

Cosmological Gravitational Waves: Their Origin and Consequences*

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Summary. We consider the origin and consequences of a cosmological background of gravitational radiation. Such a background might be *primordial*, in the sense that it goes back to the beginning of the Universe or was produced by quantum processes at the Planck time. Alternatively, it might have been *generated* at some finite time in the past by, for example, the formation of a population of black holes; the gravitational bursts generated in this way would overlap, producing a stochastic background, provided the holes formed at a cosmological redshift ($z_B > 1$) and provided their density (Ω_B in units of the critical density) satisfies the relation $\Omega_B z_B > 0.01 \Omega^2$, where Ω is the total density of the Universe. In most circumstances, cosmological gravitational waves can be described by their density (Ω_g) and a characteristic period (P_0), in which case these parameters uniquely determine their cosmological effects and observational consequences. Primordial waves would have had an important effect on the dynamics of the early Universe even if Ω_g is small and they may have rendered it chaotic. We review the various observational limits on their spectrum: the most stringent comes from cosmological nucleosynthesis considerations and suggests that Ω_g does not exceed the background radiation density $\Omega_{\gamma} \sim 10^{-4}$. Generated gravitational waves, on the other hand, need not be subject to this restriction; even though Ω_g may be large, thus affording a good chance of their detection, they would never significantly affect the dynamics of the Universe. We discuss various scenarios for the generation of a gravitational wave background and find that, in principle, Ω_g could be as large as 10^{-2} over periods in the range 10^{-3} s to 10^5 s.

Key words: cosmology – gravitational waves – black holes

1. Introduction

Much attention has focussed on possible sources for *bursts* of gravitational radiation (such as a supernovae and gravitational collapse) or *continuous* gravitational radiation (such as binaries and pulsars). Strenuous efforts to detect this radiation are already afoot, as described in recent reviews by Thorne (1978), Tyson and Giffard (1978), Epstein and Clark (1979). People have also considered the possibility of detecting a *background* of gravitational

waves. The term “background” has been used in so many different contexts, however, that there is some confusion to what it means. Accordingly, we begin by classifying the possible origins of what one would commonly call backgrounds, as shown in Fig. 1. We will be discussing these origins in more detail later. For the present it is sufficient to point out that the common feature of the associated waves is that they are continuous and, in some sense, *stochastic*. Throughout this paper, therefore, we will regard these two features as the defining characteristics of a “background”. We will not require that a background be isotropic or extragalactic, even though most background may have these properties.

Some backgrounds derive from events occurring at postgalactic epochs, as discussed by Rosi and Zimmerman (1976), but backgrounds may also derive from pregalactic events or even the beginning of the Universe. We refer to gravitational waves which originate at a redshift z exceeding 1 as *cosmological*; if the waves are also continuous and stochastic, we will say that they form a cosmological background. Even some of the backgrounds generated by postgalactic events may be classified as “cosmological”, since the redshift of galaxy formation almost certainly exceeds 1. However, some fraction of these events will usually be *local* ($z < 1$) and the waves generated by each local event will be easier to detect than the background itself. Most of the emphasis of this paper will be on the sorts of backgrounds which are pregalactic in origin and can therefore *only* be detected as cosmological waves.

Cosmological backgrounds are of particular interest because they tend to have characteristics which would make them detectable by the doppler tracking of interplanetary spacecraft (Bertotti and Carr, 1980; Hellings, 1979) or by their effects on the orbits of the moon and the planets (Bertotti, 1973; Mashhoon, 1979; Turner, 1979). Cosmological gravitational waves are also of interest because of the many contexts in which they might originate and because of the wide variety of consequences they might have. In this paper we will discuss these possible origins and consequences.

One can conceive of two sorts of cosmological background. The first possibility is that the background is *primordial* in the sense that it goes all the way back to the big bang and is part of the initial conditions of the Universe. We will also classify a background which derives from graviton production at the Planck epoch as primordial. If the waves have some characteristic (epoch-dependent) wavelength λ , i.e., if they have a wideband spectrum with $(\delta\lambda) \sim \lambda$, they can be described by just two parameters: their present mean energy density, Ω_g in units of the critical density, and their present wavelength λ_0 (or equivalently their present period P_0). These two parameters, which are in principle arbitrary, then determine an “effective” dimensionless amplitude

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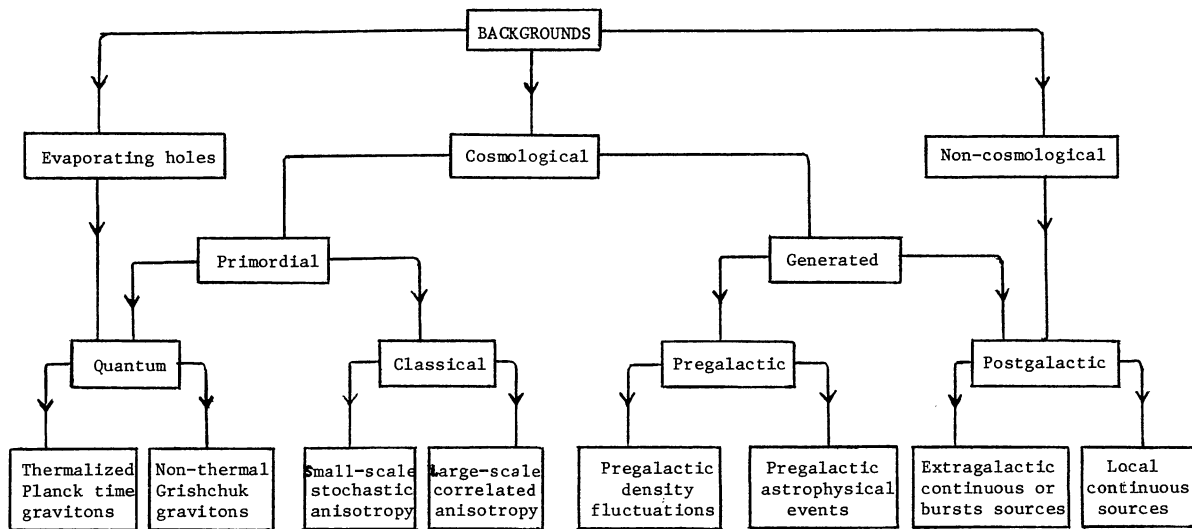


Fig. 1. This classifies the various types of gravitational background radiation. A background is classified as “cosmological” if it derives from a redshift z exceeding 1. It may either have been generated at some finite redshift or it may be “primordial” in that it goes all the way back to the Planck time. The many sorts of background into which the primordial and generated types break down are discussed in the text. There could also exist non-cosmological backgrounds, deriving from that subset of postgalactic events which occur at a redshift less than 1. Only non-cosmological events producing *continuous* radiation can form a background in the sense that one sees a superposition of many sources simultaneously; burst events could not overlap unless at a cosmological distance. The radiation from evaporating black holes could be either cosmological or non-cosmological, depending on the evaporation epoch

for the waves h_0 . If the spectrum is more complicated, a less simplistic description is required but one can still define $\Omega_g(P_0)$ and $h_0(P_0)$ as the density and amplitude of the waves with period around P_0 . We will show in Sect. 2 that a primordial background can be regarded as a stochastic superposition of gravitons, each of amplitude $h_0/\sqrt{N}$ where $N \gg 1$ is determined uniquely by Ω_g and P_0 and represents the number of gravitons per (graviton wavelength)³. The background is stochastic in the sense that the *phases* of these gravitons are assumed to be random and uncorrelated. If, in addition, the *directions* of the gravitons are uncorrelated, the background will be isotropic to within $1/\sqrt{N}$ fluctuations. Even if the gravitons come in “packets” with correlated phase and direction, the background will still be effectively stochastic and isotropic if the number of gravitons per packet is much less than N .

The second possibility is that the cosmological background is formed from the superposition of a lot of individual bursts which were *generated* astrophysically at some finite time in the past. The point here is that, if the waves were generated early enough (necessarily at redshifts larger than 1) and if the events producing them were sufficiently numerous, then the gravitational bursts from the individual events will overlap producing a stochastic background (Rees, 1977; Thorne, 1978) with an effective amplitude h_0 larger than that of the individual bursts by a factor $(\tau/\Delta\tau)^{1/2}$, where τ is the duration of the bursts and $\Delta\tau$ is their separation. If all the events generate waves with the same characteristic period, such a background can also be described in terms of “effective” values for Ω_g , P_0 , and h_0 . However, these values, instead of being arbitrary (as they were for primordial waves), are now specified by the type of process which originally produced the waves. For example, if the waves derive from a population of black holes which have a present density Ω_B and formed at a cosmological redshift z_B , and if each collapsing object converted its rest mass into gravitational radiation with an

efficiency ϵ , then one would have $\Omega_g \sim \epsilon \Omega_B z_B^{-1}$. If the holes have a mass M , one would expect the radiation to have a characteristic period today of order $10 \text{ GM} z_B / c^3$. Since each gravitational burst contains many gravitons, a generated background *necessarily* comes in packets. But provided the packets overlap ($\tau > \Delta\tau$), and provided the source-events do not lie along some preferred direction, the background should be effectively isotropic.

From the point of view of its detectability, the question of whether the cosmological background radiation is primordial or generated is not very important. All one needs to determine is whether the appropriate values of Ω_g and P_0 are observable. Indeed, were one to observe such a background, there might be no simple way of determining whether it was primordial or generated. However, in determining the effects of the background on the history of the Universe, and in assessing what values for Ω_g and P_0 one could reasonably expect, the distinction is an important one. For example, since the density of gravitational radiation generated by black holes cannot exceed the density of the holes themselves, one could never expect generated radiation to dominate the dynamics of the Universe; nor could one expect its effective amplitude h ever to be close to 1. Primordial radiation, on the other hand, will in general *control* the dynamics of the Universe at sufficiently early times: not only will it make the early Universe anisotropic (when its wavelength exceeds the particle horizon size) but also, since h increases as one goes back in time and may become of order 1, it may make the early Universe anisotropy-dominated. Thus, the fact that we can already place limits on the anisotropy of the early Universe during certain epochs already restricts the spectrum of primordial waves. In particular, if one maintains that the cosmic abundances of helium and deuterium result from cosmological nucleosynthesis, one immediately infers that Ω_g probably does not exceed the density of the 3 K background radiation, $\Omega_g \sim 10^{-4}$. (This inference could be avoided only by invoking an unconventional nucleosynthesis scenario.)

However, the same limitation would *not* apply to generated gravitational waves. For these the most interesting limits on Ω_g and P_0 would relate to the independent limits one has on the number of events producing the radiation.

The plan of this paper is as follows: In Sect. 2 we examine the effects a *primordial* gravitational wave background would have had on the dynamical history of the Universe. In Sect. 3 we discuss the possible origins of such a background and in Sect. 4 we review the various observational limits one has on its spectrum. In Sect. 5 we discuss the general features of a *generated* gravitational wave background. In Sect. 6 we review some specific scenarios for black hole formation and discuss the characteristics of the radiation they might generate.

2. The Cosmological Effects of Primordial Gravitational Radiation

Any gravitational wave will induce metric perturbations h_{jk} (“ripples”) in the background spacetime. For a single wave burst one defines the dimensionless amplitude of the wave as

$$h \equiv \frac{1}{2} \left[\frac{1}{4} \sum_{jk} (h_{jk}^{TT \max} - h_{jk}^{TT \min})^2 \right]^{1/2} \quad (2.1)$$

where the sum is over the four transverse-traceless components of the metric perturbation and the subscripts “max” and “min” specify the maximum and minimum values of those components during the passage of the wave (Misner et al., 1973). For a stochastic background such a definition is not appropriate since the perturbation lasts indefinitely. It is more appropriate to define h in terms of the power spectrum associated with h_{jk}^{TT} . One first decomposes h_{jk}^{TT} into its various directional and polarization components, each of which may be characterized as a plane wave with dimensionless amplitude A :

$$h_{jk}^{TT}(x, t) = \int d^3k \tilde{h}_{jk}(\hat{k}, \omega) e^{i\omega(t - \hat{k} \cdot x)} = \sum_{\alpha} \left[\int d\Omega A(\alpha, \hat{k}, u) e_{jk}^{\alpha}(\hat{k}) \right] \quad (2.2)$$

where α and $\hat{k}$ specify polarization and direction, $u = t - \hat{k} \cdot x$, and

$$A(\alpha, \hat{k}, u) = \int d\omega \sum_{jk} [\tilde{h}_{jk}(\hat{k}, \omega) e_{jk}^{\alpha}(\hat{k})] \omega^2 e^{i\omega u}. \quad (2.3)$$

One then takes the *finite* Fourier transform of each A and deduces its associated power spectrum:

$$\tilde{A}(\alpha, \hat{k}, \omega) = \int_{-T/2}^{T/2} A(\alpha, \hat{k}, u) e^{-i\omega u} du, \quad (2.4)$$

$$S_A(\alpha, \hat{k}, \omega) = \lim_{T \rightarrow \infty} \left\langle \frac{2}{T} |\tilde{A}(\alpha, \hat{k}, \omega)|^2 \right\rangle.$$

In terms of this, one can define the dimensionless quantity

$$h(\alpha, \hat{k}, \omega) = [S_A(\alpha, \hat{k}, \omega)]^{1/2}. \quad (2.5)$$

The total amplitude of the stochastic background h , for a given frequency ω , is then the root-mean-square value of this quantity for all α and $\hat{k}$.

This is not the only way of defining h . Alternatively, for example, one could define h in terms of the so-called Stokes parameters (Anile and Breuer, 1974; Mashhoon, 1978), which are convenient for characterizing the polarization state of weak gravitational waves. These parameters just add linearly for a superposition of waves and, for waves which are unpolarized and incoherent in phase, all but one of them are zero. h can then be defined in terms of the non-zero Stokes parameter. The value of h so obtained would be equivalent to the h defined via Eq. (2.5), but not formally the same.

Although the stress-energy carried by a gravitational wave cannot be localized within a wavelength, one can still refer to the effective stress-energy contained in regions much larger than a single wavelength. For a plane wave, in the nearly inertial frame of linearized theory, this smeared out stress-energy tensor (Isaacson, 1968) is

$$T_{\mu\nu}^G = \frac{1}{32\pi G} \left\langle \sum_{jk} (h_{jk,\mu}^{TT} h_{jk,\nu}^{TT}) \right\rangle \quad (2.6)$$

where $\langle \rangle$ denotes some appropriate average. For a background, one must decompose h_{jk}^{TT} into its plane wave components as in Eq. (2.2). The total mean density of the background can then be expressed as

$$\begin{aligned} T_{00}^G &= \int_{-T/2}^{T/2} \frac{dt}{T} \left[\frac{1}{32\pi G} \int_0^\infty d\omega \sum_{\alpha} \int d\Omega \tilde{A}(\alpha, \hat{k}, \omega)^2 \omega^2 \right] \\ &= \int_0^\infty d\omega \left[\frac{1}{32\pi G} \sum_{\alpha} \int d\Omega S_A(\alpha, \hat{k}, \omega) \omega^2 \right], \end{aligned} \quad (2.7)$$

where we have used the stochastic independence of the waves to write $\langle \tilde{A}(\alpha, \hat{k}, \omega) \tilde{A}(\alpha', \hat{k}', \omega') \rangle = \delta(\omega - \omega') \delta(\alpha - \alpha') \delta(\hat{k} - \hat{k}') \tilde{A}(\alpha, \hat{k}, \omega)^2$. The term in square brackets, which we denote as $\varrho(\omega)$, is the *spectral density*. The total density can then be written as

$$\varrho_g = \Omega_g \varrho_c = \int_0^\infty \varrho(\omega) d\omega, \quad (2.8)$$

where $\varrho_c = 3H_0^2/8\pi G$ is the critical density and H_0 is the Hubble parameter. Henceforth we will usually assume that the background is isotropic. In this case the energy flux implied by Eq. (2.6) is zero, the isotropic pressure is $p_g = \varrho_g/3$, and the integration over direction in Eq. (2.7) just gives a 4π factor.

In principle the spectrum of the background radiation could be arbitrary but in many situations it can be characterized by a single frequency ω . For example, if the gravitons have a *thermal* distribution, ω could be taken as the frequency at which the spectrum peaks. In particular, any primordial gravitons existing at around the Planck time $\sim 10^{-43}$ s might be thermalized (although a specific calculation of graviton production by Grishchuk (1977) suggests a non-thermal spectrum). More generally, one might anticipate that the spectrum of a primordial background would be broadband with a bandwidth of order the characteristic frequency. In this case $\varrho_g \sim \omega \varrho(\omega)$ and Eq. (2.6) implies that h , as defined via Eq. (2.4), is related to ϱ_g and ω by the equation

$$\varrho_g \sim \frac{h^2 \omega^2}{8G}. \quad (2.9)$$

In particular, one can write the *present* value of h as

$$h_0 \sim \Omega_g^{1/2} \left(\frac{P_0}{t_0} \right) \quad (2.10)$$

where $P_0 = 2\pi/\omega_0$ characterizes the present period of the wave and $t_0 \sim H_0^{-1}$ is the present age of the Universe $\sim 10^{17.5}$ s. Since the background curvature is ignored by linear theory and since the curvature induced by the gravitational waves $\sim h^2 c^{-2} P^{-2}$ should not exceed the background cosmological curvature, this sort of description is self-consistent only if

$$h \leq \frac{\lambda}{\mathcal{R}} \ll 1. \quad (2.11)$$

Here λ is the graviton wavelength $\sim cP$ and $\mathcal{R}$ is the radius of curvature of the background spacetime, which is of order the

horizon size $\sim ct$. This means that Eq. (2.6) is applicable only if $h \ll 1$ and $\lambda \ll ct$ (or $P \ll t$). However, it does not matter if the background curvature is *dominated* by the gravitational waves. This merely corresponds to equality on the left-hand side of Eq. (2.11).

Equation (2.10) shows that waves characterized by a single frequency may be described by just two parameters: it will be convenient to take these parameters to be the present period P_0 and density Ω_g . If the waves are *not* characterized by a single frequency, a more complicated description is required: one has to specify $h(\omega)$, or equivalently the spectral density $\varrho(\omega) \sim G^{-1} \omega h(\omega)^2$, as functions of ω . But one can still define $\Omega_g(P_0)$ as the density of the gravitons with period around P_0 : $\Omega_g(P_0) \sim \omega \varrho(\omega) / \varrho_c$ with $\omega = 2\pi/P_0$. Assuming that the gravitational waves *can* be characterized by a single period P_0 , amplitude h_0 and density Ω_g , we need to know how these quantities change in a Friedmann Universe as one goes back in time. The exact evolution equations are rather complicated (Lifshitz, 1946) but in the geometric optics approximation ($\lambda \ll R$) one just has

$$P \propto (1+z)^{-1}, \quad h \propto (1+z), \quad \varrho_g \propto (1+z)^4. \quad (2.12)$$

This is because, in such an approximation, the waves behave like a stream of zero-rest-mass gravitons and the density of a zero-rest-mass field always goes like z^4 . However, Eq. (2.12) may fail at times sufficiently early that λ exceeds ct .

Another parameter associated with a primordial background is the graviton number density $\sim \varrho_g / h\omega$ or, equivalently, the separation between gravitons $\sim (\varrho_g / h\omega)^{-1/3}$. This separation is less than the wavelength associated with the gravitons provided h exceeds ω / ω_{pl} , where

$$\omega_{pl} \sim \left(\frac{G\hbar}{c^5} \right)^{-1/2} \sim 10^{43} \text{ Hz} \quad (2.13)$$

is the Planck frequency. We will show later that h is comparable to ω / ω_{pl} only if the gravitons originate at the Planck time $t_{pl} \sim \omega_{pl}^{-1} \sim 10^{-43}$ s and are thermalized. Otherwise the condition $h > \omega / \omega_{pl}$ is easily satisfied, so the gravitons in a primordial background can be assumed to *overlap*. An important role is played by the overlap factor

$$N \sim \left(\frac{\hbar \omega_{pl}}{\omega} \right)^2 \sim \Omega_g \left(\frac{P_0}{t_{pl}} \right)^2 \left(\frac{P_0}{t_0} \right)^2 \sim 10^{120} \Omega_g \left(\frac{P_0}{t_0} \right)^4, \quad (2.14)$$

which represents the number of gravitons per (wavelength)³. N can also be interpreted as the “occupation number” of the background gravitons (i.e., the number of gravitons per quantum mechanical state). $N \gg 1$ provided the gravitons are classical in the sense that stimulated processes dominate spontaneous processes. So long as h and ω scale according to Eq. (2.12), this number is epoch-independent. The fact that it is in general very large justifies the notion that the background radiation is stochastic and isotropic: as a consequence of Eq. (2.14), one can regard a primordial background as being a superposition of N gravitons each with amplitude $h/\sqrt{N} \sim (\omega / \omega_{pl})$. The background is “stochastic” in the sense that the phases of the gravitons are assumed to be uncorrelated; provided the gravitons are not preferentially aligned, the total background perturbation should also be isotropic to within $1/\sqrt{N}$ fluctuations. We stress that there is no *a priori* reason why primordial gravitons should be uncorrelated in phase and direction; like generated gravitons, they might come in packets. However, provided the number of gravitons per packet is much less than N , the assumptions of stochasticity and isotropy will be appropriate.

Let us now ask how the gravitational waves affect the dynamical history of the Universe. Their dynamical importance depends on the value of Ω_g relative to the total matter density $\Omega_m \sim (10^{-2} \text{ to } 1)$ and the 3 K background density $\Omega_r \sim 10^{-4}$. From measurements of the cosmological deceleration parameter, Ω_g cannot exceed ~ 1 (Sandage and Hardy, 1973), but in principle it could be close to 1.

If $1 > \Omega_g > \Omega_m^2$, the Universe will be freely expanding, with its lengthscale R going like t , back to a time

$$t_f \sim t_0 \Omega_g^{1/2}. \quad (2.15)$$

Before t_f the expansion will be controlled by the gravitational radiation density and $R \propto t^{1/2}$. Also $h \sim \lambda / R$ [cf. Eq. (2.11)] in this period. However, from Eqs. (2.10) and (2.12), when one gets back to a time

$$t_a \sim \Omega_g^{1/2} P_0^2 t_0^{-1} \quad (2.16)$$

two things happen, both of which change the qualitative nature of the Universe: h becomes of order 1 and λ becomes of order the horizon size ct . (These things occur at the same time only if Ω_g exceeds Ω_r .) We have seen that the concept of gravitational waves as ripples moving through spacetime becomes meaningless in this situation. When λ becomes larger than ct , the gravitational wave is manifested as a local anisotropy in the cosmological expansion rate. Furthermore, since the shear induced by the waves, $\sigma \sim h/t$ for $t < t_a$, is of order the expansion rate at t_a , the associated shear density

$$\varrho_\sigma \sim \frac{\sigma^2}{G} \sim \frac{h^2}{Gt^2} \sim h^2 \varrho \quad (2.17)$$

is then comparable to the *total* density ϱ of the Universe. Thus before t_a the Universe is not only locally anisotropic, it is also shear-dominated. Although h is not well-defined before t_a [cf. Eq. (2.11)], it can be reinterpreted, for a given gauge, in terms of the amplitude of the associated shear fluctuation, viz. $h \sim \sigma t$. It should be appreciated that this shear fluctuation has a stochastic nature. From Fig. (2.14) every horizon volume at t_a can be regarded as containing N gravitons, each inducing a shear fluctuation of order $N^{-1/2}$. The total fluctuation on the horizon scale, $h \sim 1$, results from the random superposition of these gravitons. We stress that the gravitons do not produce a fluctuation in the density itself: a gravitational wave does not change the volume of a region, it merely distorts its shape. We discuss the evolution of the Universe before t_a later.

If $\Omega_m^2 > \Omega_g > \Omega_r$, the situation is different only in that the density of the Universe is matter-dominated rather than gravitational-radiation-dominated throughout the free-expansion phase. This means that $R \propto t$ back to a time

$$t_f \sim t_0 \Omega_m. \quad (2.18)$$

Before this, so long as ϱ_m exceeds ϱ_g , $R \propto t^{2/3}$. However, since ϱ_g goes like z^4 and ϱ_m goes like z^3 , there is a time

$$t_{eq} \sim t_0 \Omega_m^{-2} \Omega_g^{3/2} \quad (2.19)$$

before which the density of the Universe is dominated by its gravitational radiation content and $R \propto t^{1/2}$. For $t < t_{eq}$ the Universe evolves just as it did in the $1 > \Omega_g > \Omega_m^2$ case; in particular, $h \sim \lambda / R$ and the time t_a at which $h \sim 1$ and $\lambda \sim ct$ is still given by Eq. (2.16). t_a precedes t_{eq} provided $\Omega_g > (\Omega_m P_0 / t_0)^2$, which is true for all interesting values of Ω_g .

If $\Omega_g < \Omega_r$, as we conclude is most probable in Sect. 4, the Universe is freely expanding after the time t_f given by Eq. (2.18) and it is matter-dominated ($R \propto t^{2/3}$) back to a time

$$t_{eq} \sim t_0 \Omega_m^{-2} \Omega_r^{3/2} \sim 10^{11} \Omega_m^{-2} s. \quad (2.20)$$

Before this time the density of the Universe is dominated by the 3 K background radiation and $R \propto t^{1/2}$ back until λ becomes comparable to the horizon size at a time

$$t_H \sim P_0^2 t_0^{-1} \Omega_r^{1/2} \sim 10^{-19.5} P_0^2 s. \quad (2.21)$$

This time precedes t_{eq} providing $P_0 < t_0 \Omega_r^{1/2} / \Omega_m$. The value of h at this point $\sim (\Omega_g / \Omega_r)^{1/2}$ is less than 1 (in distinction to the first two situations), so in some period before t_a the Universe is locally anisotropic without being shear-dominated. However, as one goes further back in time, the shear induced on scales smaller than the wave, $\sigma \sim h/t$, will increase. In the simplest situation, it will increase like z^3 (i.e., $h \propto z$) and, in this case, Eq. (2.17) implies that the shear density will become comparable to the total density (i.e., $h \sim 1$) at a time

$$t_a \sim P_0^2 t_0^{-1} \Omega_g \Omega_r^{-1/2} \sim 10^{-15.5} \Omega_g P_0^2 s. \quad (2.22)$$

Even with $\Omega_g < \Omega_r$, therefore, one expects the presence of primordial gravitational waves to induce local anisotropy domination at some stage in the early Universe. In fact, as Grishchuk (1975) has emphasized, h does not necessarily evolve according to the “adiabatic” law $h \propto z$ in the period $t_a < t_0 < t_H$ because its evolution equation is then more complicated than indicated by Eq. (2.12). Indeed, if the Universe’s equation of state deviates from $p = \rho/3$, h may remain approximately constant. This “amplification” (relative to the adiabatic case) means that h need *not* become of order unity as one goes back in time, in which case the time t_a would not exist. However, Eq. (2.22) *does* apply in the most natural situation with $p = \rho/3$.

The description of the Universe in the period before t_a in these three situations is uncertain. Although the induced anisotropy is not global (since that would obviously be inconsistent with the assumed isotropy of the gravitational wave background today), one might hope to represent the Universe on scales less than λ by one of the standard *homogeneous* anisotropic models. In the simplest such model, Bianchi I, the average length-scale R goes like $t^{1/3}$ and σ increases with redshift like z^3 . (Thus the quantity $h \sim \sigma t$ remains constant.) Although Bianchi I is a good behavioral paradigm for more complicated anisotropic models, it should be appreciated that, in special circumstances, σ need not increase in this way as one goes back in time. The so-called “quasi-isotropic” solutions (Grishchuk et al., 1976), for example, contain gravitational waves in which h decreases as $t \rightarrow 0$. However, these solutions apply only if the perturbations have a special phase. They could not describe the early evolution of a gravitational background which is made up of lots of gravitons with uncorrelated phase. In general the gravitational-wave-induced anisotropies will grow indefinitely as $t \rightarrow 0$ and h will remain of order unity at times before t_a .

While this conclusion is certainly true *classically*, it is possible that quantum particle production effects in an anisotropic Universe may tend to reduce σ at early times (Zel’dovich, 1970). In this context, note that the number N given by Eq. (2.14) is no longer constant if the relation $h \propto z$ fails. In particular, for $t < t_a$, N increases like z^{-2} . Since N may be interpreted as the quantum mechanical occupation number, this can be taken as an indication that graviton production may then be important. We discuss this point further in Sect. 3.

3. The Origin of Primordial Gravitational Waves

Since the conclusions of the present paper are independent of how the Universe behaves before t_a , our ignorance of its state in that epoch is not very important. The qualitatively important point is that primordial gravitational waves necessarily induce anisotropy-domination at early times. However, in discussing the origin of primordial waves, some further comments are necessary. Primordial waves are *acausal* in the sense that they go back to the beginning of the classical Universe (the Planck time) and initially have a wavelength larger than the horizon. But since they cannot be recognized as waves at the Planck time, they can only be identified by virtue of the shear fluctuations they induce. Only later (at t_a or t_H) will they turn into conventional gravitons. Thus, in some sense, the origin of a primordial gravitational wave background is the initial anisotropy of the Universe. Of course, this does not mean that *any* anisotropic early Universe will give rise to a primordial background. A homogeneous anisotropic universe, for example, would never do so. Only a special form of shear fluctuations can turn into a background of gravitons and one could in principle identify the presence of a “nascent” primordial background even at the Planck time by recognizing this form. In the simplest situation, one envisages stochastic shear fluctuations superposed on a Friedmann background (cf. Tomita, 1978). But one might also conceive of more exotic cosmological models which exhibit shear motions correlated on a large scale (e.g. the Bianchi models). In this case, the resultant gravitons could be sufficiently correlated to form dissociated plane waves (Hu, 1975) and one would not then get an isotropic background.

We have stressed that the existence of primordial gravitational waves need not imply that the early Universe had large *density* fluctuations. This inference would apply only if one appealed to the so-called “equipartition hypothesis” (Zel’dovich and Novikov, 1975) which is discussed later. On the other hand, the existence of density fluctuations would inevitably generate gravitational radiation. The distinction is important because the usual train of inferences (galaxies $\Rightarrow$ primordial density fluctuations $\Rightarrow$ possibly chaotic early Universe $\Rightarrow$ cosmological gravitational waves) is not the only reason for suggesting the existence of cosmological gravitational waves. (Indeed, we will see in Sect. 4 that a large value for Ω_g actually *inhibits* galaxy formation.) One could anticipate a significant density of primordial waves even if the early Universe were devoid of density fluctuations.

The assertion that primordial waves necessarily induce anisotropy-domination at early times must be qualified because a special situation occurs if t_a coincides with the Planck time. From Eqs. (2.16), (2.22) this applies if

$$\Omega_g \sim \begin{cases} (t_0 t_{P1} / P_0^2)^2 & \text{for } \Omega_g > \Omega_r, \\ (t_0 t_{P1} \Omega_r^{1/2} / P_0^2)^2 & \text{for } \Omega_g < \Omega_r. \end{cases} \quad (3.1)$$

If $\Omega_g > \Omega_r$, this is just the condition that the gravitons have the Planck frequency at the Planck time. Having Ω_g less than the value indicated by Eq. (3.1), corresponding to $t_a < t_{P1}$, would then require a frequency higher than ω_{Pl} at t_{P1} , which may be physically implausible. If $\Omega_g < \Omega_r$, however, Ω_g could be somewhat smaller than the value given by Eq. (3.1) without having $\omega > \omega_{Pl}$ at t_{P1} .

We note from Eq. (2.14) that, if $\Omega_g > \Omega_r$, the condition $t_a = t_{P1}$ is also the condition that the overlap factor N be 1. Usually one would expect $N \gg 1$, but there is one situation of special interest in which $N \sim 1$. This applies if the background gravitons originate at the Planck time and have a thermal distribution. In this case the graviton spectrum peaks at a present period $P_0 \sim (t_{P1} t_0)^{1/2} \Omega_r^{-1/4} \sim 10^{-11} s$ and $N \sim 1$ for gravitons with this period. It can be

argued that one would *expect* such a background: simple cross-section considerations (Zel'dovich and Novikov, 1975) suggest that any gravitons existing at t_{p1} or prior to it will be thermalized by graviton-graviton interactions. The expected temperature of the black-body distribution today, using an argument similar to that of Matzner (1968), can be shown to be

$$T_g = \left(\frac{22 + 7n_\nu}{22n} \right)^{1/3} T_r \quad (3.2)$$

where $T_r \approx 3$ K is the background radiation temperature, n_ν is the number of neutrino species and n is of order the *total* number of particle species (including gravitons). The numerical factors in Eq. (3.2) come from considering the jumps in the lepton and photon entropies as the various massive particle species annihilate. n_ν is presently thought to be 6; the value of n is dependent on one's picture of particle physics but, in the quark picture, would certainly not exceed 100. One thus expects $T_g \sim 1$ K or $\Omega_g \sim 10^{-2} \Omega_r$ for such a thermal background of gravitons.

Grishchuk (1977), on the other hand, has argued that one would *not* necessarily expect gravitons originating from the Planck time to have a thermal distribution. He shows that, even in an isotropic Universe, one might expect the quantum production of gravitons as a result of classical amplification (alluded to in Sect. 2) acting upon zero-point-fluctuations present at t_{p1} . This process operates provided the equation of state of the Universe deviates from $p = \rho/3$ and provided the graviton wavelength exceeds the horizon size. One could not expect such conditions after the hadron era ($t > 10^{-5}$ s), so the process only produces gravitons with present period 10^{-11} s $< P_0 < 10^8$ s. (These period limits correspond to the redshifted timescale tz with $t \sim 10^{-43}$ s and $t \sim 10^{-5}$ s.) The spectrum of the radiation depends sensitively on the equation of state. If p is much harder than $\rho/3$, graviton production is very efficient (indeed Ω_g could exceed 1, which is excluded observationally). Furthermore, the gravitons so produced would quickly endow the Universe with a $p = \rho/3$ equation of state, at which point the process would cut off. Thus one would not expect gravitons with $P_0 \gg 10^{-11}$ s in this situation. If p is much softer than $\rho/3$, graviton production is less marked and its effect on the equation of state is never important. Thus Ω_g is small but the spectrum extends over the full range 10^{-11} s $< P_0 < 10^8$ s. Similar considerations by Starobinsky (1976), applied to graviton production in a universe which is initially collapsing and then bounces into expansion at the Planck density, predict a somewhat wider class of spectra.

We shall include a background of gravitons which derives from quantum effects like this under the classification "primordial". This seems appropriate since the gravitons go back to the Planck time (which, after all, is the beginning of the classical Universe); also they have a wavelength greater than the horizon size at the Planck time and are therefore acausal in some sense. There is, however, an important difference between a quantum primordial background and the sort of classical primordial background discussed earlier. For the classical case, $t_a \gg t_{p1}$ and the early Universe is necessarily very anisotropic; for the quantum case, the Universe is nearly isotropic ($h < 1$) all the way back to t_{p1} and the initial fluctuations have a specific origin (zero point fluctuations).

While the creation of gravitons in an isotropic Universe, as envisaged by Grishchuk, is at first sight rather surprising (because it used to be claimed, erroneously, that massless particles *cannot* be created in an isotropic expansion), it is well known that gravitons can be created by quantum processes in an anisotropic

Universe (Zel'dovich, 1970; Berger, 1974). Furthermore, the created gravitons will tend to act back upon the background and perhaps isotropize it (Lukash and Starobinsky, 1974; Hartle, 1977; Hu and Parker, 1977). Since we have seen that a non-quantum primordial background will itself render the early Universe anisotropic, it is therefore necessary to inquire whether the existence of a non-quantum background might not indirectly imply the existence of a quantum background also, and whether indeed the state presumed to exist at t_a could be a possible outcome of any preceding state of the Universe. As regards the first question, since quantum isotropizing processes are presumed to be efficient only near the Planck time, the present period of the resultant quantum gravitons ($\sim 10^{-11}$ s) would be much less than that of the classical primordial gravitons. Thus the two sorts of graviton background should be readily distinguishable. The second question is more problematic: if quantum effects dissipated all anisotropy at the Planck time, no surviving primordial background could have a density larger than the value indicated by Eq. (3.1). However, the question of how quantum effects feed back upon the cosmological expansion is very ill-understood, so such a pessimistic conclusion would hardly be warranted.

4. Limits on the Spectrum of Primordial Gravitational Waves

In this section we shall review the various observational limits on the values of Ω_g and P_0 for primordial waves. Most of the limits have been discussed before, but it is useful to bring them together. Also, by expressing them as restrictions on (Ω_g, P_0) , we shall give the limits in the form that is most useful for present purposes. The limits are summarized in Fig. 2. We shall find that some of them also pertain to *generated* gravitational waves, but only if the events producing the waves precede the epoch associated with the limit in question. Only primordial waves go back to the beginning of the Universe, so only primordial waves are subject to *all* of the limits in Fig. 2. Note that, if the primordial waves do not have a characteristic period, $\Omega_g(P_0)$ must be interpreted as the density of waves with period around P_0 .

The most interesting limit is the cosmological nucleosynthesis limit. If one believes that the observed abundances of helium and deuterium are produced through cosmological nucleosynthesis, this places a strong upper bound on Ω_g as we now show. The neutron-proton ratio freezes out when the weak interaction rate (Weinberg, 1972)

$$\sim 0.4 \left(\frac{T}{10^{10} \text{ K}} \right)^5 \text{ s}^{-1} \sim 10^{-48} z^5 \text{ s}^{-1} \quad (4.1)$$

becomes comparable to the cosmological expansion rate $\sim t^{-1}$. During the epochs of interest, the dependence of z on t is given by

$$z = \begin{cases} \Omega_g^{-1/4} t_0^{1/2} t^{-1/2} & \text{for } t_{eq} > t > t_a \\ \Omega_g^{-1/3} P_0^{-1/3} t_0^{2/3} t^{-1/3} & \text{for } t < t_a \end{cases} \quad (\Omega_g > \Omega_r)$$

$$z = \begin{cases} \Omega_r^{-1/4} t_0^{1/2} t^{-1/2} & \text{for } t_{eq} > t > t_a \\ \Omega_g^{-1/6} \Omega_r^{-1/6} P_0^{-1/3} t_0^{2/3} t^{-1/3} & \text{for } t < t_a \end{cases} \quad (\Omega_g < \Omega_r) \quad (4.2)$$

(where we have assumed Bianchi I behavior before t_a), so freeze-out occurs at a temperature

$$T_f \sim \begin{cases} 10^6 \Omega_g^{1/4} P_0^{1/2} \text{ K} & \text{for } 10^{-4} > \Omega_g > 10^{15.5} P_0^{-2} \\ 10^7 \Omega_g^{1/2} P_0^{1/2} \text{ K} & \text{for } \Omega_g > \max\{10^{10.5} P_0^{-3/2}, 10^{-4}\} \\ 10^{11} \Omega_g^{1/6} \text{ K} & \text{for } 10^{-4} < \Omega_g < 10^{10.5} P_0^{-3/2} \\ 10^{10} \text{ K} & \text{for } \Omega_g < \min\{10^{-4}, 10^{15.5} P_0^{-2}\}. \end{cases} \quad (4.3)$$

The four different (Ω_g, P_0) regions, which are shown in Fig. 3, correspond to whether the Universe is anisotropic or isotropic at the freeze-out time ($t_a > t_f$ or $t_a < t_f$) and to whether Ω_g is greater or less than Ω_r . Nucleosynthesis cannot occur until the temperature has fallen to a value of order 10^9 K (with only a logarithmic dependence on the values of Ω_m and Ω_g). From Eq. (4.2) this occurs at a time

$$t_{ns} \sim \begin{cases} 10^{10} \Omega_g^{-1/2} P_0^{-1} \text{ s} & \text{for } 10^{-4} > \Omega_g > 10^{17} P_0^{-2} \\ 10^8 \Omega_g^{-1} P_0^{-1} \text{ s} & \text{for } \Omega_g > \max\{10^{17} P_0^{-2}, 10^{-4}\} \\ \Omega_g^{-1/2} \text{ s} & \text{for } 10^{-4} < \Omega_g < 10^{17} P_0^{-2} \\ 100 \text{ s} & \text{for } \Omega_g < \min\{10^{-4}, 10^{17} P_0^{-2}\}, \end{cases} \quad (4.4)$$

the four (Ω_g, P_0) regions corresponding to whether t_a is greater or less than t_{ns} and to whether Ω_g is greater or less than Ω_r . The corresponding helium production, allowing for the β -decay of neutrons between the times t_f and t_{ns} (t_f always precedes t_{ns}), is

$$Y = \frac{2 \exp[-(t_{ns}/10^3 \text{ s})]}{1 + \exp[(m_n - m_p)/kT_f]}. \quad (4.5)$$

From Fig. 3 there are essentially six (Ω_g, P_0) regimes and in only one of these, the regime where $\Omega_g < \Omega_r$ and $t_a < t_f$, is nucleosynthesis unaffected by the gravitational waves. In the regime ($\Omega_g > \Omega_r$, $t_a < t_f$), the effect of the waves is merely to speed-up the expansion rate by a factor $\xi = (\Omega_g/\Omega_r)^{1/2}$ (Schwartzman, 1969). In the remaining four regimes, the expression for Y is more complicated because the speed-up factors at t_f and t_{ns} are different; but Y still increases with increasing Ω_g and P_0 because both t_f and t_{ns} decrease.

Now to produce a helium abundance in the range $29 \pm 4\%$ and a deuterium abundance of order 0.001% as observed, Reeves (1974) has argued that T_f must lie in the range 1 ± 0.1 times the "standard" Friedmann freeze-out temperature. This would certainly appear to exclude the $(\Omega_g > \Omega_r, t_a < t_f)$ regimes. [Actually, when ξ gets very large, the helium production falls again because the Universe is expanding so fast at nucleosynthesis that there is not enough time for all the deuterium to burn into helium. Y falls back to 0.29 for $\xi \sim 10^4$; but in this situation one produces far too much deuterium (Thorne, 1967) and there are, anyway, other limits which preclude Ω_g being large enough to give $\xi \sim 10^4$.] In the four (Ω_g, P_0) regimes with $t_a > t_f$ the situation is less straightforward because the Universe is anisotropy-dominated at the time of nucleosynthesis. However, Barrow (1976) has shown that one can also reject a Universe which is highly anisotropic at the time of nucleosynthesis (essentially because of the speed-up factor ξ it induces). One should therefore probably exclude the $t_a > t_f$ regimes also. If one invoked the "equipartition hypothesis" (discussed later) to argue that, when $h \sim 1$, the Universe would also have large density inhomogeneities and thus be "chaotic" (even though the waves do not induce such inhomogeneities directly), the situation would be less clear. Although many people have examined the effects of anisotropy and inhomogeneities in isolation (e.g., Schramm and Wagoner, 1977), it is not certain what the combined effects would be. It is conceivable that one could produce the required element abundances even with $\Omega_g > \Omega_r$, but the situation would have to be very contrived. One can probably reject any model which is chaotic at t_{ns} or t_f , and, in particular, the entire $\Omega_g > \Omega_r$ domain of Fig. 3. In this case the detection of primordial waves will not be easy.

It is possible that this rather pessimistic conclusion could be circumvented. For example, excess helium production could be suppressed by feeding into the initial conditions of the Universe a

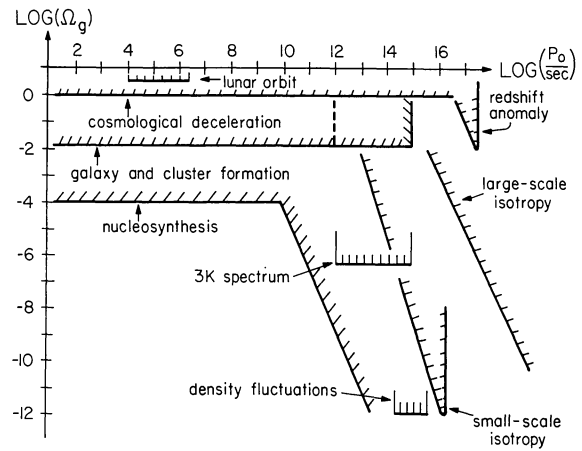


Fig. 2. This summarizes the most interesting observational limits on the values of Ω_g and P_0 for primordial gravitational waves. The shaded regions are excluded for the reasons indicated in the figure and discussed in the text. The "galaxy and cluster formation" limit depends on the total matter density of the Universe Ω_m , which we have taken to be 0.1. The "small-scale isotropy" limit depends on the redshift z_s at which the 3 K radiation was least scattered; we show the limit corresponding to $z_s \sim 10^3$. The "density fluctuations" and "3 K spectrum" limits depend on the equipartition hypothesis, which says that gravitational waves and density fluctuations have a comparable amplitude when they enter the horizon. Note that none of the limits are very severe ($\Omega_g < \Omega_r$) for waves of period less than 10^{10} s. All except the "cosmological deceleration" and "large-scale isotropy" limits could conceivably be circumvented, but only by invoking some unconventional cosmological scenario

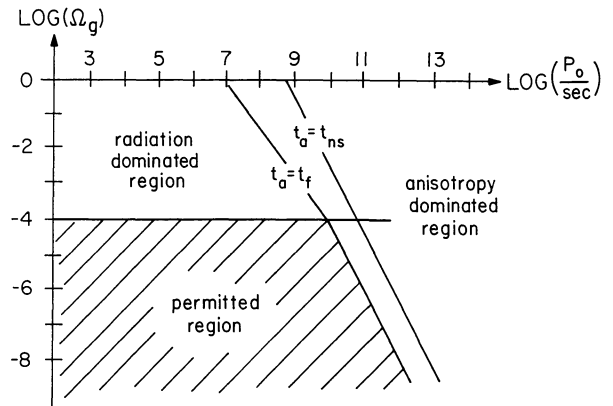


Fig. 3. This shows how cosmological helium production is affected by primordial gravitational waves. For any value of Ω_g and P_0 there is a time t_a before which the waves induce anisotropy-domination. There are six (Ω_g, P_0) regimes, corresponding to whether $\Omega_g \geq \Omega_r$, $t_a \geq t_f$ and $t_a \geq t_{ns}$, where t_f and t_{ns} are the "freeze-out" and "nucleosynthesis" times respectively. Only in the $(\Omega_g < \Omega_r, t_a < t_f)$ regime is helium production unaffected and in the $(\Omega_g > \Omega_r, t_a < t_f)$ helium would definitely be overproduced. In the remaining regimes, where the Universe is anisotropy-dominated (and perhaps chaotic) at the freeze-out time, helium would also probably be overproduced. In this case, the entire unshaded region can be excluded except in very contrived circumstances. This gives the strongest limit on the values of Ω_g and P_0 for primordial waves presently available

large number of (degenerate) electron-neutrinos. Alternatively, one could suppose that the gravitational wave spectrum is such that $h(\omega)$ peaks at a frequency $\omega_{\max}$ much less than that which gives the dominant contribution to Ω_g . In this case, the radiation might be detectable at $\omega = \omega_{\max}$ even if Ω_g is too small ($< \Omega_r$) to affect cosmological helium production. The spectrum of gravitons produced by the Grishchuk mechanism is, in fact, of the kind invoked in the second situation if the early Universe has $p \ll \rho$, although $h(\omega_{\max})$ is still very small.

If one could somehow sidestep the nucleosynthesis problem, what other limits could one place on Ω_g and P_0 ? To begin with, one can be certain that Ω_g does not exceed ~ 1 from measurements of the cosmological deceleration parameter (Sandage and Hardy, 1973), which imply that the *total* density of the Universe cannot much exceed 1 (although its precise value is very uncertain). Various dynamical arguments (Gunn, 1976) support this conclusion.

The conventional picture of galaxy formation would have serious problems if Ω_g exceeded Ω_m^2 . This is because the matter density fluctuations which evolve into protogalaxies in the conventional scenario cannot begin to grow until the time t_{eq} specified by Eq. (2.19). [If at time t_i one has an initial density fluctuation δ_i in the matter density ρ_m on some scale, then one can show (Meszaros, 1974) that, in the presence of a background radiation field of density ρ_g , δ evolves according to

$$\delta(t) = \delta_i \left(1 + \frac{3\rho_m(t)}{2\rho_g(t)} \right) \left(1 + \frac{3\rho_m(t_i)}{2\rho_g(t_i)} \right)^{-1}. \quad (4.6)$$

Thus δ stays approximately constant until t_{eq} . The reason, essentially, is that the matter density (and hence the matter density fluctuation) in a region is dynamically negligible until t_{eq} . Any fluctuation in ρ_g itself could only be of statistical origin and would be tiny. Although Eq. (4.6) was originally derived for a background of electromagnetic radiation, it is easy to show that the same analysis applies for gravitational radiation.] On the other hand, density fluctuations stop growing at the time t_f given by Eq. (2.18). Thus a period of growth exists at all only if $t_{eq} < t_f$, which corresponds to the condition $\Omega_g < \Omega_m^2$. Since Ω_m is itself less than 1, this implies $\Omega_g < \Omega_m$, i.e., the Universe's density cannot be *dominated* by gravitational waves unless one appeals to some unconventional picture of galaxy formation. This conclusion might be circumvented if the wavelength of the gravitational radiation at the time of galaxy formation was more than the size of a galaxy; this would require values for P_0 exceeding 10^{12} s. The same argument applied to clusters of galaxies would imply $\Omega_g < \Omega_m^2$ for $P_0 < 10^{15}$ s.

One problem with having primordial gravitational waves is that, as we have seen, in most circumstances one would expect the Universe to be very anisotropic and possibly chaotic at early times. This inference is invalid only if the gravitational waves are represented by one of the quasi-isotropic solutions (Grishchuk et al., 1976) or if the time t_a is comparable to t_{pr} . With these provisos, any limits on the anisotropy of the early Universe should therefore translate into limits on (Ω_g, P_0) .

One of the strongest limits on the anisotropy of the early Universe derives from considerations of cosmological nucleosynthesis effects (Barrow, 1976), which we discussed above. The isotropy of the 3 K background radiation also places limits on the values of Ω_g and P_0 (Sachs and Wolfe, 1967). Gravitational waves would induce two types of anisotropy. Firstly, there is the anisotropy generated by the waves today: the associated (quadrupole) temperature fluctuation is just of order h_0 and since,

observationally, the quadrupole anisotropy does not exceed 10^{-3} (Boynnton and Partridge, 1973), one infers

$$\Omega_g < 10^{-6} (P_0/t_0)^2 \quad \text{for } P_0 < t_0 \sim 10^{17.5} \text{ s}, \quad (4.7)$$

using Eq. (2.10). Grishchuk and Zel'dovich (1978) argue that this limit may apply if $P_0 > t_0$ (i.e., even for waves which have not yet fallen inside the horizon). Secondly, there is the (generally larger) anisotropy generated by the waves at the moment of recombination ($z_R \sim 10^3$). Waves which are falling inside the horizon at recombination (with $P_0 \sim 10^{16}$ s) will induce present-day temperature fluctuations of order their amplitude at that time h_R . Since $\Delta T/T$ on angular scales between about $1'$ and 1° does not exceed $10^{-4.5}$ (Parijskij, 1973), one deduces $h_0 < 10^{-7.5}$ or

$$\Omega_g < 10^{-12} (P_0/10^{16} \text{ s})^{-2} \quad \text{for } 10^{14} \text{ s} < P_0 < 10^{16} \text{ s}. \quad (4.8)$$

The lower limit of the period range corresponds to a wavelength which subtends an angle of about $1'$ at $z \sim 10^3$. (A wave of present period P_0 would have an angular scale $\Delta\theta \sim (P_0/t_0)$ when seen at redshifts $z \gg 1$.) However, for $P_0 \ll 10^{16}$ s, limit (4.8) is weakened because recombination is not instantaneous but takes a time of order $0.1 t_R$. Therefore, if the gravitational waves have a wavelength much smaller than ct_R , there will be many of them – each contributing a different sign to the temperature fluctuation – within the effective “surface” of last scattering (Rees, 1971; Doroshkevich et al., 1977; Dautcourt, 1974). This effect reduces $\Delta T/T$. The reduction is even more severe if the last scattering of the 3 K background occurs at a redshift z_s much less than 10^3 , as would apply, for example, if the Universe were ionized throughout the pregalactic era. The temperature fluctuation will be reduced by a factor $n^{-1/2}$ where n is the number of graviton wavelengths along the line of sight back to z_s (Anile and Motta, 1978). If $\Omega_g < \Omega_m^2$,

$$n \sim \frac{t_s(1+z_s)}{P_0} \sim \left(\frac{P_0}{t_0} \right)^{-1} z_s^{-1/2} \Omega_m^{-1/2} \quad \text{for } t_R < t_s < t_f. \quad (4.9)$$

The limit on Ω_g is consequently weakened to

$$\Omega_g < 10^{-16.5} \left(\frac{P_0}{t_0} \right)^{-3} \left(\frac{z_s}{10^3} \right)^{-1/2} \Omega_m^{-1/2} \quad \text{for } P_0 < t_s z_s. \quad (4.10)$$

If the Universe is ionized throughout the pregalactic era and $\Omega_m \sim 0.1$, one finds $z_s \sim 10$. If the Universe is *not* ionized between recombination and galaxy formation, $z_s = z_R \sim 10^3$. Even in the former case, Eq. (4.10) is an important restriction on the density of long period gravitational waves.

Another important consequence of cosmological gravitational waves would be their effect on the apparent redshifts of distant objects (Rees, 1971; Dautcourt, 1974; Burke, 1975). For $z \ll 1$ one would expect fluctuations Δz of order h_0 both transversely, on angular scales $\Delta\theta \geq (P_0/t_0)z$, and along the line of sight. If one accepts that redshifts fluctuations cannot exceed about 10% on scales 10 to 1000 Mpc, one infers

$$\Omega_g < \left(\frac{P_0}{10^{16.5} \text{ s}} \right)^{-2} \quad \text{for } 10^{15} \text{ s} < P_0 < 10^{17} \text{ s}. \quad (4.11)$$

More positively, some people have tried to *explain* various redshift anomalies by appealing to gravitational waves. A fluctuating redshift Δz , for example, would increase the apparent velocity dispersion in clusters of galaxies and for a cluster of radius R_c would lead to an apparent virial mass M_v exceeding the true mass M_c by a factor

$$M_v - M_c \sim c^2 R_c (\Delta z)^2 G^{-1} \sim \rho_g R_c (c P_0)^2 \quad \text{for } c P_0 < R_c \quad (4.12)$$

(Dautcourt, 1974). It is possible that such an effect has already been observed and the distinctive form of the predicted mass discrepancy (viz. proportional to R_c and independent of M) should make it readily identifiable. Equation (4.12) applies for wavelengths less than the cluster size. For wavelengths exceeding R_c , one could expect to see systematic redshift variations across chains of galaxies. Such an effect may have been observed in at least one cluster (Gregory and Connolly, 1973) and there are several examples of clusters which have galaxies with individual anomalous redshifts (Arp, 1971). Finally, a gravitational wave background with well-defined period P_0 could produce a clustering in redshift with period $\Delta z \sim (P_0/t_0)$. Burbidge and O'Dell (1972) claim to detect such an effect with $\Delta z = 0.03$, corresponding to $P_0 \sim 10^{16}$ s, though this claim is questionable. Furthermore, their effect is spherically symmetric, which one would not expect for a gravitational wave effect.

Gravitational waves of smaller wavelength could have more local consequences. For example, a planet with orbital period P responds to gravitational waves of period $P_0 \sim P$. It can be shown (Bertotti, 1973; Mashhoon, 1979) that in a time t the mean fractional change in the semimajor axis a (and other Keplerian elements) is given by

$$\left\langle \left(\frac{\delta a}{a} \right)^2 \right\rangle \sim G \rho_g(P) P t \sim \Omega_g \left(\frac{P}{t_0} \right) \left(\frac{t}{t_0} \right). \quad (4.13)$$

(This effect probably extends to waves with periods down to the orbital light travel time.) One can understand this result as follows: we have seen from Eq. (2.14) that a primordial background can be regarded as N overlapping gravitons of individual amplitude $N^{-1/2}h$. Each such graviton which passes the planet will induce a fluctuation $(\delta a/a)$ of order this individual amplitude. In a time t , the number of gravitons passing is Nt/P , and if one assumes that the perturbations to a add stochastically, one immediately obtains Eq. (4.13). Let us apply Eq. (4.13) in the case of the moon. One can be reasonably sure from lunar laser ranging experiments that over the last few years the fractional change in the moon's orbital radius has not exceeded $\sim 10^{-10}$. This implies that waves with $P_0 \sim 10^6$ s must have $\Omega_g < 4$. Similar limits applied to the planets have been discussed by Turner (1979). Note that, since $\Omega_g < 1$ and $P \ll t_0$, Eq. (4.13) implies that even on a cosmological timescale $(\delta a/a)$ could only be small. There might, however, be a much larger effect (Mashhoon, 1978) associated with the *phase* of the orbit.

Another set of limits on (Ω_g, P_0) can be inferred if one invokes the so-called "equipartition hypothesis" (Zel'dovich and Novikov, 1975), which says that adiabatic density fluctuations and gravitational waves on any scale have comparable amplitudes, ε_H and h_H , when that scale falls within the horizon. For example, on the scale of clusters of galaxies ($10 \text{ Mpc} < \lambda_0 < 100 \text{ Mpc}$) one believes $\varepsilon_H < 10^{-4}$, so the equipartition hypothesis implies

$$\Omega_g < 10^{-8} \Omega_r \sim 10^{-12} \quad \text{for} \quad 10^{15} \text{ s} < P_0 < 10^{16} \text{ s}. \quad (4.14)$$

One can also argue that $\varepsilon_H < 0.05$ on scales $10^{14} M_\odot > M > 10^6 M_\odot$ (or $10 \text{ kpc} < \lambda_0 < 10 \text{ Mpc}$) because such fluctuations would fall within the horizon before recombination and the heat released by their dissipation would distort the 3 K background spectrum too much (Sunyaev, 1974) unless this condition was satisfied. Thus the equipartition hypothesis implies

$$\Omega_g < 10^{-2.5} \Omega_r \sim 10^{-6.5} \quad \text{for} \quad 10^{12} \text{ s} < P_0 < 10^{15} \text{ s}. \quad (4.15)$$

More generally, if $\Omega_g < \Omega_r$, Eq. (2.21) implies $h_H \sim (\Omega_g/\Omega_r)^{1/2}$ and so one infers $\Omega_g \sim \varepsilon_H^2 \Omega_r$. If $\Omega_g > \Omega_r$, the considerations of Sect. § 2

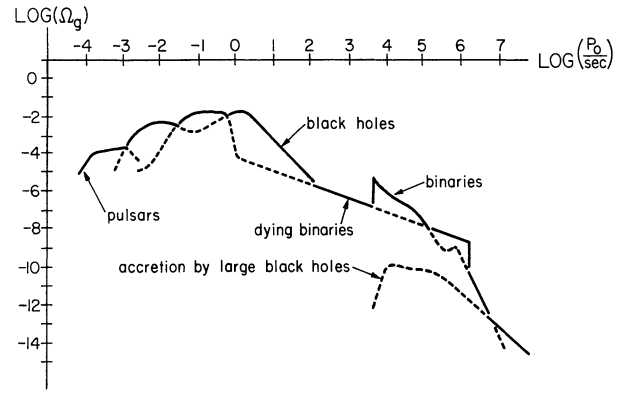


Fig. 4. This illustrates the sort of gravitational wave background one might expect to be generated by various postgalactic processes. The spectrum is taken from Rosi and Zimmerman (1976). Note that the contribution to the mean wave density Ω_g is dominated by sources at a redshift $z \sim 1$ and, in this sense, the Rosi-Zimmerman background can be described as cosmological. However, the *local* density of gravitational waves (generated by sources in our own galaxy) is comparable to the mean density because of the local source concentration. This means that gravitational bursts generated by events in our own galaxy (which would not overlap) may be individually detectable even if the background is not

imply that $h_H \sim 1$ which, if one believes the spectral distortion argument, would be immediately excluded by the equipartition hypothesis for scales larger than $10^6 M_\odot$. One might exclude $h_H \sim \varepsilon_H \sim 1$ even on scales *below* $10^6 M_\odot$ because otherwise one might get an overproduction of primordial black holes (Carr, 1975) on such scales. However, since the Universe is also anisotropy-dominated when $h \sim 1$, this inference is far from certain (Barrow and Carr, 1978). In any case, the status of the equipartition hypothesis is unclear, so limits (4.14) and (4.15) are not definitive.

5. Generated Gravitational Waves

All sorts of processes associated with galaxies may have generated a gravitational wave background: the formation of black holes through stellar collapse, black hole accretion, the formation and subsequent evolution of pulsars, supernova explosions, binary stars, "dying" binary systems, the collapse of the protogalaxies themselves, the activity associated with galactic nuclei. All such *postgalactic* sources have been analyzed by Rosi and Zimmerman (1976). The associated background spectrum is sensitive to galactic evolution and, in particular, to the assumed mass function of the stars. Figure 4 shows the most likely spectrum, expressed in terms of Ω_g and P_0 , as calculated by Rosi and Zimmerman. There are contributions over the entire range $10^{-4} \text{ s} < P_0 < 10^8 \text{ s}$, with a peak ($\Omega_g > 10^{-3}$) in the range $10^{-3} \text{ s} < P_0 < 10 \text{ s}$ due to stellar collapses and dying binaries. There may be an extra peak due to quasars but we do not show it in Fig. 4 since it is model-dependent and very uncertain. At extremely long wavelengths, $10^{16} \text{ s} > P_0 > 10^{12} \text{ s}$, there might also be a contribution generated by density fluctuations in the early Universe [cf. Eq. (4.14)] but the associated Ω_g would be small unless the Universe was chaotic before decoupling (Rees, 1971). Other sources (e.g., white dwarfs, stellar

pulsations, black hole vibrations, star formation, black hole collisions, explosions in galactic nuclei) are either too weak or too infrequent to produce a significant background density.

The sorts of generated waves discussed by Rosi and Zimmerman can be described as “backgrounds” provided the gravitational waves from individual sources overlap. Rosi and Zimmerman do not discuss this question explicitly since they regard a background as anything *extragalactic* in origin; from our point of view, this is not a very appropriate use of the term. Nevertheless most of their “backgrounds” do satisfy the overlap condition. This is certainly the case for the “continuous” sources (like binary stars). It also applies for most of the “burst” sources (like stellar collapse) since the interval between successive bursts is typically less than their individual duration, though only marginally so. However, as we show later, the Rosi-Zimmerman mean background density is dominated by bursts at $z \sim 1$ and it is only these cosmological bursts which overlap. The bursts generated at $z \ll 1$, such as in our own galaxy, would *not* overlap.

In the rest of this section we shall emphasize backgrounds of gravitational waves which may have been generated by *pregalactic* events. These were not discussed by Rosi and Zimmerman, although they have been alluded to by Thorne (1978) and Blandford (1979). The most interesting sources in this context would be pregalactic black holes. We shall discuss specific scenarios for pregalactic black hole formation in Sect. 6. In this section we discuss the general characteristics of the background radiation which might have been generated by such scenarios on the assumption that the holes have a specific mass M , density Ω_B (in units of the critical density) and formation redshift z_B . The discussion will be sufficiently general to cover the production of a background by *postgalactic* events as well, so we shall end this section by pointing out some crucial differences between pregalactic and postgalactic backgrounds.

We confine attention to the generation of a gravitational background by black holes, not because these are the *only* possible generators, but because they offer the most interesting range of scenarios. For example, there could exist a population of pregalactic neutron stars; but because neutron stars lie in such a narrow mass range, any scenario involving them will be very similar, differing only with respect to epoch. There might also exist pregalactic binaries; but our knowledge of what their characteristics would be is so limited that one can hardly begin to formulate a specific scenario. Scenarios involving black holes, however speculative, offer both variety and specificity.

Three processes associated with black holes might generate gravitational waves with a reasonable efficiency: their formation, their accretion and collisions between them. With conventional ideas about the state of the Universe before galaxy formation, the gas density would be too low for the holes to even double their mass through accretion; also the holes would be too far apart to collide with an appreciable frequency. Thus one's best hope lies in the individual collapses themselves. One might hope that the collapse of an object of mass M would produce a burst of gravitational radiation with a reasonably high efficiency

$$\varepsilon \equiv (\text{energy carried off by the burst})/Mc^2. \quad (5.1)$$

The value of ε is very uncertain, although optimistically it could be as high as 0.1; we shall leave ε unspecified in the following calculations. Most of the gravitational radiation will probably be emitted in a short initial broadband burst (generated by the imploding matter) with period of order 10 times the light travel time across the black hole (Thorne, 1978). There will also be

radiation associated with the subsequent narrow-band ringing (due to the hole's normal mode pulsations) but the energy content of this should be considerably smaller. One would thus expect the spectrum of the gravitational radiation to peak at a present period

$$P_0 \sim 10^{-4} \left(\frac{M}{M_\odot} \right) (1+z_B) \text{ s}, \quad (5.2)$$

where the last factor allows for the cosmological redshift effect. This should also be the duration of the burst τ_0 to an order of magnitude. For $P > P_0$, we shall assume that the spectrum falls off as some power of P , but we leave the exponent unspecified.

Rees (1977) has pointed out that the collapse of a very massive object would not necessarily be sufficiently sudden to generate an energetic gravitational pulse: a non-rotating supermassive star becomes unstable (Chandrasekhar, 1964) at a radius which exceeds its Schwarzschild radius by a factor $(M/10^2 M_\odot)^{1/2}$, so its infall would be spread out over a timescale $(M/10^2 M_\odot)^{3/4}$ longer than the timescale τ_0 indicated above. Were this the case, the spectrum would extend to periods $P \gg P_0$ and ε would be reduced by a factor $(M/10^2 M_\odot)^{-3/4}$ for $P \sim P_0$. However, one could still expect a large value for ε if the region fragmented into two during the collapse, because the fragments might generate a lot of gravitational radiation as they spiralled into each other. For large holes this is probably the *only* way of getting a large value for ε .

The amplitude h of the gravitational waves generated by the collapse of an object of mass M at a distance R is (Thorne, 1978):

$$h = (2\varepsilon)^{1/2} \frac{GM}{c^2 R} \approx 4 \cdot 10^{-24} \left(\frac{\varepsilon}{0.1} \right)^{1/2} \left(\frac{M}{M_\odot} \right) \left(\frac{R}{c/H_0} \right)^{-1}. \quad (5.3)$$

Here H_0 is the present value of the Hubble parameter $\sim t_0^{-1}$ and R is the “circumference” function of the Friedmann cosmology, given by

$$\frac{H_0 R}{c} = \frac{1 - q_0 + q_0 z - (1 - q_0)(1 + 2q_0 z)^{1/2}}{q_0^2(1 + z)} \sim \begin{cases} z/(1 + z) & \text{if } q_0 = 1 \\ z(1 + \frac{1}{2}z)/(1 + z) & \text{if } q_0 \ll 1, z \ll q_0^{-1} \\ q_0^{-1} & \text{if } q_0 \ll 1, z \gg q_0^{-1} \end{cases} \quad (5.4)$$

where q_0 is the cosmological deceleration parameter. For non-cosmological sources ($z \ll 1$) h goes like z^{-1} . For cosmological sources ($z \gg 1$) the dependence on redshift is only weak because one is looking out most of the way to the particle horizon anyway: R is just of order the horizon size $\sim ct_0$, though it may be larger by an order of magnitude if q_0 is only ~ 0.1 . Note that, if one were observing at an earlier epoch, but later than t_B , R would merely be the *current* horizon size. Only at the epoch t_B itself could one hope to see events closer than the horizon.

Let us now consider the combined effect of gravitational radiation from all the holes which formed at z_B . If these holes have a present density Ω_B , their number density is

$$n_B = \frac{\Omega_B \rho_c}{M} = \frac{3\Omega_B H_0^2}{8\pi G M}. \quad (5.5)$$

In a time t one sees waves coming from a (present) volume $4\pi R^2 ct$, where R is given by Eq. (5.4). The time-integrated energy flux from each hole is $c^3 h^2 / GP_0$, so the total energy flux observed on Earth is

$$F \sim \frac{c^3}{G} \cdot \frac{h^2}{P_0} \cdot 4\pi R^2 \cdot cn_B \quad (5.6)$$

and the corresponding mean gravitational wave density is

$$\varrho_g \sim \left(\frac{F}{c^3}\right) \sim \frac{4\pi c}{G} \cdot \frac{(hR)^2}{P_0} \cdot \frac{\Omega_B}{M} \cdot \varrho_c. \quad (5.7)$$

From Eq. (5.3) the product hR just depends on M and ε ; using Eq. (5.2) for P_0 , the present gravitational wave density (in units of the critical density) is thus

$$\Omega_g \sim \varepsilon \Omega_B (1 + z_B)^{-1}. \quad (5.8)$$

This, of course, is exactly what one would expect from energy considerations. As one goes back in time: hR is constant, n_B increases like z^3 and P decreases like z^{-1} . Thus ϱ_g increases like z^4 , which is also what one would expect.

Now from Eqs. (5.4) and (5.5) the interval between bursts as observed today is

$$(\Delta\tau)_0 = \frac{1}{4\pi c R^2 n_B} = \frac{2GM}{3c^3 \Omega_B} \times \begin{cases} \Omega^2/4 & \text{for } z \gg \Omega^{-1} \\ 4z^{-2} & \text{for } z \ll \Omega^{-1} \end{cases} \quad (5.9)$$

where $\Omega = 2q_0$ specifies the total matter density of the Universe, whereas the duration τ_0 of each burst is of order the value P_0 given by Eq. (5.2). The ratio of these two times is

$$\left(\frac{\tau}{\Delta\tau}\right)_0 \sim 10^2 \Omega_B \times \begin{cases} 0.1 z_B^2 & \text{for } z_B < 1 \\ 0.1 z_B^3 & \text{for } 1 < z_B \ll \Omega^{-1} \\ z_B \Omega^{-2} & \text{for } z_B \gg \Omega^{-1} \end{cases} \quad (5.10)$$

with no M dependence. The coefficients in the bracketed terms are not precise, but the coefficient in the expression for P_0 is only correct to an order of magnitude anyway. $(\tau/\Delta\tau)_0$ can exceed 1 in the $z_B < 1$ regime only if $\Omega_B > 0.1$; but since $\Omega_B < \Omega < 1$, it could only do so marginally. Thus bursts can appreciably overlap only if they form at a cosmological redshift ($z_B > 1$). In the $z_B > \Omega^{-1}$ regime, the holes must also be sufficiently numerous that $\Omega_B > 10^{-2} z_B^{-1} \Omega^2$, a condition which is necessarily satisfied if the holes dominate the density of the Universe ($\Omega_B \sim \Omega$). If the bursts do overlap, they will add stochastically to give an effective present-day amplitude

$$\tilde{h}_0 \sim \left(\frac{\tau}{\Delta\tau}\right)^{1/2} h. \quad (5.11)$$

In terms of this “effective” amplitude, the gravitational wave density [from Eqs. (5.7) and (5.9)] is just

$$\Omega_g \sim \tilde{h}_0^2 \left(\frac{P_0}{t_0}\right)^{-2}. \quad (5.12)$$

This is the same relation which applies for primordial gravitational radiation [cf. Eq. (2.10) with h_0 replaced by $\tilde{h}_0$]. The quantity $(\tau/\Delta\tau)_0$ thus has a similar stochastic interpretation as the number N defined by Eq. (2.14). However, whereas $(\tau/\Delta\tau)_0$ refers to the number of overlapping bursts, N refers to the number of overlapping gravitons. Since each burst produces $(hP_0\omega_{P1})^2$ gravitons per (wavelength)³, it is the product $(\tau/\Delta\tau)_0 (hP_0\omega_{P1})^2$ which is analogous to N . If the wave bursts do *not* overlap, $(\tau/\Delta\tau)_0$ is less than 1 and the effective amplitude $\tilde{h}_0$ is less than the individual burst amplitude h .

Even if the gravitational bursts overlap now, they need not always have done so. If one goes back to a time t and redshift z , the duration of each burst decreases like z^{-1} , whereas from Eq. (5.9) the interval between bursts goes like $t^{-2}z^{-3}$. Thus,

assuming $\Omega_g < \varepsilon \Omega^2$ (which implies that the holes form before t_f),

$$\left(\frac{\tau}{\Delta\tau}\right) \sim \left(\frac{\tau}{\Delta\tau}_0\right) \left(\frac{t}{t_0}\right)^2 z^2 \sim \begin{cases} 10 \Omega_B z_B^3 & \text{for } t > t_f \\ 10^2 \Omega_B z_B \Omega^{-2} \left(\frac{t}{t_f}\right)^{2/3} & \text{for } t_B < t < t_f \end{cases} \quad (5.13)$$

(If $\Omega_g > \varepsilon \Omega^2$, t_B exceeds t_f and $\tau/\Delta\tau$ never changes.) The ratio remains constant during the Universe’s free expansion phase but it decreases when one goes back to times earlier than t_f . Indeed $(\tau/\Delta\tau)$ becomes of order 1 (that is, the waves dissociate) at a time

$$t_* \sim t_f (10^2 \Omega_B z_B \Omega^{-2})^{-3/2}. \quad (5.14)$$

The ratio of the time t_B at which the waves are generated to the time t_* is

$$\frac{t_B}{t_*} \sim \left(\frac{10^2 \Omega_B}{\Omega^3}\right)^{3/2}. \quad (5.15)$$

Thus generated waves are dissociated for some time after t_B provided $\Omega_B < \min(10^{-2} \Omega^3, \varepsilon \Omega^2)$.

An observer’s description of the background becomes more complicated when he gets back to the epoch t_B at which the bursts are actually generated, because at this time he can see sources at distances much less than the current horizon size. This, in particular, applies to an observer seeing a background produced by events at the *present* epoch. The background density must now be expressed as an integral over R :

$$\begin{aligned} \left(\frac{\text{background}}{\text{radiation density}}\right) &= \int \frac{1}{c} \times \left(\frac{\text{flux per event}}{\text{at distance } R}\right) \\ &\times \left(\frac{\text{number density of}}{\text{events at distance } R}\right) \times (4\pi R^2 dR) \sim \int \left(\frac{h^2 n(R) R^2}{G P_0^2}\right) dR. \end{aligned} \quad (5.16)$$

Let us first assume that the events are uniformly distributed (with n independent of R). Then, using Eqs. (5.2) and (5.3), one sees that the biggest contribution to the integral comes from $z \sim 1$, provided the redshift z_B at which the events start to occur is itself of order 1. Only if the events began very recently ($z_B < 1$) would the dominant contribution to Ω_g derive from non-cosmological redshifts. It is important to appreciate the relationship between Eqs. (5.16) and (5.8). Equation (5.8) assumes that the generating events are *only* occurring during the epoch t_B ; in this case, if $z_B \gg 1$, there is no contribution to the background density from $z \sim 1$.

If the events are *not* uniformly distributed but confined, say, to galaxies, one can still deduce that the *mean* radiation density (i.e., the density averaged over all points of space) will be dominated by sources at a redshift of 1 if $z_B \sim 1$. However, the *local* radiation density may now be larger than the mean density and therefore need not be dominated by cosmological sources. For example, if the sources are clustered in galaxies with a concentration factor (ϱ_g/ϱ_u) typical of that of galactic material in general, one has:

$$\left(\frac{\text{radiation density in galaxy}}{\text{mean radiation density}}\right) \sim \frac{R_g}{R_u} \times \frac{\varrho_g}{\varrho_u} \sim \frac{M_g}{M_u} \times \left(\frac{R_u}{R_g}\right)^2 \quad (5.17)$$

where subscripts “g” and “u” refer to “galaxy” and “Universe” characteristics respectively. As it happens M_u/M_g and $(R_u/R_g)^2$ are both of order 10^{11} , so the flux ratio is close to 1. (This is just a coincidence.) A consequence of this is that the amplitude associated with an individual galactic event is much larger than the mean amplitude $\tilde{h}_0$ of the background:

$$\left(\frac{h_{\text{gal}}}{\tilde{h}_0}\right) \sim \left(\frac{R_u}{R_g}\right) \left(\frac{\tau}{\Delta\tau}_0\right)^{-1/2} \sim \left(\frac{R_u}{R_g}\right) (10 \Omega_B)^{-1/2} > 10^5, \quad (5.18)$$

where we have used Eq. (5.10) and assumed $z_B \sim 1$. This means that individual galactic bursts are much easier to detect than the background itself, although there might be a considerable interval between such bursts. Similar considerations apply to continuous sources of gravitational radiation.

Finally, we stress that the value of $\tilde{h}$, even though it goes like z , can never get as large as 1 as one goes back in time. This is in distinction to the situation with *primordial* waves, where h becomes of order 1 at the time t_a specified by Eq. (2.16) or (2.22). This reflects the fact that the density of generated waves can never exceed the density of the holes which produce them, so they can never be dynamically important.

6. Scenarios for Black Hole Formation

In this section we review six possible scenarios for black hole formation. In the first four the holes are pregalactic; in the last two, they are postgalactic. It is not clear that *any* of these scenarios actually occurred and some of them are mutually exclusive, but one evidently wants to know the range of possibilities.

Scenario A

One possibility, if the Universe started off with large density fluctuations, is that black holes may have formed within the first few moments of the big bang. Such primordial holes would have of order the horizon mass at formation (Hawking, 1971). Although this is rather unlikely in the conventional hot scenario, because one requires the density fluctuations to have a special form (Carr, 1975), it is in principle possible. In order to avoid the very strong upper limits one has on the number 10^{15} g evaporating black holes (Hawking, 1974; Chapline, 1975; Page and Hawking, 1976) pertaining also to larger holes, one has to postulate that something prevents primordial black hole formation occurring at *very* early times. Two possibilities have been suggested: the Universe may have had a stiff ($p = \rho$) equation of state before 10^{-4} s because of strong interactions (Lin et al., 1976), in which case the black hole mass spectrum would begin at $\sim 10 M_\odot$; or it may have been highly anisotropic (Barrow and Carr, 1978) until some time t_s , in which case the mass spectrum would begin at $\sim 10^5 t_s M_\odot$. (In principle the value of t_s is arbitrary, but in the favored scenario of Barrow and Carr it is about 1 s since this is the time at which one would expect anisotropy to be dissipated by neutrino viscosity. If one believes that helium is produced through cosmological nucleosynthesis, t_s cannot be *larger* than 1 s.) Primordial black holes could conceivably have a critical density ($\Omega_B \sim 1$). Although their density would be dominated by the smallest ones, the integrated mass spectrum only falls off slowly (Carr, 1975).

Scenario B

The production of primordial black holes would be much more plausible in less conventional cosmologies. For example, prolific black hole formation would seem to be inevitable if the early Universe was cold (photonless) or tepid (without *all* of the present background photons) even if the initial density fluctuations were small (Carr, 1977; Carr and Rees, 1978). These models have the distinctive feature that the 3 K background is actually generated through black hole accretion and, from simple energy considerations, this requires that the holes have a near-critical density and form after 10^4 yr. If one also requires that the holes form

before the time $\sim 10^6$ yr which conventionally corresponds to decoupling (to ensure thermalization of the generated radiation), the density fluctuations have to lie in a fairly narrow range. Carr (1977) has argued that the first regions to collapse are the largest ones that can do so without fragmenting: in this case they have a mass $M \sim 10^{15} \delta_H^{5/3} M_\odot$ and form at a time $t_B \sim 10^3 \delta_H^{-5/3}$ s, where δ_H is the amplitude of the fluctuations when they enter the horizon. To ensure $10^4 \text{ yr} < t_B < 10^6 \text{ yr}$, one needs $10^{-5} > \delta_H > 10^{-6}$, which in turn implies $10^7 M_\odot > M > 10^5 M_\odot$. However, it is conceivable that generated radiation could be thermalized long after 10^6 yr (Rees, 1978), in which case z_B might be as low as 10^2 (if $\delta_H \sim 10^{-7}$) and M might be as small as $10^3 M_\odot$. The spectrum of holes could extend down to $10 M_\odot$, although most of the mass would probably be in the largest ones.

Scenario C

Returning to the conventional hot scenario, it is possible that a lot of black holes may have formed after decoupling, when the Jeans mass would have dropped to $10^6 M_\odot$ and fluctuations on that scale would have easily become bound. The time at which the $10^6 M_\odot$ regions bind would depend on the form of the initial density fluctuations: if the fluctuations decrease with scale and if galactic scales are to bind by 10^9 yr, it would certainly need to be very early; extrapolating the inferred galactic fluctuations down to $10^6 M_\odot$ scales (Peebles, 1974) suggests it could be of order 10^6 yr. It is usually assumed that such bound regions would fragment into globular cluster type objects (Peebles and Dicke, 1968); but some fraction of them may also have collapsed to black holes, either directly or via a supermassive star phase. A *necessary* condition for fragmentation is that the collapse timescale of a region should be more than the cooling timescale when it first binds, and in the context of $10^6 M_\odot$ regions binding after decoupling this condition is satisfied. (On the other hand, it is not satisfied for the large regions which form black holes in scenario B; essentially because the regions bind well before decoupling and therefore have a large density.) However, this timescale requirement is not a *sufficient* condition for fragmentation and in this scenario we assume that the $10^6 M_\odot$ regions collapse directly. In fact, if this is true for $10^6 M_\odot$ regions, it may also be true for much larger regions. This suggests that the holes which form after decoupling could extend to a mass range well above $10^6 M_\odot$ and the largest holes might not form until quite late.

Scenario D

The question of whether fragmentation can be avoided in scenario C is obviously very uncertain. However, even if the $10^6 M_\odot$ regions which bind after decoupling *do* fragment, they may fragment into very massive rather than solar-type stars; these stars could then pursue their usual main-sequence evolution and end up as black holes well before galaxy formation. The possibility that a lot of massive stars may have formed in the period $10 < z < 10^3$ is accentuated by the evidence that there may have been a burst of heavy element enrichment prior to galaxy formation (Truran and Cameron, 1971) and by the near-solar iron abundance in intergalactic space (Mitchell et al., 1976). If the $10^6 M_\odot$ regions *do* fragment in this way, the mass range and formation time of the resultant holes would depend on the details of the fragmentation process (e.g., Hartquist and Cameron, 1977). A good guess would be that M lies between 10 and $10^3 M_\odot$, and that t_B is of order the corresponding main-sequence lifetime or the binding time of the $10^6 M_\odot$ region, whichever is larger.

Scenario E

Another possibility is that a lot of black holes may have formed *after* galaxy formation as the endpoint of evolution of a first generation of massive population III stars (Truran and Cameron, 1971). It has been suggested that such remnants could have $\Omega_B \sim 0.1$ and provide the “invisible” mass in galactic halos. However, Ω_B can be this large only if the mass of the postgalactic holes are large enough: the stars from which they derive would need to have a mass in the range above $10^2 M_\odot$ because otherwise they would produce too much background light during their main-sequence phase (Thorstensen and Partridge, 1975; Eichler and Solinger, 1976). Stars bigger than $10^2 M_\odot$ would be radiation pressure dominated and could avoid this limit. The extent to which this scenario generates gravitational waves has already been discussed by Rosi and Zimmerman (1976), although they do not impose the condition that the population III stars be bigger than $100 M_\odot$ and they assume the particular mass spectrum derived by Truran and Cameron.

Scenario F

Finally, there is the possibility that entire galaxies may have collapsed to black holes. There is reason to believe that collapsed galaxies must have a mass less than $10^8 M_\odot$ else they would have been detected by their tidal effects in, for example, the Virgo cluster (Van den Bergh, 1969). Such collapsed “dwarf” galaxies might in principle have $\Omega_B \sim 0.1$ and dominate the density of the Universe. However, there is no reason for expecting dwarf galaxies to collapse. Indeed, were $10^8 M_\odot$ holes discovered in clusters, it would be most natural to interpret them as deriving from scenario C.

In all these scenarios, one expects the resultant black holes to have a mass in the range 10 to $10^8 M_\odot$ and a density which, a priori, could be close to the critical density. Carr (1978) has reviewed the various observational limits on the values of Ω_B and M . For holes confined to galaxies or clusters of galaxies, one can be reasonably sure – from the lack of perturbation to the Hubble flow in clusters – that Ω_B is less than 0.1 (Gott et al., 1974). Unclustered holes might in principle have $\Omega_B \sim 1$, but only *pregalactic* holes (such as are associated with scenarios A to D) could avoid clustering like galaxies and they would not necessarily avoid it. The limits on the mass range in which Ω_B could be large depend on the holes’ environment. The strongest limit for holes inside galaxies or galactic halos is associated with their tidal effects and requires that M be less than $10^5 M_\odot$ (Carr, 1978). The strongest limit for holes inside clusters of galaxies is also associated with tidal effects: as already mentioned, observations of the Virgo cluster (Van den Bergh, 1969) imply that such holes must be smaller than $10^8 M_\odot$. The strongest limit for unclustered holes comes from the lack of image-doubling of distant quasars (Press and Gunn, 1973) and implies that such holes must be smaller than $10^{12} M_\odot$.

In order to translate these limits on (Ω_B, M) into limits on (Ω_g, P_0) , one needs to specify the redshift at which the holes form. Holes which have $\Omega_B > 0.1$ would need to be pregalactic and hence have $z_B > 10$. Holes which have ordinary stars as progenitors would need $z_B > 100 \Omega_B$ in order to avoid the stars producing more background light than is observed (Eichler and Solinger, 1976). By comparison, one expects $z_B > 10^{10}$ in scenario A, $10^4 > z_B > 10^2$ in scenario B, and $10^3 > z_B > 10$ in scenarios C and D. Thus the condition on z_B is necessarily satisfied in the pregalactic scenarios, though it may not be in the postgalactic scenarios. Note that, of the limits shown in Fig. 2, only the “lunar orbit”, “cosmological

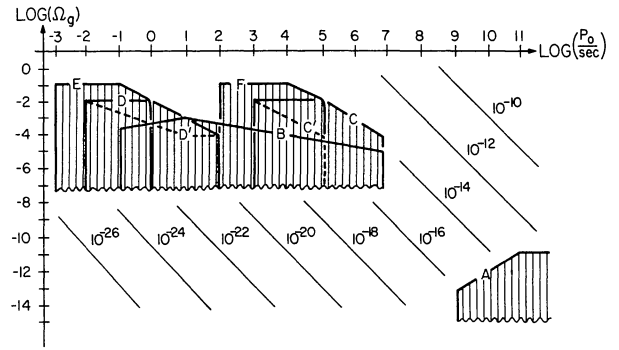


Fig. 5. This illustrates the range of (Ω_g, P_0) values which may have been generated by the formation of pregalactic and postgalactic black holes. The upper boundary applies if ε , the efficiency of gravitational wave generation, is 0.1 and if the holes have a density $\Omega_B \sim 1$. The six scenarios for black hole formation (A to F) are discussed in the text. The (Ω_g, P_0) region spanned by each scenario allows for every possible variant of the scenario; any particular variant will not necessarily cover all the region indicated. The figure also shows the lines which correspond to various values for the effective amplitude $\tilde{h}_0$. This exceeds the amplitude associated with the burst from any *individual* black hole because the bursts, being at cosmological distances, overlap. We stress that one cannot be certain that black holes really formed in *any* of the scenario indicated and, even if they did, ε and Ω_B may be considerably smaller than their optimal values

deceleration”, “redshift anomaly” and “large-scale isotropy” limits pertain to all generated backgrounds. The “galaxy and cluster formation” limit pertains only to pregalactic backgrounds; the “small-scale isotropy” and “nucleosynthesis” limits pertain only to backgrounds which exist before decoupling and 100 s, respectively.

For any particular scenario for black hole formation, one can specify the density $\Omega_B(M)$ and the formation redshift $z_B(M)$ of the holes with mass M . From Eqs. (5.2) and (5.8), for a given efficiency parameter ε , this determines the density Ω_g and period P_0 of the associated generated gravitational waves as functions of M . From Eqs. (5.3), (5.4), and (5.11) one can also determine the effective amplitude of the waves:

$$\tilde{h}_0(M) \approx 10^{-23} \left(\frac{\varepsilon}{0.1} \right)^{1/2} \left(\frac{M}{M_\odot} \right) [\Omega_B(M) z_B(M)]^{1/2} \quad \text{for } z_B(M) \gg \Omega_B^{-1} \quad (6.1)$$

We note that $\tilde{h}_0$ is independent of Ω . In general the scenario will specify a mass spectrum for the holes and hence $\Omega_B(M)$ as a function of M and the *total* black hole density Ω_B . (The mass which dominates Ω_B need not also be associated with the dominant contribution to Ω_g .) The dependence of z_B on M , which implies a relationship between P_0 ($\propto M z_B$) and z_B , thus specifies $\Omega_g \sim \varepsilon z_B(M)^{-1} \Omega_B(M)$ as a function of P_0 for given values of ε and Ω_B . Thus in the (Ω_g, P_0) plane, each scenario can be represented by a line (for given ε and Ω_B), the extent and slope of this line corresponding to the mass spectrum of the holes and the $z_B(M)$ relationship. In some scenarios the specification of the mass spectrum may also depend on some extra “cosmological” parameter (such as the time t_s in scenario A or the amplitude of the density fluctuations δ_H in scenario B). In this case, as the parameter varies, the (Ω_g, P_0) line will trace out an *area* in the (Ω_g, P_0) plane. Finally, as one varies ε and Ω_B , the line or area spanned by the scenario will move up and down in the Ω_g -direction.

Table 1. A summary of the possible scenarios for black hole formation and the characteristics of the background gravitational radiation they would have generated. The “ M ” column specifies the mass range spanned by the holes; the mass spectrum, where it can be determined, is indicated in the text. The “ z_B ” and “ P_0 ” columns specify the range of values for the formation redshift of the holes and the present period of the waves. The “ $\tilde{h}_0$ ” and “ Ω_g ” columns specify the range of values for the present “effective” amplitude of the waves and their density in units of the critical density on the optimum assumptions that $\Omega_B \sim 1$ and $\varepsilon \sim 0.1$.

Scenario	Description	$M/M_\odot$	z_B	$P_0(s)$	$\tilde{h}_0$	Ω_g
A	Primordial black hole formation in an early Universe which is anisotropic or has a stiff equation of state	$>(10^{-10^5})$	$<(10^{12}-10^{10})$	$>(10^9-10^{11})$	$>(10^{-16}-10^{-13})$	$10^{-13}-10^{-11}$
B	Pregalactic black hole formation in a cold or tepid early Universe	10 to (10^3-10^7)	10^2-10^4	$10^{-1}-10^7$	$10^{-22}-10^{-14}$	$10^{-3}-10^{-5}$
C	Direct collapse of the objects which bind after decoupling	10^6-10^8	$10-10^3$	10^3-10^7	$10^{-16.5}-10^{-13.5}$	$10^{-2}-10^{-4}$
D	Collapse of the pregalactic stars into which objects binding at decoupling fragment	$10-10^3$	$10-10^3$	$10^{-2}-10^2$	$10^{-21.5}-10^{-18.5}$	$10^{-2}-10^{-4}$
E	Black hole formation associated with a first generation of galactic stars	$10-10^3$	$1-10$	$10^{-3}-1$	$10^{-22}-10^{-19.5}$	$10^{-2}-10^{-1}$
F	Black hole formation associated with the collapse of dwarf galaxies	10^6-10^8	$1-10$	10^2-10^5	$10^{-17}-10^{-14.5}$	$10^{-2}-10^{-1}$

Figure 5 indicates the (Ω_g, P_0) regimes spanned for each of the scenarios A to F discussed earlier, on the optimum assumption that $\varepsilon \sim 0.1$ and $\Omega_B \sim 1$. The figure also indicates how the regimes extend for lower values of ε and Ω_B . The boundaries of the extended regimes specify the range covered by all possible variants of the particular scenario. Any one variant may only cover a part of the regime indicated, but no variant can give values for (Ω_g, P_0) outside the regime. The optimum condition $\Omega_g \sim 1$ is feasible in all scenarios involving pregalactic holes. In scenarios E and F, which involve postgalactic holes, one probably requires $\Omega_B < 0.1$, although this condition is not explicitly assumed in Fig. 5. The condition $\varepsilon \sim 0.1$ may be unreasonable for scenarios which involve holes larger than $10^2 M_\odot$; as discussed in Sect. 5, ε may then be reduced by a factor $(M/10^2 M_\odot)^{-3/4}$. One feature which is not included in Fig. 5 is the fact that the spectrum of radiation from collapsing objects may extend well beyond the characteristic period P_0 given by Eq. (5.2), especially for $M > 10^2 M_\odot$. Allowing for this feature could extend the domains shown in Fig. 5 further to the right, but only where the spectrum falls off above P_0 more slowly than the boundary presently indicated.

Some of the information in Fig. 5 is displayed in a different form in Table 1, which shows the range of values for M , z_B , P_0 , Ω_g , and $\tilde{h}_0$ in each of the scenarios. The value of $\tilde{h}_0$ is significant because it usually determines whether a particular detector can observe the background. We emphasize that $\tilde{h}_0$ only exceeds the individual burst amplitude if the ratio $(\tau/\Delta\tau)_0$, given by Eq. (5.10), exceeds 1. The values of Ω_g and P_0 do not directly determine whether this condition is satisfied, but the values of Ω_g and z_B do. The condition may be satisfied in all the scenarios but only if Ω_B exceeds about 0.1 in the two postgalactic ones.

In order to justify Fig. 5, some elaboration on the various scenarios is required:

Scenario A

For any primordial hole $M \propto t_B$, so $P_0(M) \propto M z_B \propto M^{1/2}$ (where we are assuming a $p = q/3$ equation of state at early times, i.e., $z \propto t^{-1/2}$). For a given total density, $\Omega_B(M) \propto M^{-1/2}$ and so $\Omega_g(M) \propto \Omega_B(M) z_B^{-1}$ is independent of M . The lowest value for P_0 is determined by the time $t_{\min}$ at which the smallest holes form. In no variant of the scenario could $t_{\min}$ exceed 1 s, but it might be considerably smaller. The highest value for P_0 is determined by the upper cut-off in the mass spectrum, which in principle could be very large.

Scenario B

Holes of mass M form on the cooling timescale at their instability radius, $t_B \propto M^{-1}$, so $P_0(M) \propto M^{5/3}$ (where we are assuming a $p = 0$ equation of state, i.e., $z \propto t^{-2/3}$). The mass spectrum extends up to $M_{\max} \sim 10^{15} \delta_H^{5/3} M_\odot$ and one expects $\Omega_B(M) \propto M$ (Carr, 1977). Thus most of the mass is in the largest holes and $\Omega_g(M) \propto M z_B^{-1} \propto M^{1/3}$. One would not expect holes to form smaller than $10 M_\odot$ because the equation of state is hard before 10^{-4} s on account of strong interactions and degeneracy effects.

Scenario C

If only the $10^6 M_\odot$ regions collapse to holes, at a redshift z_B in the range 10 to 10^3 , one has $P_0 \propto z_B$ and $\Omega_g \propto z_B^{-1}$. This defines the broken boundary (C') in Fig. 5. In principle larger regions may also collapse, though presumably they would have to collapse after the $10^6 M_\odot$ objects and well before galaxy formation. The dependence of z_B on M and the mass spectrum of the holes are uncertain. The solid boundary indicates the domain spanned for all variants of the scenario with $\Omega_B < 1$, $10^6 M_\odot < M < 10^8 M_\odot$, $10 < z_B < 10^3$.

Scenario D

The crucial feature of this scenario is the mass M of the stellar fragments. For a given M , if the $10^6 M_\odot$ regions bind at 10^6 yr, we take t_B to be the main-sequence lifetime of a star with this mass, viz. $t_B \sim (10^{10} M^{-2}, 10^6) \text{ yr}$ or $z_B \sim (M^{4/3}, 10^3)$ for ($M < 10^2 M_\odot$, $M > 10^2 M_\odot$). In this case $P_0 \propto M z \propto (M^{7/3}, M)$ and $\Omega_g \propto z^{-1} \propto (M^{-4/3}, M^0)$. This defines the broken boundary (D') in Fig. 5, where we have assumed that the fragments span a mass range between 10 and $10^3 M_\odot$. If one envisages the $10^6 M_\odot$ regions binding long after decoupling, the (Ω_g, P_0) regime is extended. For binding redshifts between 10 and 10^3 , the domain spanned is as shown by the solid boundary.

Scenario E

Since the stars from which the holes derive are postgalactic, they necessarily form at $z_B < 10$ and their characteristic mass spectrum is dependent on fragmentation details. In the Truran and Cameron (1971) scenario, $\Omega_B(M) \propto M$ and the spectrum extends up to $\sim 10^3 M_\odot$. In this case, for a given z_B , $P_0 \propto M$ and $\Omega_g \propto M$. We consider a more general situation, viz. all variants of the scenario with $\Omega_B < 1$, $10 M_\odot < M < 10^3 M_\odot$, $1 < z_B < 10$. The boundary depends sensitively on the upper mass of the progenitors stars: if this were increased, the regime would extend further to the right. Note that, since the holes in this scenario would have to be in galaxies, one probably requires $\Omega_B < 0.1$. This would bring the (Ω_g, P_0) regime down a decade.

Scenario F

Again $z_B < 10$ and, from the Virgo tidal limit, we infer $M < 10^8 M_\odot$. The mass spectrum is obviously uncertain: dwarf galaxies might conceivably dominate the matter density and their spectrum might extend all the way down to $10^6 M_\odot$. Figure 5 indicates the (Ω_g, P_0) regime spanned by all variants of the scenario with $\Omega_B < 1$, $1 < z_B < 10$, $10^6 M_\odot < M < 10^8 M_\odot$. As in scenario E, one probably requires $\Omega_B < 0.1$.

One other, very speculative, scenario for graviton production via black holes deserves mention. Any primordial black holes smaller than 10^{15} g would have evaporated by now (Hawking, 1974) and about 1% of their mass-energy might have been channelled into black-body gravitons. A hole of mass M would have a lifetime and temperature given by

$$t_L \sim 10^{10} \left(\frac{M}{10^{15} \text{ g}} \right)^3 f(M) \text{ yr}, \quad \theta \sim 10^{12} \left(\frac{M}{10^{15} \text{ g}} \right)^{-1} \text{ K} \quad (6.2)$$

where $f(M)$ accounts for the number of particle species which can be emitted and is normalized such that $f=1$ for $M \sim 10^{15} \text{ g}$. If the holes would have had a present density $\Omega_B(M)$, had they not evaporated, the present period and density of the associated gravitons would be

$$\left. \begin{aligned} P_0 &\sim 10^{-22} \left(\frac{M}{10^{15} \text{ g}} \right)^{-1} \text{ s} \\ \Omega_g &\sim 10^{-2} \Omega_B(M) \left(\frac{M}{10^{15} \text{ g}} \right)^2 \end{aligned} \right\} \text{ for } M > 10^{13} \text{ g}. \quad (6.3)$$

For holes with $M < 10^{13} \text{ g}$, which evaporate in the radiation era, the dependence of P_0 and Ω_g on M is somewhat different. For $M \sim 10^{15} \text{ g}$, observations of the 100 MeV photon background imply $\Omega_B < 10^{-8}$, so $\Omega_g < 10^{-10}$. (In scenario A we avoided this limit by presuming that primordial holes do not form on scales as small as 10^{15} g .) Carr (1975) has argued that $\Omega_B(M)$ should go like

$M^{-1/2}$, so the contribution to Ω_g should be even smaller for $M < 10^{15} \text{ g}$. However, if the mass spectrum of the holes had an upper limit below 10^{15} g , this restriction would not apply and all one could directly infer would be $\Omega_g < 1$. In particular, if the entire Universe was contained in 10^{-5} g black holes at the Planck time, their evaporation could have generated the sort of 1 K background of thermal gravitons mentioned in Sect. 3.

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