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A Model of a Thin Accretion Disk Around a Black Hole

by

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ABSTRACT

An "alpha disk" model is presented with a radial heat flow due to accretion flow, and a radial pressure gradient taken into account. Even though our model ignores some important physical effects it removes infinities present in a standard thin disk model at the inner disk boundary at $r_{in} = 3r_g$. Our model has its inner boundary somewhat closer to the black hole, and the radial flow is transonic at that place.

1. Introduction

A theory of thin accretion disks was developed in classical papers by Pringle and Rees (1972), Shakura and Sunyaev (1973), Novikov and Thorne (1973) and Lynden-Bell and Pringle (1974). The relativistic theory was refined by Cunningham (1975, 1976), who allowed for the heating of the disk by radiation comming from other reagions of the disk surface. This effect is possible even for a thin disk because of space curvature close to a black hole.

The conditions at the inner disk boundary depend on the nature of the central object. We shall restrict ourselves to the simplest case when the disk accretes onto a black hole. In all the classical papers the inner disk boundary coincided with the marginally stable orbit of a test particle. For a Schwarzschild black hole this orbit has a radius of $3\,r_g$, where $r_g = 2GM/c^2$ is the gravitational radius. For $r > r_g$ the disk rotation was Keplerian and the radial pressure gradients were neglected. The disk was in a hydrostatic equilibrium. For $r < r_g$ the matter was falling freely into the black hole. It was natural to adopt a no torque boundary condition at the inner disk radius, $r_{in} = r_{ms} = 3r_g$.

Stoeger (1976) pointed out that classical models gave infinite gas density at r_{in}, but he failed to recognize the reason for that phenomenon. A theory of thick accretion disks (Abramowicz *et al.* 1978, Kozłowski *et al.* 1978, Paczyński and Wiita 1980, Jaroszyński *et al.* 1980) helped to understand the conditions at r_{in}. It was found that a thick disk is in a hydrostatic equilibrium, and the radial drift velocity is subsonic for $r > r_{in}$, while gas is freely falling towards the black hole with a supersonic velocity for $r < r_{in}$. Even though the disk may be geometrically thick for $r > r_{in}$ it forms a thin cusp at the transition radius, and flows as a thin supersonic stream for $r < r_{in}$. The transition radius is located between the marginally bound and marginally stable particle orbits. For a Schwarzschild black hole we have

$$2r_g = r_{mb} < r_{in} < r_{ms} = 3r_g. \tag{1}$$

A transonic nature of the radial flow at the inner disk boundary was recently analysed by Kozłowski *et al.* (1978), Jaroszyński *et al.* (1980), Liang and Thomson (1980), Abramowicz and Żurek (1981), and Loska (1981).

Disk accretion is driven by angular momentum transfer due to some viscosity. Unfortunately, the physical nature of this viscosity is not understood. Nevertheless, some general statements are possible. In the "alpha disk" model the parameter α should not exceed unity. That means that viscous processes cannot transfer angular momentum with a supersonic speed. The stream falling towards the black hole cannot communicate back with the disk across the sonic point located at r_{in}. Therefore, we should apply a no torque boundary condition at r_{in}, just like it was done in classical papers. A detailed structure of the cusp region for a thick disk case is studied by Loska (1981). The aim of this paper is to present a simple model of a thin disk in the region close to the cusp. Our model is somewhat refined in comparison with classical one. In particular we take into account small radial pressure gradients, and we allow for the TdS/dr term in the heat balance equation.

2. A Model

The angular momentum balance equation for a steady-state accretion within the "alpha disk" framework may be written as

$$\dot{M}(l-l_{in}) = 4\pi r^2 z\alpha P, \tag{2}$$

where $\dot{M} > 0$ is the accretion rate, $l(r)$ is the specific angular momentum at a radial distance r, l_{in} is the specific angular momentum at the inner disk radius r_{in}, $z(r)$ is the half thickness of the disk, and $P(r)$ is the total pressure in the equatorial plane. We define the dimensionless parameter α with this equation.

The heat balance equation at any point $r > r_{in}$ may be written as

$$Q^+ + Q_{flow} + Q_{rad} = Q^-, \tag{3}$$

where

$$Q^+ = \dot{M}(l-l_{in})\left(\frac{-\mathrm{d}\Omega}{\mathrm{d}r}\right), \tag{4}$$

is the heat input due to viscous processes,

$$Q_{flow} = B_1 \dot{M} T \mathrm{d}S/\mathrm{d}r, \tag{5}$$

is the heat input due to radial accretion flow,

$$Q_{rad} = B_2 4\pi r z \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4acT^3}{3\varkappa\varrho}\frac{\mathrm{d}T}{\mathrm{d}r}\right), \tag{6}$$

is the heat input due to radial diffusion of radiation, and

$$Q^- = 4\pi r F^- = B_3 4\pi r \frac{acT^4}{3\varkappa\varrho z}, \tag{7}$$

is the heat loss due to radiation leaving the disk surface. Various symbols have the following meaning: T, ϱ, S, $\varkappa$ are the temperature, density, specific entropy and opacity, respectively, all specified in the equatorial plane, Ω is the angular velocity, a and c are the standard physical constants, and B_1, B_2, B_3 are dimensionless parameters of the order of unity, that depend on details of vertical disk structure. A standard thin disk theory adopts a local heat balance, *i.e.* the Eq. (3) is written as $Q^+ = Q^-$. In this paper we adopt an approximation $Q_{rad} = 0$, but we take the Q_{flow} term into account.

The equation of hydrostatic equilibrium in the radial direction may be written as

$$\frac{1}{\varrho}\frac{\mathrm{d}P}{\mathrm{d}r} = r(\Omega^2 - \Omega_K^2), \tag{8}$$

where Ω_K is the Keplerian angular velocity. The equation of hydrostatic equilibrium in the vertical direction may be written as

$$z^2 \Omega_K^2 = B_4 \frac{P}{\varrho}, \tag{9}$$

where B_4 is a dimensionless parameter of order unity, that depends on details of vertical disk structure. For a polytropic model with $P = K\varrho^{1+1/n}$ we have

$$B_4 = 2(n+1), \tag{10}$$

(cf. Paczyński 1978).

In a standard thin disk model with a dominant radiation pressure and electron scattering opacity the radiative flux from the disk surface may be calculated as

$$F^- = \frac{c}{\varkappa_e} g_z = \frac{c}{\varkappa_e} \Omega_K^2 z, \tag{11}$$

where g_z is the gravitational acceleration at the surface, and $\varkappa_e$ is the electron scattering opacity. Combining Eqs. (7) and (9) with the equation of state, $P = aT^4/3$, we obtain

$$F^- = \frac{B_3}{B_4} \frac{c}{\varkappa} \Omega_K^2 z \frac{aT^4}{3P}. \tag{12}$$

In case of dominant radiation pressure the Eq. (11) will coincide with Eq. (12) provided $B_3 = B_4$. If the vertical disk structure can be approximated with a polytrope with an index between $n = 1.5$ and $n = 3$ then B_4 is between 5 and 8 (cf. Eq. 10). For these reasons we adopt

$$B_3 = B_4 = B = 6, \tag{13}$$

throughout this paper.

As our model is rather crude we decided to use a simple pseudo-Newtonian potential to simulate a black hole (cf. Paczyński and Wiita 1980) rather than full general relativity. This potential is given as

$$\Phi = -\frac{GM}{r - r_g}, \tag{14}$$

where M is the black hole mass. The Keplerian rotational velocity, angular velocity and angular momentum are given as

$$v_K = \left(r \frac{\mathrm{d}\Psi}{\mathrm{d}r}\right)^{1/2} = \left(\frac{GM}{r}\right)^{1/2} \frac{r}{(r - r_g)},$$

$$\Omega_K = \frac{v_K}{r} = \left(\frac{GM}{r^3}\right)^{1/2} \frac{r}{(r-r_g)}, \tag{15}$$

$$l_K = v_K r = (GMr)^{1/2} \frac{r}{(r-r_g)}.$$

We used the following equation of state and opacities

$$P = P_g + P_r = \frac{k}{\mu H} \varrho T + \frac{a}{3} T^4 = 1.34 \times 10^8 \varrho T + 2.523 \times 10^{-15} T^4,$$

$$\varkappa = 0.34(1 + 6 \times 10^{24} \varrho T^{-3,5}), \tag{16}$$

(all in c.g.s. units), which are good approximations for a normal chemical composition: $X = 0.7$, $Y = 0.27$, $Z = 0.03$. Our basic thermodynamic functions were $\log T$ and $\log \varrho$. We have

$$\mathrm{d}\ln P = (4-3\beta)\,\mathrm{d}\ln T + \beta\,\mathrm{d}\ln \varrho,$$

$$T\mathrm{d}S = \frac{P}{\varrho}[(12-10.5\beta)\,\mathrm{d}\ln T - (4-3\beta)\,\mathrm{d}\ln \varrho], \tag{17}$$

where

$$\beta = \frac{P_g}{P}. \tag{18}$$

Our third unknown function was

$$y = \ln \frac{\Omega}{\Omega_K}, \tag{19}$$

i.e. y was a measure of non-Keplerian rotation.

As we neglected the term Q_{rad} in Eq. (3) our disk structure was described with three coupled equations

$$\dot{M}(l - l_{in}) = 4\pi r^2 z \alpha P,$$

$$\dot{M}\left[(l-l_{in})\left(-\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) + B_1 T \frac{\mathrm{d}S}{\mathrm{d}r}\right] = B \frac{4\pi r a c T^4}{3\varkappa \varrho z}, \tag{20}$$

$$\frac{\mathrm{d}\ln P}{\mathrm{d}r} = \frac{\varrho r}{P} \Omega_K^2 (e^{2y} - 1),$$

with $B = 6$, $B_1 = 0.5$, and

$$l_{in} = l_{in,K} = (GMr_{in})^{1/2} \frac{r_{in}}{(r_{in} - r_g)},$$

$$l = e^y (GMr)^{1/2} \frac{r}{(r-r_g)}, \tag{21}$$

$$-\frac{d\Omega}{dr} = e^y \left(\frac{GM}{r^3}\right)^{1/2} \frac{r}{(r-r_g)} \left[\frac{(3r-r_g)}{2r(r-r_g)} - \frac{dy}{dr}\right],$$

where r_{in} is the inner disk radius.

At very large radii, $r \gg r_{in}$, we could adopt the standard assumptions: the disk was Keplerian, *i.e.* $y = 0$, and the heat balance was local, *i.e.* $\dot{M} dS/dr = 0$. In that case our three Eqs. (20) were all algebraic, and could be solved if the parameters M, $\dot{M}$, α and r_{in} were specified. We did not require $r_{in} = 3r_g$, but rather we treated r_{in} as an eigenvalue of the problem, and allowed it to be less than $3r_g$. Therefore it was necessary to allow for a non-Keplerian rotation and for a radial pressure gradient near the inner disk radius.

3. Computations

The disk model was specified with the values of black hole mass M, accretion rate $\dot{M}$, and the parameter α. A guess of r_{in} had to be made. The integrations were started at $r_{out} \gg r_{in}$. At the starting point the disk was assumed to be Keplerian, and the heat balance was assumed to be local. Therefore, the derivatives dy/dr, $d\ln T/dr$, $d\ln\varrho/dr$ were neglected, as well as $y = 0$ was adopted. The set of three Eqs. (20) was algebraic and could be solved at $r = r_{out}$. Later the Eqs. (20) were integrated towards $r = r_{in}$. This turned out to be difficult. Various integration schemes were tried out, and finally one worked. An integration step was taken as

$$\Delta r = (r - r_{in}) \cdot f, \tag{22}$$

with a constant $f = 0.02$. All the functions which appear in the Eqs. (20) were evaluated as mean of their values at r and $r - \Delta r$. As the equations were nonlinear a number of iterations were required at every step. The independent variable was r, while y, $\log T$ and $\log\varrho$ were calculated. To stabilize the solutions it was necessary to put $dy/dr = 0$ in the equation of heat balance in the expression for $d\Omega/dr$ (cf. Eq. 21), while allowing y to vary with r in all other terms and equations. Obviously, this approach introduced some error into our solutions.

As an example we present the results for

$$\begin{aligned} M &= 10 M_\odot = 1.99 \times 10^{34}\,\mathrm{g}, \\ \dot{M} &= \dot{M}_{cr} = \frac{64\pi GM}{c\varkappa_e} = 2.62 \times 10^{19}\,\mathrm{g/s}, \\ \alpha &= 0.001, \end{aligned} \tag{23}$$

with $\varkappa_e = 0.2(1+X) = 0.34$. The integrations were started at $r_{out} = 100 r_g$. The results of three integrations, with $r_{in}/r_g = 2.880$, 2.885, and 2.890 are shown in Fig. 1. The variation of disk thickness with radius and the variation

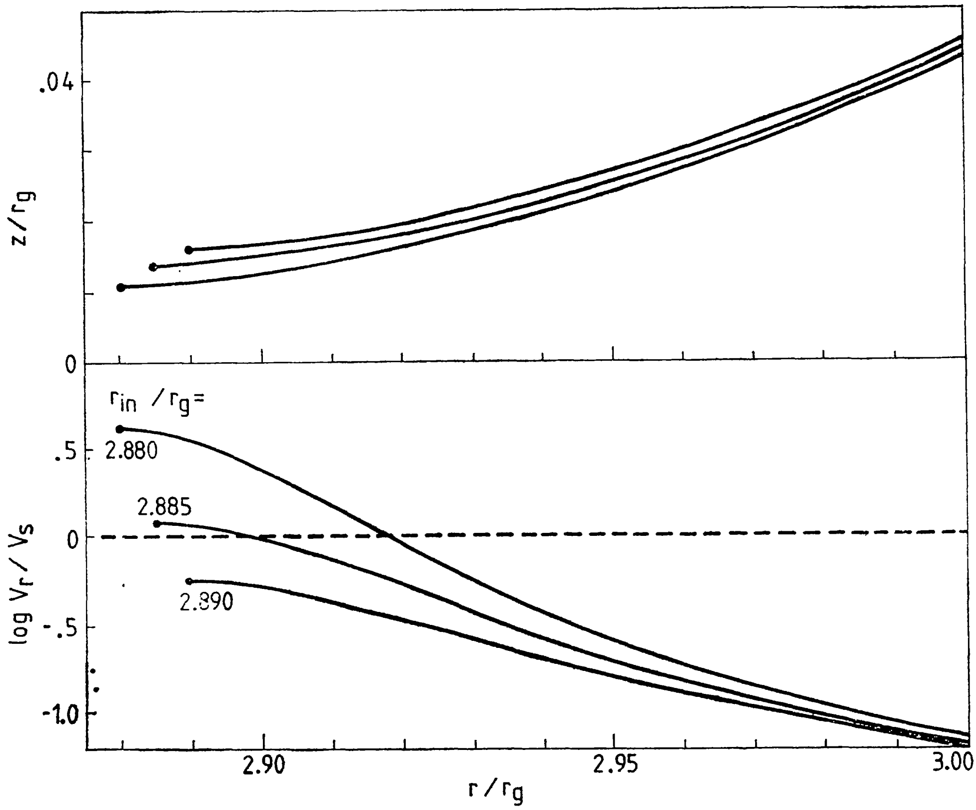

Fig. 1. The variation of disk thickness z, with radius r is shown for a model with a black hole mass $M = 10\, M_\odot$, an accretion rate $\dot{M} = \dot{M}_{cr}$ and $a = 0.001$ for three values of the inner disk radius, $r_{in}/r_g = 2.880$, 2.885, and 2.890. Also shown is the variation of the ratio of radial drift velocity v_r to the speed of sound v_s.

of the radial drift velocity to the speed of sound ratio v_r/v_s with radius is shown. Our equations are not valid when $v_r > v_s$, as we assume that hydrostatic equilibrium holds. Therefore, we cannot decide which of the curves in Fig. 1 describes the correct model. To do this we would have to include the dv_r/dr term and find the transonic solution going through a critical point (cf. Liang and Thomson 1980, Loska 1981). However, we may expect that the correct solution is transonic close to the inner disk radius, *i.e.* we should have $v_r = v_s$ close to $r = r_{in}$. For this reason the middle curve in Fig. 1 is probably the best. It can be readily seen that the solution far from r_{in} is not sensitive to a precise location of the sonic point.

The variation of the disk thickness with radius is shown in Fig. 2 for $1 < r/r_{in} < 7.6$. Our solution (solid line) deviates a little from the solution obtained with a standard thin disk approximation (broken line). The main reason is a large amount of heat carried with an accretion flow to the inner disk region, where the viscous heat input is small, as $l \approx l_{in}$. In our case up to 90 percent of the radiation emitted from the inner disk was generated farther out, as follows from the Q_{flow}/Q^+ curve in Fig. 2. The

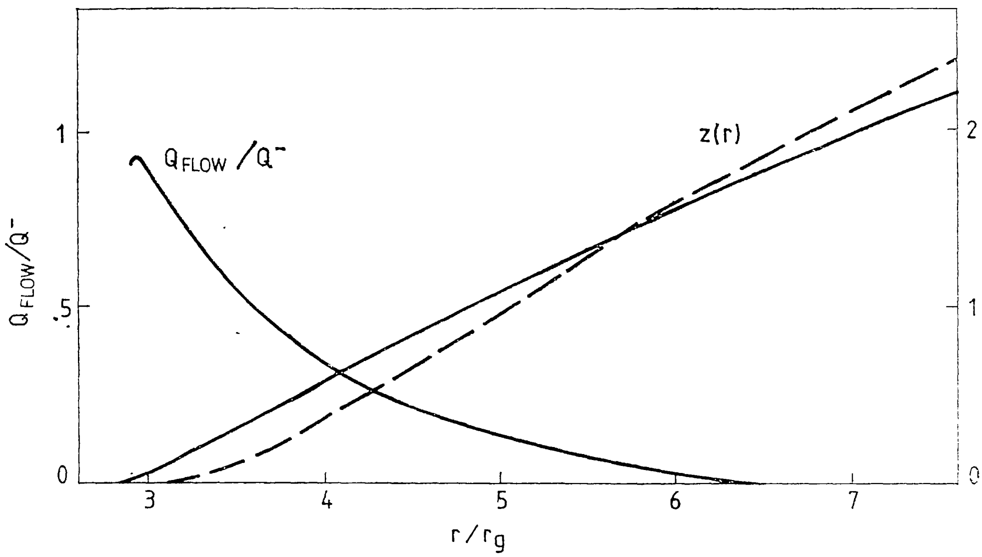

Fig. 2. The variation of disk thickness z and the ratio Q_{flow}/Q^- (cf. eqs. 3,5, 7) with radius r is shown with a solid line for a model with $M = 10\ M_\odot$, $\dot{M} = \dot{M}_{cr}$, $a = 0.001$, $r_{in}/r_g = 2.886$. The variation of disk thickness with radius is also shown with a broken line for a standard thin disk model with $\dot{M} = \dot{M}_{cr}$ and dominant radiation pressure and electron scattering opacity, a local heat balance, Keplerian rotation, and of course $r_{in} = 3r_g$.

other detail of our model that is worth noticing is the location of r_{in}, which is noticeably closer to the black hole than the marginally stable orbit, which is located at $3r_g$ if the pseudo-Newtonian potential is used (cf. Paczyński and Wiita 1980). This effect is possible only because the radial pressure gradient is allowed for.

4. Discussion

Our models did not show any singularity close to r_{in}, unlike the classical thin disk models (Stoeger 1976). This is so because we included two important terms in the disk structure equations: Q_{flow} in the heat balance equation and dP/dr in the hydrostatic equilibrium equation. Unfortunately, our model is not compete even within the "alpha disk" framework. We did not include the dynamical term dv_r/dr, and therefore we could not integrate the disk structure equations through the sonic point all the way into the stream region at $r < r_{in}$. We did not include the radial heat diffusion term Q_{rad} (cf. Eq. 6). In our model shown in Figs. 1 and 2 this term would be close to 10 percent of the Q^{-} term in the inner disk region, *i.e.* it could noticeably affect the solution. Finally, because of stability problems we were forced to calculate the $\mathrm{d}\Omega/\mathrm{d}r$ term as if it was equal to the $\mathrm{d}\Omega_K/\mathrm{d}r$ term. It turned out that the difference between the two terms did not exceed 1 percent level throughout the disk model displayed in Figs. 1 and 2. The difference may be larger in models with supercritical accretion rates, but most likely this approximation is very good for all subcritical disks.

We did not study the stability of our model. In fact no models with $r_{in} < r_{ms}$ were ever tested for stability. This is certainly a serious defect of those models.

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