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Thick Accretion Disks and Supercritical Luminosities

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Summary. When accretion rates exceed a critical value accretion disk around a black hole must become thick, vitiating an assumption necessary in the construction of most models. Even for subcritical rates, the inner regions of standard accretion disks are believed to be unstable and are expected to puff up. By replacing the usual assumption of local energy balance with a global conservation requirement, and by taking the local radiated flux to be critical, we construct families of consistent thick accretion disks. These have cusps at their inner edges, which can lie between the marginally bound and marginally stable orbits, and depend upon the angular momentum distribution specified but are not directly dependent on assumptions about the viscosity law. They can be matched onto relatively thin disks at a "transition radius". The accretion rates can become very large, and the total luminosities can exceed the nominal Eddington luminosity by substantial factors due to geometrical effects. Such luminosities produce huge radiative energy densities in the cusp region, so that the formation of well collimated beams is a distinct possibility. The calculations use a pseudo-Newtonian potential which reproduces many of the salient features of the Schwarzschild solution.

Key words: accretion – black holes – active galactic nuclei

1. Introduction

If one attempts to build a model of an accretion disk where the accretion rate becomes very large, it is clear that the disk must become thick, at least in the region close to the compact object. According to standard models, the maximum value of the ratio of the thickness of the disk to its radius is equal to the ratio of the accretion rate to a critical rate (Shakura and Sunyaev, 1973). Since these standard "α models" assume that the disk is relatively thin everywhere, they cannot treat this situation consistently. Even when the accretion is subcritical, it has been recognized that the inner portions of standard disk models around black holes are subject to various instabilities (e.g., Lightman and Eardley, 1974; Shakura and Sunyaev, 1976). These instabilities probably cause such disks to puff up and become geometrically thick, thus vitiating the assumptions of thinness and hydrostatic equilibrium which provide the foundations of these models. By making various assumptions, different authors have constructed models in which winds are driven off the disk's surface, or in which two streams of material are ejected along the axes of the disk (e.g., Shakura and Sunyaev, 1973; Shapiro et al., 1976; Callahan, 1977; Piran, 1977; Lynden-Bell, 1978; Meier, 1978) thus attempting to allow for the instabilities and the strong radiation pressure encountered in the central region of the disk. However Shakura and Sunyaev (1976) have shown that the faster growing instabilities are thermal, and are predominantly due to the assumption that Q_-, the rate at which energy is radiated per unit surface area, is equal to Q_+, the rate at which it is generated by friction within the disk. When the disk is both geometrically and optically thick, this equality cannot in general be true, for the energy produced at some point inside the bloated disk will not just diffuse vertically, but may emerge from essentially any part of the surface. We are thus led to consider how this assumption can be replaced by a more physical one, and if possible to construct models of thick inner regions of accretion disks that have a consistent time-averaged stationary behaviour.

The approach we have chosen was inspired by the recent general relativistic treatment of the structure of perfect fluid disks around a black hole given by Abramowicz et al. (1978) and Kozłowski et al. (1978); see also Fishbone and Moncrief (1976). These authors have shown that a cusp exists at the inner edge of such disks (or really rings) and that fat disks can be constructed whose inner edges extend down to the marginally bound circular orbit, r_{mb} ($r_{mb}=2r_g=4GM/c^2$ for a Schwarzschild black hole). Thus, we investigate disks whose inner edge lies somewhere between r_{mb} and the last stable circular orbit, r_{ms} ($r_{ms}=3r_g$ for the Schwarzschild case); the portion of the disk inside r_{ms} is supported by a non-Keplerian angular momentum distribution, and therefore does not immediately plummet into the black hole. However, unlike Abramowicz et al. (1978) we will allow for the generation of energy by viscous stresses and for its radiation. We will also match our thick disks onto thin disks at suitably large radii, thus giving the solutions more physical significance.

However our treatment will not be a correct general relativistic one, for we shall employ a pseudo-Newtonian potential, $\psi=-GM/(R-r_g)$, that correctly reproduces the positions of both r_{ms} and r_{mb}, and yields efficiency factors in close agreement with the Schwarzschild solution. One great advantage of our approach is that it is not explicitly dependent upon the assumed form of the viscosity law, which is subject to considerable uncertainty; thus we only have to bring in the "α model" (Novikov and Thorne, 1973; Shakura and Sunyaev, 1973) to provide a consistency check on our results (Sect. 4c). There are however many assumptions and approximations in our model that make the quantitative conclusions inexact. We assume that: the equation of state is barytropic, so that the specific angular momentum is constant on

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cylinders; red-shift effects can be neglected; reabsorption of radiation and evaporation of the disk can be ignored (cf. Shakura and Sunyaev, 1973; Cunningham, 1975, 1976; Rees, 1978 and references therein). Because each of these approximations is crude our treatment is an idealized one. We hope to loosen some of these restrictions in subsequent work, and to perform a fully relativistic treatment. However, the removal of the barytropic assumptions would require solving a much more complicated set of differential equations.

Apart from being independent of assumptions about the viscosity, our approach has other advantages. One is simplicity, both conceptually and in the computations, allowing different angular momentum distributions to be investigated. Another is that the salient effects of general relativity are included via our pseudo-Newtonian potential. We are able to construct thick disks whose inner radii lie within r_{ms} and the match onto more standard thin disks. The accretion rates can exceed the nominal critical rates, and because of the bloated shape of our disks, the total luminosities can exceed the nominal Eddington luminosity. We feel that these basic conclusions are likely to stand even if the details are inaccurate because of the nature of our idealized model, and we feel that this new approach does yield new insight.

In Sect. 2 we present the basic equations we use in studying stationary accretion disks of arbitrary thickness. The assumptions made and the specialized equations needed by our approach in obtaining the shape and luminosity of thick disks are given in Sect. 3. Specific cases are calculated and our results summarized in Sect. 4. In Sect. 5 we draw conclusions and point out modifications that should be included in more refined calculations.

2. Equations for Stationary Accretion Disks

We now summarize the important relations needed for our analysis. Naturally, most of these are standard and appear in earlier papers (Lynden-Bell, 1969; Pringle and Rees, 1973; Shakura and Sunyaev, 1973; Novikov and Thorne, 1973) but we derive them again here for two reasons. Firstly, we use a somewhat different approach in obtaining the equations, and also a slightly different notation. Secondly, we shall be using non-Newtonian potentials and non-Keplerian angular momentum distributions, and thus we must use a rather more general formulation of the equations.

We use cylindrical coordinates (r, z, φ) centered on the black hole. Azimuthal symmetry is assumed and the spherical radius $R=(r^2+z^2)^{1/2}$. The generalized potential ψ must satisfy the following conditions: $\psi(R)<0$; $\psi(R)\to 0$ as $R\to\infty$; $d\psi(R)/dR>0$.

The typical assumption of a thin disk has $z \ll r$ so that $r \approx R$. The circular velocity, angular velocity, and angular momentum per unit mass (which we henceforth refer to simply as angular momentum) are given, assuming Keplerian orbits, by

$$v=\left(r\frac{d\psi}{dr}\right)^{1/2}, \quad \Omega=\left(\frac{1}{r}\frac{d\psi}{dr}\right)^{1/2}, \quad l=\left(r^3\frac{d\psi}{dr}\right)^{1/2}. \tag{1}$$

The total mechanical energy per unit mass is given in general by

$$e=v^2/2+\psi, \tag{2}$$

as long as the radial and vertical velocities are much less than the azimuthal velocity; $e<0$ implies that a mass element is bound.

Let η be the viscosity at some point in the disk, whose half thickness is given by z_0. The heat released due to friction can be expressed as (in erg cm^{-3} s^{-1})

$$\varepsilon=\left(r\frac{d\Omega}{dr}\right)^2\eta. \tag{3}$$

The torque applied by frictional forces at a given radius can be written as

$$g=2\pi r^3\left(\frac{-d\Omega}{dr}\right)\int_{-z_0}^{z_0}\eta dz. \tag{4}$$

Typical definitions of surface density and accretion rate are taken:

$$\Sigma=\int_{-z_0}^{z_0}\varrho dz, \quad \dot{M}=\int_{-z_0}^{z_0}2\pi r\varrho v_r dz. \tag{5}$$

The total couple gives us the angular momentum flux,

$$\dot{J}=\dot{M}l+g. \tag{6}$$

The radial energy flux can be expressed as

$$\dot{E}_r=\dot{M}\cdot e+g\Omega, \tag{7}$$

whilst the vertical energy flux is usually taken to be

$$\frac{\partial\dot{E}_z}{\partial r}=2\pi r\cdot 2F, \tag{8}$$

where F is the energy flux from the surface.

In this notation the conservation of mass, angular momentum, and energy can be written as

$$\frac{\partial}{\partial t}(2\pi r\Sigma)+\frac{\partial\dot{M}}{dr}=0, \tag{9}$$

$$\frac{\partial}{\partial t}(2\pi r\Sigma l)+\frac{\partial\dot{J}}{\partial r}=0, \tag{10}$$

$$\frac{\partial}{\partial t}(2\pi r\Sigma e)+\frac{\partial\dot{E}_r}{\partial r}+\frac{\partial\dot{E}_z}{\partial r}=0. \tag{11}$$

We now employ the mass and angular momentum conservation laws directly, since they remain valid for thick disks as long as Ω is taken as a function of r, only i.e., if meridional circulation and radial velocities can be ignored. Substituting Eqs. (5) and (6) into Eqs. (9) and (10) immediately yields

$$\dot{M}\frac{\partial l}{\partial r}+\frac{\partial g}{\partial r}=0, \tag{12}$$

which is also true for thick disks (Loska, private communication). At this point we use the assumption of stationarity and take $\dot{M}=$const. Eq. (12) can then be integrated to give

$$g=g_0+(-\dot{M})(l-l_0), \tag{13}$$

and we have accretion if $\dot{M}<0$. We now evaluate g_0 and l_0 at the inner edge of the disk, r_0, and make the following reasonable assumptions about the boundary conditions near a black hole: the torque at r_0, $g(r_0)=g_0=0$; the angular momentum at r_0, l_0, is given by the Keplerian value, and since the disk is thin at r_0 (it has a cusp there), Eq. (1) is adequate. For convenience we define

$$S=\int_{-z_0}^{z_0}\eta dz, \tag{14}$$

and then use Eq. (4) to write

$$S=\frac{g}{2\pi r^3}\left(\frac{-d\Omega}{dr}\right)^{-1},$$

which, using Eq. (13) with $g_0=0$ can be expressed as

$$S=(-\dot{M})\left(\frac{-d\Omega}{dr}\right)^{-1}(2\pi r^3)^{-1}(l-l_0). \tag{15}$$

The total energy generated within a given column of the disk is

$$Q_+=\int_{-z_0}^{z_0}\varepsilon dz=\left(r\frac{d\Omega}{dr}\right)^2\int_{-z_0}^{z_0}\eta dz=\left(r\frac{d\Omega}{dr}\right)^2 S, \tag{16}$$

where we have made use of Eqs. (3) and (14). We can eliminate the viscosity from our expression for the energy generation by inserting Eq. (15) into Eq. (16) to obtain a key relationship

$$Q_+=(-\dot{M})(2\pi r)^{-1}\left(-\frac{d\Omega}{dr}\right)(l-l_0). \tag{17}$$

If we make the usual assumption, valid for Newtonian disks, that $\psi=-GM/r$, and further assume a thin disk so that $Q_-\equiv 2F\equiv Q_+$, we immediately recover from Eq. (17), using Eq. (1), the well known expression for the flux,

$$F=(-\dot{M})\frac{3}{8\pi}\frac{GM}{r^3}\left[1-\left(\frac{r_0}{r}\right)^{1/2}\right].$$

However, we are interested in the more general situation where neither Eq. (1), nor $Q_-=Q_+$, is acceptable.

At this point we summarize the forms certain important quantities take on for the specific potential we use,

$$\psi=\frac{-GM}{(R-r_g)}. \tag{18}$$

In the regions of the disk where it *is* thin, and thus $r\approx R$:

$$v=\left(\frac{GM}{r}\right)^{1/2}\left[\frac{r}{r-r_g}\right],\qquad \Omega=\left(\frac{GM}{r^3}\right)^{1/2}\left[\frac{r}{r-r_g}\right],$$
$$l=(GMr)^{1/2}\left[\frac{r}{r-r_g}\right]; \tag{19}$$

$$-\frac{d\Omega}{dr}=\tfrac{3}{2}\left(\frac{GM}{r^5}\right)^{1/2}\left[\frac{(r-\frac{1}{3}r_g)r}{(r-r_g)^2}\right],$$
$$\frac{dl}{dr}=\tfrac{1}{2}\left(\frac{GM}{r}\right)^{1/2}\left[\frac{(r-3_g)r}{(r-r_g)^2}\right]; \tag{20}$$

$$e=\left(\frac{-GM}{2r}\right)\left[\frac{(r-2r_g)r}{(r-r_g)^2}\right],$$
$$\frac{de}{dr}=\left(\frac{GM}{2r^2}\right)\left[\frac{r^2(r-3r_g)}{(r-r_g)^3}\right]. \tag{21}$$

In each of the above equations the correction due to the inclusion of the r_g term is enclosed in brackets. Inspection of Eq. (21) shows that e, the binding energy, vanishes at $r=2r_g$, and thus we identify $r_{mb}=2r_g$, as for Schwarzschild geometry. Likewise, from Eq. (20), we see that $dl/dr=0$ and $de/dr=0$ when $r=3r_g$, and as an orbit can only be stable when $dl/dr\geq 0$ and $de/dr\geq 0$, we conclude that $r_{ms}=3r_g$, and no *Keplerian* disk can exist inside this radius. The efficiency of energy conversion is given by $\eta'=e/c^2$, and we note that at $r=r_{ms}$, Eq. (21) yields the result $\eta'=0.0625$, whilst the correct result for the Schwarzschild metric is 0.057. At smaller radii the relative agreement is even closer, so we expect that our estimation of luminosities for given mass fluxes will not be more than 10% too high because of the overestimation of η', although the uncertainty is increased by other factors to be discussed later.

3. An Approach to Thick Disks

a) Basic Assumptions and Specialized Equations

The first important approximation, or idealization we make is to assume a barytropic equation of state, $P=P(\varrho)$. It then follows (von Zeipel's theorem) that $\Omega=\Omega(r)$, and this simplification is very helpful. However this approximation must eventually fail in the inner regions where the flow is almost spherical. In this barytropic situation we can define the enthalpy by $dH=dP/\varrho$, and in the equatorial plane of the disk ($z=0$), we denote the density, enthalpy and gravitational potential as

$$\varrho_c(r),\quad H_c(r),\quad \psi_c(r)=\psi(r). \tag{22}$$

We take the boundary conditions on the surface of the disk to be the vanishing of the enthalpy and density, so that on the surface, defined by $\pm z_0$, we have

$$\varrho_0(r)=0,\quad H_0(r)=0,\quad \psi_0(r)=\psi(r^2+z^2)^{1/2}. \tag{23}$$

Our next key assumption is that the equations of hydrostatic equilibrium hold. This certainly fails to be true to some extent in the very innermost region, close to the cusp, where both radial and vertical velocities will not actually be negligible. However, within our formulation the problems are not as severe as when an attempt is made to construct a consistent thin disk (e.g. Bisnovatyi-Kogan and Blinnikov, 1977), and we hope to lossen this restriction in a future paper. The equilibrium equations can be written as

$$\frac{1}{\varrho}\nabla P=\nabla H=\nabla\left(-\psi+\int_{r_0}^{r}\Omega^2 r dr\right). \tag{24}$$

Using the notation defined in Eqs. (22) and (23), the vertical component of Eq. (24) integrates to

$$H_c(r)=\psi_0(r)-\psi(r). \tag{25}$$

Differentiating Eq. (25) with respect to r and comparing with the radial component of Eq. (24) yields the relation

$$r\Omega^2=\frac{dH_c}{dr}+\frac{d\psi}{dr}=\frac{d\psi_0}{dr}. \tag{26}$$

As we have already required that $\Omega=\Omega(r)$ we have the following basic equations that determine the angular quantities in terms of the potential at the surface, or *vice versa*:

$$\Omega=r^{-1/2}\left(\frac{d\psi_0}{dr}\right)^{1/2},\quad l=r^{3/2}\left(\frac{d\psi_0}{dr}\right)^{1/2},\quad v=r^{1/2}\left(\frac{d\psi_0}{dr}\right)^{1/2}. \tag{27}$$

With this interpretation, all the Eqs. (3–17) are just as valid for thick disks as for thin ones. When we make the further basic (and nearly universal) assumption that the self-gravity of the disk is negligible, we may choose Eq. (18) as the potential, with M being taken as the mass of the black hole. [For investigations into accretion disks where self-gravity may be dominant see Paczyński (1978a, 1978b) and Kozłowski et al. (1979).

In order to progress further we must decide upon the best way of finding the flux radiated from the surface of the disk. It is important to recall that the typical "α disk" models should puff up in the inner regions, where radiation pressure dominates, electron scattering provides the bulk of the opacity, and where the disk is optically thick. Thus it is natural to assume that the disk is radiating critically, just as a stellar atmosphere under the same conditions would (Paczyński, 1978a). We thus take the power

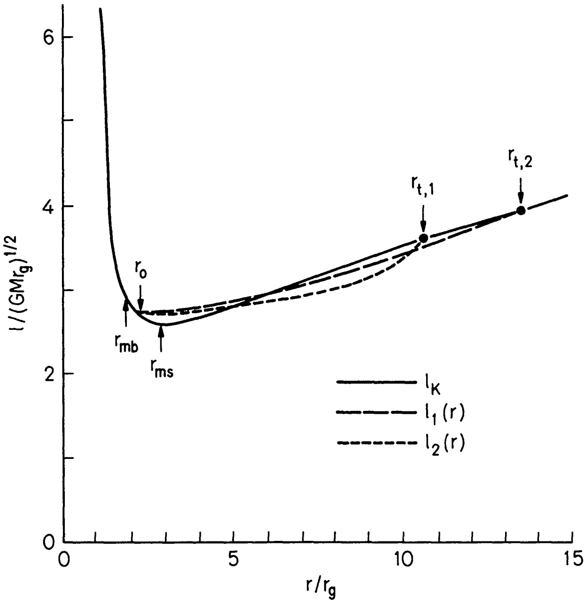

Fig. 1. The Keplerian specific angular momentum curve for the pseudo-Newtonian potential, $l_K(r)$, and two arbitrary angular momentum distributions that satisfy stability requirements. The marginally bound r_{mb} and marginally stable r_{ms} orbits are indicated, as are the inner radius r_0 and transition radii $r_{t,1}$, $r_{t,2}$ for the thick portions of the accretion disks

emitted per unit surface area as

$$F_{\text{rad}}=\frac{c}{K}g_{\text{eff}}, \tag{28}$$

where K is the opacity per gram and g_{eff} is the effective acceleration of gravity, including the centrifugal force. It is obvious that g_{eff} is normal to the surface of the disk and that we have

$$g_z=g_{\text{eff}}\cos\theta, \tag{29}$$

where $g_z=\frac{\partial\psi_0}{\partial z}$ and θ is the instantaneous angle that the surface of the disk makes with the radial direction, i.e., $\cos\theta=\left[1+\left(\frac{dz_0}{dr}\right)^2\right]^{-1/2}$. Thus we may use Eq. (29) in Eq. (28) to obtain

$$F_{\text{rad}}=\frac{c}{K}\frac{\partial\psi(r,z_0)}{\partial z}\left[1+\left(\frac{dz_0}{dr}\right)^2\right]^{1/2}. \tag{30}$$

b) Angular Momentum Distributions and the Form of the Disk

The time has now come to specify how we construct our disk models, and the interplay of angular momentum distributions becomes crucial. In Fig. 1 we the pseudo-Keplerian $l_K(r)$ curve given by Eq. (19) as well as two other arbitrary angular momentum distributions that characterize possible disks, and which must have the following properties: (1) $l(r_0)=l_K(r_0)\equiv l_0$; (2) $dl/dr\geqq 0$, for $r\geqq r_0$; (3) $l(r)=l_K(r)$ at two values of $r>r_0$. The first condition is merely our boundary condition; the second is needed to ensure a stable disk (cf. Abramowicz et al., 1978). We impose the third so that the disk may again be thin at the second intersection, the transition radius r_t, in the sense that $z_0(r_t)\ll r_t$. At that point we can allow the disk to have once again a Keplerian distribution of angular momentum, for such a distribution will be stable for $r>r_{ms}$. If we have chosen $l(r)$ in such a way that $z_0(r_t)\ll r_t$ then the region $r>r_t$ can be adequately described by standard thin disk models because in that region $\psi_0(r)=\psi(R)\approx\psi(r)$. However, for $r_0\leqq r<r_t$ the ratio z_0/r may not be negligible, and $\psi_0(r)\not\approx\psi(r)$, so the additional freedom allowed by a thick disk is necessary.

We can find the thickness of this thin disk at the transition radius if we assume that it is radiating critically there. It may not satisfy this assumption at larger radii and it will be necessary to check *a posteriori* that the physical preconditions for this assumption are valid (see Sect. 4c). If the disk is thin and radiating critically, then Eq. (30) can be replaced by

$$F_{\text{rad}}=\frac{c}{K}g_z=\frac{c}{K}\frac{\partial\psi}{\partial r}\frac{z_0}{r}=Q_-/2. \tag{31}$$

The energy generated within the disk is Q_+ which is given by Eq. (17). Since the disk *is* thin at this large radius we may take $Q_+=Q_-$, and for a thin disk under these conditions we have

$$z_0(r)=\left(\frac{-\dot M}{4\pi}\right)\frac{K}{c}\left(\frac{\partial\psi}{\partial r}\right)^{-1}\left(\frac{-d\Omega}{dr}\right)(l(r)-l_0). \tag{32}$$

When the derivatives and angular momentum are evaluated at r_t we obtain a value of z_t that depends only on $\dot M$, which in turn is totally determined by $l(r)$, as we shall see in the next subsection [Eq. (46)].

The basic constraint upon a consistent disk is the requirement that the thickness of the fat portion, given by Eq. (37) below, matches that of the thin portion, determined from Eq. (32, when both are evaluated at r_t. However we stress that the calculations in the next subsection are independent of this matching *via* Eqs. (31) and (32), so if a different method is proposed to obtain the thickness of the thin disk, the procedure we are about to describe could also be employed to satisfy it.

c) The Shape of the Thick Portion

It is not very difficult to obtain the differential equation that yields the shape of the disk as a function of the radius and the angular momentum distribution. If we define a total potential $\phi\equiv\psi-\int\Omega^2rdr$, the surface of the disk is a generalized equipotential on which ϕ is constant. Thus we must have

$$d\phi=0=\left(\frac{\partial\psi}{\partial r}-\Omega^2r\right)dr+\left(\frac{\partial\psi}{\partial z}\right)dz \tag{33}$$

on the surface, or

$$\frac{dz}{dr}=\left(\Omega^2r-\frac{\partial\psi}{\partial r}\right)\left(\frac{\partial\psi}{\partial z}\right)^{-1}. \tag{34}$$

Substituting in Eqs. (18) and (23) this is transformed to

$$\frac{dz}{dr}=\frac{r}{z}\left\{\frac{\Omega^2}{GM}(r^2+z^2)^{1/2}[(r^2+z^2)^{1/2}-r_g]^2-1\right\}. \tag{35}$$

By a simple manipulation this takes the form

$$\frac{dR^2}{dr^2}=\frac{\Omega^2}{GM}R(R-r_g)^2,$$

whose solution is

$$R^2=\left[(r_0-r_g)\left(1-\frac{r_0-r_g}{GM}\int_{r_0}^{r}\Omega^2rdr\right)^{-1}+r_g\right]^2, \tag{36}$$

where we have used the boundary condition $R(r_0)=r_0$. Thus the surface of the disk is given by

$$z_0=\left\{\left[\frac{r_0-r_g}{1-\frac{r_0-r_g}{GM}\int\limits_{r_0}^{r} l^2(r)r^{-3}dr}+r_g\right]^2-r^2\right\}^{1/2}, \tag{37}$$

where we have employed the relation $l=\Omega r^2$.

d) The Luminosity and Accretion Rate

As we have presumed the disk to be radiating critically it is a simple matter to calculate the total energy radiated by the thick portion of the disk. When the disk is not nearly flat we must use the fact that an element of surface area will be given by the expression $d\sigma=(rd\psi)(\sec\theta\, dr)$, *so that*

$$L_{\rm rad}=2\int\limits_{0}^{2\pi}\int\limits_{r_0}^{r_t} F_{\rm rad}d\sigma, \tag{38}$$

where both surfaces of the disk have been included. When we insert Eq. (30) into Eq. (38) and evaluate $\sec\theta$ in the expression for $d\sigma$ we obtain

$$L_{\rm rad}=\frac{4\pi c}{K}\int\limits_{r_0}^{r_t}\frac{\partial\psi_0}{\partial z}\left[1+\left(\frac{dz_0}{dr}\right)^2\right]rdr. \tag{39}$$

But now that we have the solution for $z_0(r)$, this can be expressed as

$$L_{\rm rad}=\frac{4\pi cGM}{K}\int\limits_{r_0}^{r_t}\left[\frac{l^4R(R-r_g)^2}{G^2M^2zr^5}-\frac{2l^2}{GMzr}+\frac{rR}{z(R-r_g)^2}\right]dr. \tag{40}$$

Let us now find the total amount of energy that is generated within the thick portion of the disk by intergrating the columnal energy generation over the appropriate region. We shall derive a general expression for the energy generated between any two radii and then specialize to our particular case.

$$L_{\rm gen}(r_1,r_2)=\int\limits_{r_1}^{r_2} Q_+2\pi r dr. \tag{41}$$

Using Eq. (17) this becomes

$$\begin{aligned}L_{\rm gen}(r_1,r_2)&=(-\dot M)\int\limits_{r_1}^{r_2}\left(-\frac{d\Omega}{dr}\right)(l-l_0)dr\\&=(-\dot M)\int\limits_{\Omega_2}(l-l_0)d\Omega\\&=(-\dot M)\left[\int\limits_{\Omega_2}^{\Omega_1} r^2\Omega d\Omega-l_0(\Omega_1-\Omega_2)\right],\end{aligned}$$

where we first change variables and then use the definition of l.

Integrating by parts we have

$$L_{\rm gen}(r_1,r_2)=(-\dot M)\left[\tfrac{1}{2}r^2\Omega^2\Big|_{\Omega_2}^{\Omega_1}+\int\limits_{r_1}^{r_2}\Omega^2 rdr-l_0(\Omega_1-\Omega_2)\right].$$

We now use the definition of v and Eq. (27) to change this to

$$L_{\rm gen}(r_1,r_2)=(-\dot M)[\tfrac{1}{2}v_1^2-\tfrac{1}{2}v_2^2+(\psi_0)_2-(\psi_0)_1-l_0(\Omega_1-\Omega_2)].$$

If the disk is thin enough *at both relevant radii*, then $\psi_0\approx\psi$, and with a little more manipulation we have

$$L_{\rm gen}(r_1,r_2)=(-\dot M)[(e_2-e_1)+\Omega_1(l_1-l_0)-\Omega_2(l_2-l_0)], \tag{42}$$

where the definition of the binding energy, Eq. (2), has been employed.

The special case we are interested in is $r_1=r_0$ and $r_2=r_t$, but we first obtain the well known general relation that the total energy produced within the disk is

$$L_{\rm gen,\,total}=(-\dot M)(-e_0), \tag{43}$$

which comes from Eq. (42) because $e_\infty=0$ and $\Omega_\infty\cdot l_\infty=0$. Returning to the total energy produced in the thick region of the disk:

$$L_{\rm gen}(r_0,r_t)=(-\dot M)[(e_t-e_0)-\Omega_t(l_t-l_0)]. \tag{44}$$

At this point we must make one more critical assumption; that the material flowing into the disk at r_t and flowing out of it at r_0 carries negligible internal energy in comparison with its mechanical energy. If this is true we can replace the local condition $Q_+=Q_-$ with the global one

$$L_{\rm gen}=L_{\rm rad}, \tag{45}$$

which is physically justifiable. A given distribution $l(r)$ will imply a radius r_t, which, along with r_0, enables us to integrate Eq. (40) to find $L_{\rm rad}$. We then know $L_{\rm gen}$ and everything on the right hand side of Eq. (44) except the accretion rate, which we solve for:

$$-\dot M=L_{\rm rad}[(e_t-e_0)-\Omega_t(l_t-l_0)]^{-1}. \tag{46}$$

4. Calculations and Results

a) Procedure for Constructing Models

We now draw together the threads of the previous section and show how we weave complete thick disk models. From now on we present most of the results in terms of non-dimensional units, since everything scales directly with the mass of the black hole. We take $GM=1$, and r_g as our unit of length. As a first step we choose a potential and write Eq. (18) as

$$\psi=-1/(R-1). \tag{47}$$

The second key step is to choose a value of $r_0\in(2,3]$. Since angular momentum can be expressed in units of $(GMr_g)^{1/2}$ we have

$$l_K(r)=r^{3/2}/(r-1), \tag{48}$$

with $l_0=r_0^{3/2}/(r_0-1)$ as the point from which we start the assumed angular momentum distribution $l(r)$, chosen as step three in our procedure.

The first computational step is to find the two secondary intersections of $l(r)$ and $l_K(r)$. If they do not exist then that distribution of $l(r)$ is inadequate and a new one is chosen. This fourth step is followed by the calculation of the shape and luminosity of the disk between r_0 and r_t. The solution for z_0 is written

$$z_0(r)=\left\{\left[\frac{r_0-1}{1-(r_0-1)\int\limits_{r_0}^{r} l^2(r)r^{-3}dr}+1\right]^2-r^2\right\}^{1/2}. \tag{49}$$

Defining $L_{\rm edd}\equiv4\pi cGM/K$ as the Eddington, or critical, luminosity (using a Newtonian potential and spherical symmetry) we may write

$$L_{\rm rad}=L_{\rm edd}I(r_0,r_t), \tag{50}$$

where $I(r_0,r_t)$ is the non-dimensional form of the integral appearing in Eq. (40). Although the effective gravity at any point on the

Table 1. Parameters of thick disks for $\beta=1.0$

r_0	$A_{1.0}$	r_t	$\frac{z(r_t)}{r_t}$	$\left[\frac{z(r)}{r}\right]_{\max}$	L	$-\dot{M}$	η^*	η'	L_{thin}
2.05	0.015407	3847.6	0.135	4.406	1.19 (39)	1.19 (20)	0.0112	0.0113	2.07 (37)
2.10	0.027969	1078.3	0.202	2.490	9.20 (38)	5.12 (19)	0.0200	0.0207	3.01 (37)
2.20	0.047920	319.8	0.252	1.425	6.07 (38)	2.07 (19)	0.0326	0.0347	3.95 (37)
2.30	0.063301	162.6	0.281	1.021	4.54 (38)	1.25 (19)	0.0404	0.0444	4.50 (37)
2.40	0.075585	102.6	0.289	0.796	3.48 (38)	8.62 (18)	0.0449	0.0510	4.74 (37)
2.50	0.085599	72.7	0.277	0.648	2.69 (38)	6.35 (18)	0.0472	0.0556	4.77 (37)
2.60	0.093951	55.4	0.273	0.542	2.16 (38)	5.01 (18)	0.0480	0.0586	4.79 (37)
2.70	0.100989	44.3	0.265	0.462	1.75 (38)	4.10 (18)	0.0476	0.0606	4.76 (37)
2.80	0.106970	36.6	0.252	0.399	1.44 (38)	3.43 (18)	0.0465	0.0617	4.68 (37)
2.90	0.112134	31.1	0.243	0.348	1.19 (38)	2.94 (18)	0.0448	0.0623	4.60 (37)
3.00	0.116419	26.9	0.238	0.306	9.92 (37)	2.58 (18)	0.0428	0.0625	4.55 (37)

surface of the disk will be less than the purely spherical attraction to the central mass at that distance, and thus the critical flux from the disk's surface would be less than that from a spherical surface at the same radius, we will find that our disks may have surface areas far in excess of such equivalent spheres. Thus there is no reason to require that $I(r_0,r_t)\leqq 1$, and it need not surprise us if the total emitted energy exceeds the Eddington luminosity. [Geometrical effects needed in computing the ratio of gravitational to radiative pressure forces were also considered by Bisnovatyi-Kogan and Blinnikov (1977), but in the framework of test particles near a thin disk.]

We can now proceed to the sixth step, which is the calculation of $\dot{M}$ by equating the total energy generated with the total radiated luminosity. We first define a critical accretion rate in the Newtonian fashion,

$$-\dot{M}_{cr}=\frac{2r_0}{GM}L_{\text{edd}}=8\pi c r_0/K \tag{51}$$

and then define the reduced accretion rate as:

$$\dot{m}=(-\dot{M})/(-\dot{M}_{cr})\,. \tag{52}$$

We rewrite Eq. (44) for the energy generated

$$\begin{aligned}L_{\text{gen}}(r_0,r_t)&=(-\dot{M})\frac{GM}{r_g}\left\{\frac{1}{2}\left[\frac{r_0-2}{(r_0-1)^2}-\frac{r_t-2}{(r_t-1)^2}\right]-\frac{l_t}{r_t^2}(l_r-l_0)\right\}\\&\equiv(-\dot{M})GMK(r_0,r_t)/r_g\,,\end{aligned} \tag{53}$$

so that Eq. (46) is replaced by

$$\dot{m}=I(r_0,r_t)/(2r_0K(r_0,r_t))\,. \tag{54}$$

Our final step, the seventh, is to compute $z_0(r_t)$ for the thin disk using Eq. (32) and see if it matches the value for the thick disk given by Eq. (49). If they agree to within 1% we consider that a consistent match has been found, although it must be confirmed that the errors caused by the finite value of $z(r_t)/r_t$ are small. On the other hand, if they do not agree, we must modify the angular momentum distribution (step three) and repeat stages three through seven until a suitable match is found.

b) Specific Results

For simplicity we chose to look at the most basic forms of angular momentum distributions that were likely to fit the various necessary conditions. We examined the simply two parameter family,

$$l(r)=l_0+A_\beta(r-r_0)^\beta\,, \tag{55}$$

with $A_\beta\geqq 0$ so that $dl/dr\geqq 0$ and the disk is not unstable. The requirement that S be non-negative implies that we must have $d\Omega/dr\leqq 0$ [cf. Eq. (15)] which imposes a complex constraint on the combination of A_β and β, but certainly requires $\beta\geqq 1$, for otherwise (with $A_\beta\neq 0$) $d\Omega/dr\to+\infty$ as $r\to r_0$. When these simple relationships are chosen, the integral,

$$\int_{r_0}^{r} l^2(r)r^{-3}\,dr \tag{56}$$

which appears in the expression for $z_0(r)$ can be performed analytically. Thus, in this case, the only integral we must evaluate numerically is $I(r_0,r_t)$ of Eq. (40).

If A_β is chosen too large there is no hope of a solution, as $l(r)>l_K(r)$ everywhere. As we reduce A_β two intersections with the Keplerian curve are found, but the integral (56) grows too quickly and $z_0(r)\to\infty$ for some $r<r_t$. If A_β is taken too small, r_t becomes very large and we get $z_0(r)=0$ at some $r<r_t$. We could actually find values of $A_\beta=A_\beta^0$ that yield $z_0(r_t)=0$, but these accretion rings (like the solutions of Kozłowski et al., 1979) would not match onto exterior "thin" disks, and the value of A_β must be taken somewhat larger than A_β^0 if we are to obtain an equality between Eqs. (49) and (32).

Let us consider linear angular momentum distributions first, where $\beta=1.0$. Eleven values of r_0, ranging from 2.05 to 3.00 were considered, and for each the value of $A_1(r_0)$ that yielded an acceptable solution was obtained, using the iterative procedure we have just described. Our basic results are summarized in Table 1, where we have listed for each r_0 the corresponding $A_1(r_0)$, r_t, $z(r_t)/r_t$, and the maximum value of the ratio $z_0(r)/r$ in the first five columns. We also give the values of L and $-\dot{M}$ (in cgs units) per solar mass of $\dot{M}$, where K was taken as $0.34\,\text{cm}^2\,\text{g}^{-1}$, appropriate to solar envelope material. From these we calculate an efficiency factor η^* using the relation

$$L=\eta^*(-\dot{M})c^2 \tag{57}$$

and compare it with the theoretical value, η', found from Eq. (21). Because the total generated energy is given by Eq. (43), the luminosity radiated in the outer, thin, portion of the disk is

$$L_{\text{thin}}=(\eta'-\eta^*)(-\dot{M})c^2 \tag{58}$$

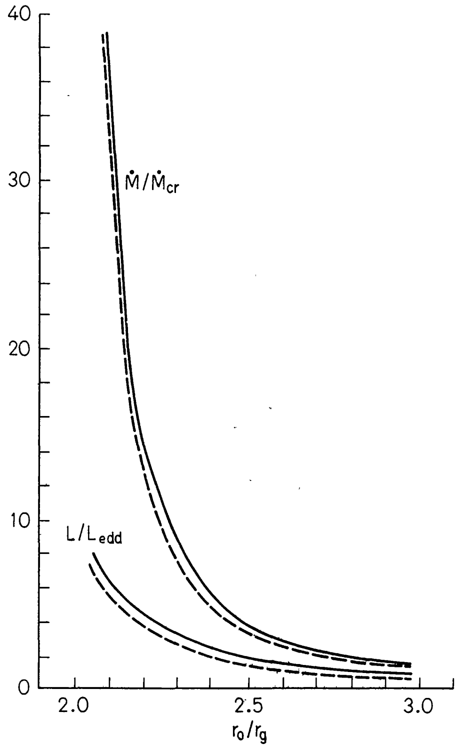

Fig. 2. Curves of L/L_{edd} and $\dot{M}/\dot{M}_{cr}$ vs. r_0 for two families of angular momentum distributions, $\beta=1.0$ and $\beta=1.1$

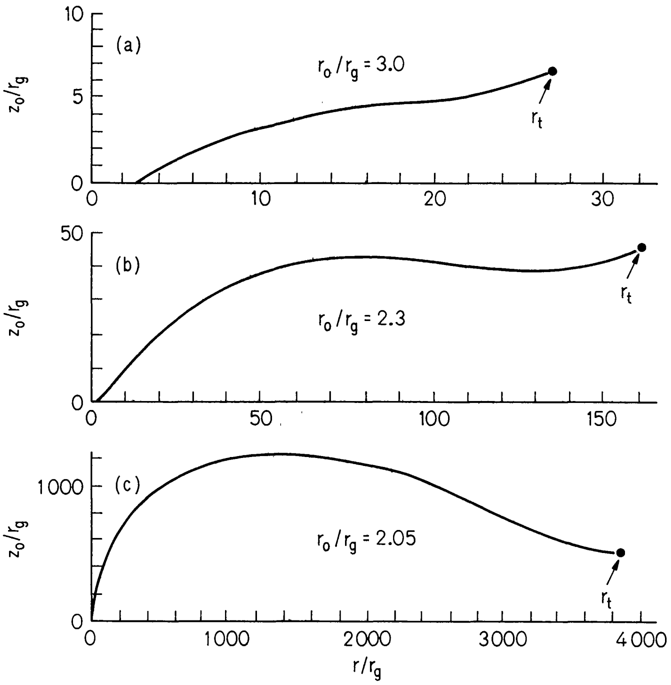

Fig. 3. **a** The shape of the thick portion of a disk (one quadrant only) constructed with $l(r)=l_0+A(r-r_0)$ and $r_0=3.0\,r_g$. **b** The same as **a** for $r_0=2.3\,r_g$. **c** The same as **a** for $r_0=2.05\,r_g$

and is tabulated in the last column of Table 1. We note that the fraction of the total luminosity contributed by the thick portion drops from 0.983 for $r_0=2.05$ to 0.686 for $r_0=3.00$. We also present the ratios L/L_{edd} and $\dot{m}$ graphically in Fig. 2. The shapes of our thick disks are shown for three values of r_0 in Fig. 3; note the different scales in the sections of this figure.

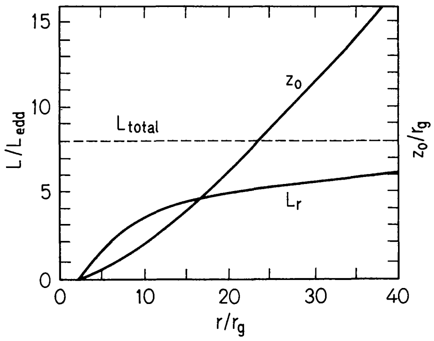

Fig. 4. The region near the cusp for the case $r_0=2.05\,r_g$ and $\beta=1.0$, expanded from Fig. 3c. Also shown is the luminosity emitted between r_0 and r, as well as the total luminosity of this thick disk model

One result is that $\dot{m}$ increases dramatically as r_0 decreases below 3, but this is to be expected as η^* is bounded above by η', which vanishes at $r_0=2$. A far more interesting result is that we find $L>L_{edd}$ for $r_0<2.8$ (when $\beta=1.0$); most of this excess is real and can be understood in terms of the geometrical effects mentioned in the previous subsection, other reasons for it are discussed below. We also note that as r_0 decreases, r_t increases, so that the thick portions of the disk become very large indeed. Not only do these thick disks become longer, but they also become fatter as the maximum value of z/r increases rapidly as r_0 approaches r_{mb}.

It is of interest to check from which portions of the disk the bulk of the energy emerges. As $r_0 \to r_{mb}$ a larger fraction of the total luminosity is emitted from a relatively small fraction of the surface close to the cusp where the disk resembles a tunnel or funnel down towards the black hole. The shape of the innermost portion of our most extreme case, $r_0=2.05$ and $\beta=1.0$ is given in an expanded scale in Fig. 4. Also shown in Fig. 4 is the fraction of the luminosity $L(r)$, emitted between r_0 and r. Even though $r_t=3848$, and $L(r_t)/L_{edd}=8.07$, we see that 25% of the energy is emitted for $r<10$, 50% for $r<14$, 75% for $r<40$, and 90% for $r<290$.

We next considered a different family of disks by constructing models with $\beta=1.1$ in Eq. (55). Such disks have $dl/dr=0$ at r_0, and this may have some physical justification (Kozłowski et al., 1978). Our results for these models are summarized in Table 2, where the same quantities given for the linear angular momentum case in Table 1 are tabulated for these steeper disks. The $\beta=1.1$ case differs uniformly from the $\beta=1.0$ case in the following ways: $\dot{m}$, L, and r_t are all reduced, but $z(r_t)/r_t$ is increased, thus implying less accuracy in these models. These values of $\dot{m}$ and L/L_{edd} are also given in Fig. 2. Here the thin portion of the disk contributes a relatively greater part of the emission, and the fraction of the luminosity radiated from the thick part drops from 0.953 to 0.446 as r_0 goes from 2.05 to 3.00. All of these trends continue in the few cases we have calculated for higher values of β.

However, we found it impossible to construct disks for values of $\beta \gtrsim 1.5$, since the values of r_t become quite small while $z(r_t)$ was greater than r_t. The nominally large values of $\dot{M}$ led to the contradictory situation where the thickness evaluated on the "thin" side of the transition radius was always greater than that evaluated on the "thick" side.

c) Tests and Caveats

The first obvious check is to examine the validity of our assumption that $z/r \ll 1$ at r_t. For the $\beta=1.0$ case this ratio ranged between

Table 2. Parameters of thick disks for $\beta=1.1$

r_0	$A_{1.1}$	r_t	$\frac{z(r_t)}{r_t}$	$\left[\frac{z(r)}{r}\right]_{max}$	L	$-\dot{M}$	η^*	η'	L_{thin}
2.05	0.012287	1344.8	0.354	3.315	1.09 (39)	1.12 (20)	0.0101	0.0113	5.36 (37)
2.10	0.022334	453.5	0.409	1.980	8.02 (38)	4.66 (19)	0.0192	0.0206	6.34 (37)
2.20	0.038649	156.5	0.429	1.171	5.16 (38)	1.88 (19)	0.0306	0.0347	6.95 (37)
2.30	0.051643	86.1	0.431	0.843	3.70 (38)	1.11 (19)	0.0372	0.0444	7.18 (37)
2.40	0.062313	55.7	0.409	0.654	2.73 (38)	7.51 (18)	0.0405	0.0510	7.11 (37)
2.50	0.071312	40.3	0.387	0.527	2.09 (38)	5.60 (18)	0.0416	0.0556	7.04 (37)
2.60	0.078977	31.0	0.357	0.435	1.61 (38)	4.36 (18)	0.0411	0.0586	6.84 (37)
2.70	0.085781	24.7	0.341	0.366	1.27 (38)	3.57 (18)	0.0395	0.0606	6.75 (37)
2.80	0.096250	20.3	0.311	0.311	9.91 (37)	2.96 (18)	0.0372	0.0617	6.53 (37)
2.90	0.097000	17.0	0.290	0.290	7.81 (37)	2.54 (18)	0.0342	0.0623	6.44 (37)
3.00	0.105000	12.2	0.332	0.332	6.30 (37)	2.51 (18)	0.0279	0.0625	7.82 (37)

0.13 and 0.29; however, we can see that these values are sufficiently small that the approximation $\psi_0(r_t)=\psi(r_t)$ remains accurate to better than 4% and the error this implies for $\dot{M}$ is even smaller. The larger values of z/r allowed when $\beta=1.1$ mean that the difference between $\psi_0(r_t)$ and $\psi(r_t)$ can be as big as 9%, so that larger values of β would yield very substantial errors.

The physical assumption that the disk is radiating critically [Eq. (28)] can also be partially verified *a posteriori*. In order for this to be true we must certainly have electron scattering as the dominant source of opacity and it is also essential that radiation pressure dominates, at least near the surface of the disk. However our models have made no detailed assumptions or predictions about the equation of state or viscosity mechanism, so that the only way to check this point is by indirect comparison with some other models. Here we consider the "α disk" models investigated by Shakura and Sunyaev (1973), where the boundary between "zone *a*" (radiation dominating pressure, electron scattering dominating opacity) and "zone *b*" (gas pressure dominating, still electron scattering) is given approximately by

$$r_{ab}=89\,r_0(\alpha m)^{2/21}\dot{m}^{16/21}, \tag{59}$$

where $\alpha<1$ is the canonical measure of viscosity, $m=M/M_\odot$, and we have renormalized Shakura and Sunyaev's expression using our definition of $\dot{m}$ [Eq. (52)]. Then $r_t<r_{ab}$ is a necessary condition for the consistency of our disks and this can be translated into a bound on the product αm. We find that this becomes only slightly restrictive for small r_0 and β, for when $\beta=1.0$ and $r_0=2.05$ we must demand $\alpha m>0.023$, which is, however, satisfied for most values of α. At larger r_0 values this condition is almost certainly met, for at $r_0=2.10$ it is $\alpha m>2.8\,10^{-5}$ and at $r_0=3.0$ it is $\alpha m>4.2\,10^{-12}$. Increasing β decreases r_t, so even for the "worst" case that we have looked at $\beta=1.1$, $r_0=2.05$, this condition is $\alpha m>5.6\,10^{-7}$. Of course $m>1$ is expected for all black holes, so the above bounds are even less strict when applied to α alone.

Another basic assumption, that the self-gravity of our disks is negligible, is connected with our choice of Eq. (18) or Eq. (47) as the potential. Because $\dot{m}$ does get very large it is not obvious that this is true for all our models, and it is difficult to estimate the total mass of the disk within our framework. It is likewise difficult to check that the flux of internal energy is negligible compared to mechanical energy, and thus that our global condition Eq. (45) is valid, without making further assumptions about the equation of state and viscosity. Preliminary estimates indicate that neither of these assumptions is bad for $r_0\gtrsim 2.2$, but the very high accretion rates and fluxes predicted when $r_0\approx r_{mb}$ will certainly be reduced by the latter condition breaking down.

It should also be mentioned that while we have deliberately built our models so that the thickness is continuous at the transition radius, the flux F_{rad} need not be, since the slope dz_0/dr is discontinuous across this boundary. If we went to the additional trouble of requiring that $\left.\frac{dl(r)}{dr}\right|_{r_t}=\left.\frac{dl_K(r)}{dr}\right|_{r_t}$ then this minor problem could be alleviated, but we would have to make the postulated angular momentum variation substantially more complex. We feel that little additional light would be thrown on the subject by adding this refinement.

Yet another source of inaccuracy is our neglect of the red-shift of the emitted radiation, for our pseudo-Newtonian potential introduces a gravitational shift that mimics that of the Schwarzschild metric. The basic effect would be to reduce the observed flux from the innermost portions of the disk, both by decreasing the frequency of the photons emitted and by lengthening the observed interval between them. This gravitational red-shift will be coupled with a rotational one which will also be strong in the inner portions of the disk and may either increase or decrease the observed flux, depending upon the observer's orientation. One further effect that has been neglected, but is certainly not negligible for the central part of the disk, is the capture of radiation by the black hole, for a fair amount of the radiation emitted from the inward facing portion will be swallowed. Only a fully relativistic treatment can solve the question of what the disk will look like to an external observer (cf. Sikora, 1980).

5. Discussion and Conclusions

By making some simple physical assumptions we have been able to construct models of the bloated inner portions of accretion disks around black holes. These results are essentially independent of assumed viscosity laws and they satisfy several self-consistency tests. The procedure we have employed could be generalized to the Schwarzschild metric without much difficulty and it is possible that other extensions could be made.

Very large amounts of mass can be swallowed, and a nominally stable disk can be formed as we do not require $Q_+=Q_-$. "Supercritical luminosities" are possible because we assume that the disk is radiating at the local equivalent of the Eddington flux, and even though the effective gravity is smaller, the bloated shape provides a very large surface area that allows the disk to dissipate much more energy than is possible under conditions of spherical symmetry. The reality of these "supercritical luminosities" will have to be verified in the future with fully general relativistic models.

We have only calculated the shapes and luminosities for two simple families of angular momentum distributions. However, within the power law form [Eq. (55)] we have come close to exhausting the valid possibilities, because $\beta<1$ yields positive angular velocity gradients implying that angular momentum cannot be transferred outwards, while for $\beta\gtrsim 1.5$ no matching thin solution can be found. Thus, working within this framework, we have a fairly narrow relationship between L or $\dot{M}$ and r_0. Because the case $\beta=1.0$ is at an analytical limit and also yields the maximum values for L and $\dot{M}$, we can be fairly confident that we have found approximate absolute maxima of these quantities. Previous calculations have assumed that the lowest allowed value for r_0 is r_{ms} and in this case we do find $L\lesssim 0.66\,L_{\mathrm{edd}}$, in agreement with earlier workers (e.g. Bisnovatyi-Kogan and Blinnikov, 1977).

The disk shapes we have discovered range from quite bloated ones that thin substantially before r_t (for small r_0), to disks that increase monotonically in thickness until r_t is reached, but which can still be considered to be relatively thin there (for large r_0). We did not find disks which matched onto external portions which became *physically* thin, i.e., z/r_g was always greater than unity at r_t.

The calculations presented here are just an early stage in our understanding of the effects that cusps on the inner edges of accretion disks can produce. Only if other angular momentum distributions are considered will we be confident that $\beta=1.0$ really does yield maximum values. In general, we expect the cusp to open up somewhat and the accretion will resemble material flowing through the inner Lagrangian point in a binary system. Including this effect will allow estimates to be made of the infall velocities and the amount of internal energy gobbled up by the hole. The simple approximation of Kozłowski et al. (1978) could be applied to our models to determine these quantities approximately. Related to this feature is the expectation that hydrostatic equilibrium in the inner region will be violated to some extent, and that this would tend to reduce our $\dot{M}$ and L values.

As mentioned previously, red-shift effects and the trapping of radiation by the hole must be considered in a more accurate treatment. We should also note that the inward facing parts of the disk will be subjected to strong radiative bombardment from the opposite portion of the surface. This will evaporate part of the disk and cause it to puff up even more. The cavities produced on either side of the black hole will contain huge radiation energy densities and could provide collimated beams for quasars and radio galaxies, a prospect that has been suggested many times previously in one form or another (Lynden-Bell, 1978; Rees, 1978 and references therein). The extreme thickness of the disk outside the hole, and the sharp drop down towards the cusp, where the bulk of the energy is radiated, suggest a crude analogy with an ordinary star into which two tunnels have been bored, allowing us to see directly the high radiation densities in its core. This approach to the structure of the inner region may be very fruitful in the investigation of the details of such collimation. By combining a fully general relativistic approach to the tracing of photon trajectories (Sikora, 1980) with an iterative attack on the reflection of radiation in the disk, the appearance of such a disk to a distant observer could be found. Finally, more general equations of state and geometries ought to be investigated by a suitable generalization of our method.

All of the above problems deserve further attention and development, and we plan to report on such elaborations in the future.

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Note added in proof. A theory of non-barotropic thick disks in general relativity has been recently published by M. Jaroszyński, M. Abramowicz and B. Paczyński, 1980, *Acta Astron.* **30**, 1.

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