01$, $e_1^0 = e_2^0 = 0.15$, $\pi_1^0 = 1$, $\pi_2^0 = 4$, $\nu_1^0 = 1$, $\nu_2^0 = 2$.

In synodic coordinates (rotating about the central body with an angular velocity equal to the angular velocity of the first particle) the orbit of the second particle does not encompass the central body, and in the case of initially circular heliocentric orbits it has a cresent form (the synodic orbit encompasses one triangular libration point) or a horseshoe from (the orbit encompasses both triangular libration points). Rabe[6–8] made a numerical investigation of such orbits within the framework of the plane, restricted, circular, three-body problem for the earth–moon $\left(\mu\approx 0.01,\ \mu=\frac{m_1}{m_0+m_1}\right)$ and sun–Jupiter ($\mu\approx 0.001$) systems, while Weissman and Wetherill[9] made one for the sun–earth system. All these authors sought the periodic solutions in the form of time series, using the recurrent equations obtained by Rabe[6] to determine the coefficients of these series. We were interested not in periodic solutions per se, but the evolution of the orbits of planetesimals over not very long time intervals with different initial data. The range of values of μ which we investigated was far wider.[2] Besides the restricted problem we also considered the case of $m_1 = m_2$.

The equations presented below were obtained for the restricted, circular, three-body problem, but as the results of numerical investigations show, the conclusions then obtained are basically valid also for the case of $m_1 = m_2$. Let us set $a_1^0 = 1$, $a_2^0 = 1 + \alpha_0$, $m_0 + m_1 = 1$, $m_1 = \mu$, and $\varphi \in (-\pi, \pi)$, where φ is the angle between the directions toward the particles from the apex at the central body and m_0 is the mass of the central body. We will assume that in the course of evolution the orbits of the particles are nearly circular, while the values of φ are not very small. Then, investigating the equations of motion of the particle of infinitely small mass analytically in polar coordinates rotating together with the first particle, one can show[2] that the maximum value of $\alpha(\varphi) = a_2 - 1$ is reached at $|\varphi| \approx \pi/3$ and equals

$$\alpha_* \approx \sqrt{\alpha_0^2 + \frac{8}{3}\mu M(|\varphi_0|)},$$

where $M(\varphi) = (1-\cos\varphi) + [2(1-\cos\varphi)]^{-0.5} - \frac{3}{2}$ $(0<\varphi\leqslant\pi)$. We note that $M(\varphi)\geqslant 0$, $M\left(\frac{\pi}{3}\right)=0$, $M(\pi)=1$. In the course of evolution the values of α and φ satisfy the relation

$$\alpha^2(\varphi) = \alpha_*^2 - \frac{8}{3}\mu M(|\varphi|).$$

As the results of numerical experiments show, for values of α_* less than some value $\alpha_*^{(1)}$ the synodic orbits have a crescent shape, while for α_* greater than some value $\alpha_*^{(2)}$ the evolution of the orbit belongs to a second type which will be discussed below. For $\alpha_*^{(1)} < \alpha_* < \alpha_*^{(2)}$ the synodic orbits have a horseshoe shape. The theoretical values are $\alpha_*^{(1)} \approx 1.6\sqrt{\mu}$, $\alpha_*^{(2)} \approx 0.3 m_1^{0.25}$. In the case of $m_1 = m_2$, or for different values of φ_0, $\alpha_*^{(1)}$ and $\alpha_*^{(2)}$ can differ somewhat from the values given above. The region of α_* corresponding to crescent-shaped orbits is about $[5\mu^{0.25}]^{-1}$ times smaller than the entire region of α_* corresponding to the evolution of orbits of the first type. As $\mu\to 0$ the fraction of crescent-shaped orbits approaches zero (the fraction of horseshoe-shaped orbits approaches 100%), while for $\mu > 10^{-3}$ it is close to unity. In the case of $\mu > \frac{9-\sqrt{69}}{18} \approx 0.04$, as is known, a triangular libration point is unstable in the Lyapunov sense. When $\alpha_* < \alpha_*^{(2)}$ one can find the minimum value of $|\varphi|$ in the course of evolution from the conditions $\frac{3\alpha_*^2}{8\mu} \approx M(\varphi_{min})$ and $0<\varphi_{min}<\frac{\pi}{3}$, while when $\alpha_* < \alpha_*^{(2)}$ one can determine φ_{max} from the conditions $\frac{3\alpha_*^2}{8\mu} \approx M(\varphi_{max})$ and $\frac{\pi}{3}<\varphi_{max}<\pi$. When $\alpha_*^{(1)} < \alpha_* < \alpha_*^{(2)}$ we have $|\alpha_{min}| = |\alpha(\pi)| \approx \sqrt{\alpha_*^2 - \frac{8}{3}\mu}$.

Therefore, in variants with the same values of $\frac{(a_2^0 - a_1^0)^2}{(a_1^0)^2 m_1}$ and $\Delta\varphi_0 = \varphi_2^0 - \varphi_1^0$ the synodic orbits are seen at the same angle from the central body (i.e., the values of φ_{min} and φ_{max} are the same). In this case when $\varphi_{max} < \pi$ the period of the crescent-shaped orbits is proportional to $m_1^{-0.5}$

If $\alpha_* < \alpha_*^{(1)}$ then graphs of the time dependence of the semimajor axes a of the heliocentric orbits are "И"- or "N"-shaped. When $\alpha_*^{(1)} < \alpha_* < \alpha_*^{(2)}$ the graphs of a are "M"-shaped (see Fig. 1a) if $\alpha(\pi)$ differs appreciably from α_* and "П"-shaped if $\alpha(\pi) \approx \alpha_* \approx \alpha_0$. As $\mu\to 0$ the graph of a approaches a "П" shape. The period of the "П"-shaped orbits is close to the synodic period.

In the circular, restricted, three-body problem if the sidereal orbit of the second particle (of infinitely small mass) is almost circular while the distance between the first and second particles is not small then the synodic trajectory of the second particle is close to the zero-velocity line. If the minimum distance between the zero-velocity line and the first particle is small then the relative velocity of the particle of infinitely small mass increases as it approaches the first particle (since the force of gravitational interaction between the particles increases), the motion no longer takes place near the zero-velocity line, and the synodic trajectory of the second particle encompasses the central body (the second type of evolution, which will be discussed below).

With initially eccentric sidereal orbits in the stationary coordinate system the distances of the particles from the central body vary from $a(1-e)$ to $a(1+e)$ during one revolution about the central body. In this case, therefore, the synodic orbits have a more complicated form than with initially circular sidereal orbits.[2,3]

In the case of initially circular heliocentric orbits the minimum distance between the particles exceeds the radius of the Hill sphere by several times. It decreases

TABLE I

$e^0_1 = e^0_2 = e_0$	$\Delta\varphi_0$ or π^0_1, π^0_2, ν^0_1, ν^0_2	Δe
0	$\Delta\varphi_0 = 4$	$2\cdot 10^{-4}$
0.05	$\pi^0_1 = 1$, $\pi^0_2 = 4$, $\nu^0_1 = 1$, $\nu^0_2 = 2$	0.1

with an increase in the initial eccentricities, but it still remains rather large. Therefore, bodies moving along such orbits cannot collide.

The second type of evolution is characterized by relatively strong aperiodic variations in the orbital elements. In this case the minimum distance between the particles may be small. The evolution of orbits of the second type for certain initial data was analyzed earlier in Ref. 10 within the framework of the restricted three-body problem. A number of authors (Dole,[11], Giuli,[12] Kiladze,[13,14] and Kozlov[15] and Éneev) have investigated the relative motion of particles over small time intervals within the framework of the restricted three-body problem. These investigations were conducted mainly to study the formation of the axial rotations of the planets, and the authors were interested in the relative motion of the particles in the case when the distance between particles was small. Variations in the elements of the sidereal orbits were not considered in these reports.

In the analysis of the evolution of initially circular heliocentric orbits through numerical integration of the equations of motion on a computer over sufficiently long time intervals the limits of variation of the semimajor axes and eccentricities were mainly equal to $\Delta a \approx (20\text{–}30) a_1^0 m_1^{0.4}$ and $\Delta e \approx (10\text{–}15) m_1^{0.4}$ (for m_1 from 10^{-9} to 10^{-5}), respectively. This is seen from Fig. 2, in which the quantities Δa and Δe are given for variants with different values of Δa_0, m_1, m_2, and e_0 ($e_1^0 = e_2^0 = e_0$). The variants of Fig. 2 are marked by the following arbitrary designations as a function of the values of m_1 and e_0:

m_1	10^{-3}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	e_0	0.01	0.05	0.15
	1	2	3	4	5	6		7	8	9

Variants in which $m_1 = m_2$ are marked by a bar (10) below the main designation. Cases of $m_2 = 0.01 m_1$ and of $e_0 = 0$ are not additionally marked. In all the variants $\Delta\varphi_0 = 4$ for $e_0 = 0$ and $\pi_1^0 = 1$, $\pi_2^0 = 4$, $\nu_1^0 = 1$, and $\nu_2^0 = 2$ for $e_0 \neq 0$.

When $m_1 \approx 10^{-3}$ the particle of smaller mass can move rather rapidly into a strongly elliptical or hyperbolic orbit. The values of Δa and Δe have a tendency to grow with an increase in the initial eccentricities. For example, in the case of ($m_1 = 10^{-5}$, $m_2 = 10^{-7}$, $e_1^0 = e_2^0 = e_3$) the average (over the variants under consideration) values of Δa and Δe are

e_0	0	0.05	0.15
Δa	$25 a_1^0 m_1^{0.4}$	$40 a_1^0 m_1^{0.4}$	$70 a_1^0 m_1^{0.4}$
Δe	$12 m_1^{0.4}$	$16 m_1^{0.4}$	$25 m_1^{0.4}$

The limits of variation of the orbital elements may be greater than the above-indicated values if very close encounters of the particles took place in the course of evolution.[2] For actual bodies such approaches correspond mainly to collisions of the bodies and can be collisionless only for bodies very distance from the sun.

Strong changes in the orbital elements take place at relatively small distances (on the order of the radius of the Hill sphere) between the particles. At small values of m_1 ($m_1 \le 10^{-7}$) the mutual gravitational influence of the particles at large distances is slight, while the time intervals between relatively close encounters are rather long. Therefore, in these cases graphs of the time dependence of the orbital elements have a stepwise character, i.e., the orbital elements are almost constant for a certain time and then very abruptly.

In the case of sufficiently large initial eccentricities ($e_1^0 = e_2^0$) in the variants under consideration (the method of

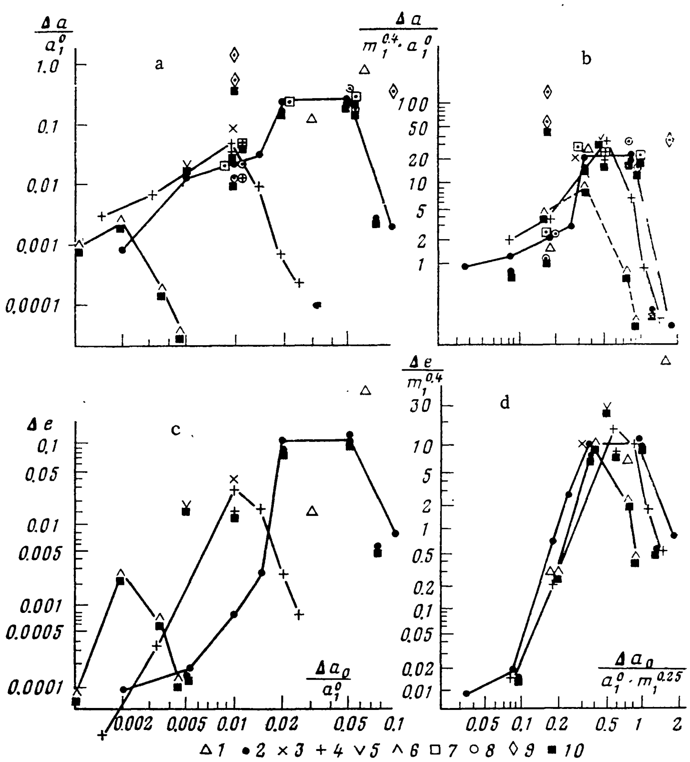

FIG. 2. Some results of numerical integration of the equations of motion of the plane three-body problem (the sun and two planetesimals). Dependence of the limits of variation of the semimajor axes and eccentricities of the orbits of the planetesimals or particles on the masses and initial orbits of the planetesimals.

TABLE II

i	a_i^0	π_i^0	ν_i^0	e_i^0	m_i
1	1.0	1	1	0.05	10^{-8}
2	1.005	4	2		

spheres of action[4,5] was used along with numerical integration of the equations of motion in obtaining the results of the present section) the limits of variation of the semimajor axes and eccentricities of the orbits were determined mainly by the values of the initial eccentricities rather than by the masses of the particles, as in the case of initially circular orbits, if $m_1 : m_2 =$ const and $m_1 \leq 10^{-5}$ (see § 1.6 of Ref. 5). In the case of ($a_1^0 = 1$, $a_2^0 = 1.01$, $e_1^0 = e_2^0 = 0.15$, $\pi_1^0 = 1$, $\pi_2^0 = 4$), for example, the maximum value e_{max} of the orbital eccentricities of the particles was somewhat greater than 0.3, both for $m_1 = m_2 = 10^{-5}$ and for $m_1 = m_2 = 10^{-7}$. In this case the time of evolution increased with a decrease in the masses of the particles. In the variants under consideration[3,4] $e_{max} \approx 2e_0$ for $10^{-9} \leq m_1 = m_2 \leq 10^{-5}$ and $0.05 \leq e_1^0 = e_2^0 \leq 0.15$. For the same values of $m_1 \leq 10^{-5}$ and $e_1^0 = e_2^0 = e_0 > 0$ and other conditions being equal, the values of e_{max} were larger in the case of $m_1 \gg m_2$ than in the case of $m_1 = m_2$. For example, for $e_0 = 0.05$ they reached 0.2, while for $e_0 = 0.15$ they could exceed 0.5.

In the case of particles of equal masses the graphs of the time dependence of e_1 are very similar to the graphs of e_2, while for small values of $m_1 = m_2$ the graphs of e_1 and e_2 practically coincide (see Fig. 1c). Only in the variant with the initial data given in Table II is the graph of e_1 "symmetrical" to the graph of e_2 relative to the straight line $e_0 = 0.05$ (see Fig. 4.7 of Ref. 3). In the variants under consideration[2,3] initially circular orbits and often eccentric orbits also of particles of equal masses became such, in the course of evolution, that the pericenter of one orbit and the apocenter of the other lay along the same ray with the apex at the central body. In this case the longitude of the pericenter π increased almost monotonically in some variants, i.e., it moved in the same direction as the particles themselves (see Fig. 1c). In the case of $m_1 \gg m_2$ and $e_1^0 \neq 0$ the variation in π_1 was small, while π_2 did not vary monotonically. Graphs of the time dependence of the semimajor axes of the orbits of two gravitationally interacting particles are "symmetrical" by virtue of the energy integral $\delta a_1 \approx -\delta a_2 \left(\frac{a_1^0}{a_2^0}\right)^2 \frac{m_2}{m_1}$, if $\delta a_1 = a_1 - a_1^0$, $\delta a_2 = a_2 - a_2^0$, and m_1 and m_2 are small.

In a number of variants the semimajor axes of the orbits oscillate periodically with a small amplitude for a rather long time. For $m_1 \geq 10^{-5}$ such "steady" sections corresponded mainly to cases of commensurabilities between the mean motions of the particles and were observed rather often.[3] With time (for $m_1 = 10^{-5}$ usually after several hundred revolutions of one of the particles about the central body) the resonance relations were disrupted and the particles went out of resonance. In the majority of these resonances ($T_2 : T_1$ = 2 : 1, 12 : 5, 13 : 5, 4 : 5, 7 : 5; T_i is the period of revolution of the i-th particle about the central body) the semimajor axes and eccentricities of the orbits varied periodically with a small amplitude, while the variations of π_i were small. In these cases the motions about the resonances are oscillations about periodic solutions which consist of closed curves in synodic coordinates and have been investigated by many authors (see Chaps. 8 and 9 of Ref. 8, for example). In the 6 : 5 and 5 : 4 resonances in the variant presented in Fig. 1d, the semimajor axis of the orbit of the particle of smaller mass varied slightly, the eccentricity increased almost monotonically, while the longitude of the pericenter decreased. In these cases the curve about which the small oscillations took place was not closed in synodic coordinates. For $m_1 = 10^{-5}$ and $e_1^0 = e_2^0 \neq 0$ the mean motions of the particles were commensurable for more than 200 revolutions of the first particle ($m_1 \geq m_2$) in 6 out of the 10 variants examined.[3] Therefore, bodies (such as comets) whose orbits cross the orbit of some giant planet can be in resonance with this planet for a considerable part of the time if the perturbations from other planets are relatively small. We recall that the body Chiron is in 5 : 3 resonance with Saturn.

The third type of evolution of two gravitationally interacting particles moving about a massive central body is characterized by relatively weak perturbations of the semimajor axes of their orbits. In this case the orbits of the particles do not cross in the course of evolution. In the case of nonresonance orbits the limits of the variations of the semimajor axes and eccentricities of the orbits in the variants under consideration[16] increased with an increase in the initial eccentricities and with a decrease in the absolute difference between the initial values of the semimajor axes of the orbits of the particles. The longitudes of the pericenters of nonresonance orbits varied almost monotonically in the course of evolution if the orbital eccentricities were not small. The evolution of resonance orbits differs strongly from the evolution of nonresonance orbits. In the majority of the resonances examined the limits of variation of the semimajor axes and eccentricities of the orbits were greater than for neighboring nonresonance orbits.[10,16]

2. INFLUENCE OF INITIAL DATA ON THE TYPE OF EVOLUTION

In the case of initially circular orbits when the initial angle $\Delta\varphi_0$ between the directions toward the particles is not small the first type of evolution takes place when $\Delta a_0 = |a_2^0 - a_1^0| \leqslant 0.3\, a_1^0 m_1^{0.25}$, the second when $0.3\, a_1^0 m_1^{0.25} \leqslant \Delta a_0 \leqslant 0.9\, a_1^0 m_1^{0.25}$, and the third when $\Delta a_0 \geqslant 0.9\, a_1^0 m_1^{0.25}$ (see Fig. 2). When $m_1 \approx 10^{-5}$, $0.9 m_1^{0.25} a_1^0 \approx 5 R_{sa}$, where R_{sa} is the radius of the sphere of action of the body of mass m_1. The ratio $a_1^0 m_1^{0.25}/R_{sa}$ increases with a decrease in m_1. The larger m_1, the more strongly the boundary values of Δa_0 depend on $\Delta\varphi_0$ and on the time interval under consideration. In a number of cases almost periodic motions of the first and third types can change with time into motions of the second type. With fixed masses of the bodies and $e_1^0 = e_2^0 = 0$ the region of values of Δa_0 corresponding to the first type of evolution is largest for $|\Delta\varphi_0| \approx \pi/3$ and decrease as $|\Delta\varphi_0| \rightarrow \pi$ and especially as $|\Delta\varphi_0| \rightarrow 0$. Evolution of the first type cannot occur when $\Delta\varphi_0 = 0$. With an increase in the initial eccentricities the region of values of $\Delta\varphi_0$ corresponding to the first type of evolution decreases while the region corresponding to the second type of evolution increases. For example, in the case of $m_1 = 10^{-5}$ with $e_1^0 = e_2^0 = 0.05$ ($a_2^0 =$

1.01) the evolution of the orbits was of the first type while when $e_1^0 = e_2^0 = 0.15$ it is of the second type; when $e_1^0 = e_2^0 = 0$ ($a_2^0 = 1.1$) it is of the third type and when $e_1^0 = e_2^0 = 0.15$ ($a_2^0 = 1.1$) it is of the second type. Dole evidently showed for the first time that initially circular orbits are unstable if the distance between them does not exceed $rm_1^{0.25}$ (Ref. 17, p. 497).

3. INTERACTION OF PLANETESIMALS IN A PROTOPLANETARY CLOUD

From the results obtained above it follows that the maximum eccentricities e_{max} which are acquired by initially circular orbits of small bodies (if their mutual gravitational influence is not taken into account) due to the gravitational influence of a separate massive planetary embryo of mass m moving along a circular orbit are approximately equal to $15m^{0.4}$, i.e., e_{max} is 0.08–0.09 for the earth and Venus and 0.03–0.04 for Mercury and Mars. Assuming that the width of the supply zone of an isolated planetary embryo moving along a circular orbit of radius r_0 does not exceed $\pm 2r_0 e_{max} \approx \pm 30 r_0 m^{0.4}$, [1] we find[3] that in the supply zone of planets of the terrestrial group the number of planets formed should be greater than the actual number of planets (four). The number of planets formed decreases if the orbital eccentricities of the planetary embryos are taken as nonzero and the mutual gravitational influence of the planetesimals and certain other factors are taken into account.

CONCLUSION

An investigation of the mutual gravitational influence of two planetesimals or particles whose orbits cross in the course of evolution allowed us to isolate and study in detail the elementary process lying at the foundation of the complex overall process of evolution of a ring of gravitating bodies.

On the basis of an analysis of a numerical solution of the equations of motion of the plane three-body problem on a computer we obtained equations permitting one to determine the character and in some cases also the limits of variation of the orbital elements of two gravitationally interacting planetesimals from the initial data. Estimates of the sizes of the supply zones of the planets are given.

It was shown that with strong changes in the orbital elements in the course of evolution the mean motions of the planetesimals can become commensurable, and these commensurabilities are retained for a rather long time.

Through an analytical investigation of the plane, restricted, circular, three-body problem we obtained equations characterizing the motion of the particles about the triangular libration points.

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