ISSN 0004-6337

Astron. Nachr., Bd. 299, H. 5 (1978)

On the Evaluation of an Integral Connected with the Thermonuclear Reaction Rate in Closed Form

H. J. HAUBOLD and R. W. JOHN, Potsdam

Zentralinstitut für Astrophysik der Akademie der Wissenschaften der DDR

(Received 1978 February 27)

Dedicated to the Member of the Academy of Sciences of G.D.R. Professor Dr. habil. Dr. e. h. HANS-JÜRGEN TREDER at his 50th birthday

The standard expression of the reaction rate for low-energy, nonresonant nuclear reactions in nondegenerate plasma contains a parameter-dependent integral which in all previous calculations with physical or astrophysical background is considered as not capable of being evaluated in a closed form. So one usually resorts to approximation methods concerning large values of the parameter. At first we point out that CONSUL (1964) has given a series representation of the integral which was identified with a MEIJER's G-function by MATHAI (1971). Next, in view of a physically more exact determination of the reaction rate formula, especially in connection with calculations concerning stellar energy generation, we consider a more general integral containing the mentioned one as special case and give an approximation-free representation by means of MEIJER's G-function. The G-function so obtained may be conceived as complex-valued continuation of CONSUL's series representation of a certain class of integrals contained in the considered one. From the series we extract a small parameter approximation of the special integral.

Скорость реакции для низкоэнергетических нерезонансных ядерных реакций в невырожденной плазме содержит интеграл, который зависит от параметра. Во всей имеющейся в наличии физической и астрофизической литературе этот интеграл рассматривается лишь приближённо, для больших значений параметра. Было обращено внимание на тот, что Консул (1964) представил интеграл бесконечным рядом, который Матай (1971) отождествил с G-функцией Мейера. Затем, ввиду физически более точного определения формулы скорости реакции. в особенности в связи с вычислениями, касающимися генерации энергии внутри звёзд, был рассмотрен более общий интеграл, содержащий как частный случай вышеупомянутый интеграл, и показано его представление в замкнутом виде с помощью G-функции Мейера. Так полученная G-функция даёт комплекснозначное продолжение ряда Консула для класса интегралов рассматриваемого типа. Из ряда было выделено приближение специального интеграла для случая малых значений параметра.

1. Introduction

The theoretical determination of the reaction rate for low-energy nonresonant thermonuclear reactions in nondegenerate nuclear plasma carried out by means of conceptions from nuclear physics and kinetic theory of gases leads to a parameter-dependent integral. Till now, in connection with nuclear physics this integral was approximately evaluated only, for large values of the parameter (cf. e.g. COX and GIULI 1968 or, more recently, MITTLER 1977). Proceeding in this way, the mathematically more exact calculation of the nuclear reaction rate requires consideration of reaction specific correction factors to compensate for the error of mathematical approximation (SALPETER 1954, cf. COX and GIULI 1968). We draw the attention to CONSUL's series representation (1964) of the integral, which was identified with a MEIJER's G-function by MATHAI (1971). Then we consider a more general integral

$$L_\alpha(x) = \int_0^\infty \mathrm{d}y\, y^\alpha \exp\left[-y - xy^{-1/2}\right]$$

comprehending the special integral $L_0(x)$ of the reaction rate, refer to CONSUL's series representation (1964) of the integral in the case of non-negative integer α, and positive real x, and give a closed expression of the integral by means of MEIJER's G-function so continuating CONSUL's series to complex values of α and x. Giving a computable closed expression for the nuclear reaction rate in such a kind is advantageous with regard to several points. On the one hand, CONSUL's series expansion of the integral $L_0(x)$ allows to investigate the reaction rate quantitatively in the whole range of the parameter. On the other hand, because of the extraordinary functional properties of MEIJER's G-function the representation of the rate by such a function is qualitatively advantageous for using the reaction rate in an analytical framework resulting from the mathematical description of energy generation by fusion reactions, e.g. in stellar interiors, where in case of using an approximate expression the approximation error propagates in a complicated manner. One immediately obtains the energy generation rate for thermonuclear reactions in stellar interiors from the reaction rates of the individual nuclear reactions. In phases of hydrogen and helium burning (pp-chain and CNO-cycle) the energy loss by neutrino emission of the star is proportional to the number of proceeding fusion processes and therefore one can compute neutrino fluxes with the aid of nuclear reaction rates. Furthermore, by using the mathematically precise expression for the reaction rate involved in an analytical framework the influence of possible theoretic modifications of the framework on the result can be considered more exactly.

Moreover, the mathematical formalism of MEIJER's G-function, a generalization of the hypergeometric function, appears usefully to us for investigating possible modifications of the reaction rate formula itself for reasons from the kinetic theory of gases, quantum mechanics and nuclear physics, especially in connection with astrophysical circumstances. To determine the rate for an individual nuclear reaction is quite difficult both experimentally and theoretically. The physically more exact determination of the reaction rate, particularly under the state conditions in stellar interiors, causes alterations of the analytical structure of the rate integral.

In the first instance, in section 2 we are dealing with the theoretical determination of the low-energy nonresonant part of the thermonuclear reaction rate for exothermic nuclear reactions. In section 3 we are occupied with the exact evaluation of the general integral $L_\alpha(x)$. 3.1. contains the evaluation via CONSUL's solution of the associated differential equation problem for non-negative integer α and positive real x, in 3.2. we evaluate the integral by means of representation as LAPLACE transform of a MEIJER's G-function. Finally, in the appendix we approximately calculate the integral for large values of the parameter x.

2. The nonresonant thermonuclear reaction rate

We are concerned with an ensemble of interacting particles of types a and X of relative velocity v. As a result of thermally induced collisions among the particles proceeds the exothermic reaction

$$a + X \rightarrow Y + b\,.$$

We calculate the reaction rate r presenting how many reactions proceed per unit volume and per unit time as a function of density and temperature. $n(v)\,\mathrm{d}v$ is the fraction of the number of particles of the type a or X, whose velocity relative to a particle of the other type lies between v and $v + \mathrm{d}v$. N_a and N_X are the number densities of particles a and X, respectively, and $\sigma(v)$ is the cross section of the considered nuclear reaction. Hence, the thermonuclear reaction rate is obtained by averaging the reaction cross section over the normalized distribution function of relative velocities of the particles (cf. COX and GIULI 1968):

$$r = (1 - \tfrac{1}{2}\delta_{aX})\, N_a N_X \int_0^\infty \mathrm{d}v\, n(v)\, \sigma(v)\, v = (1 - \tfrac{1}{2}\delta_{aX})\, N_a N_X \langle \sigma v \rangle\,. \tag{1}$$

δ_{aX} is the KRONECKER symbol. The bracketed quantity $\langle\sigma v\rangle$ is the probability per unit time that two particles a and X confined to a unit volume will react with each other. In the following we suppose the nuclei to be in thermodynamical equilibrium with regard to their velocities (not with regard to their mass abundances) — that is usually assumed for thermonuclear reactions in stellar interiors. In the case of nondegenerate and nonrelativistic gas the distribution function of the relative velocities of the nuclei is a MAXWELL-BOLTZMANNian:

$$n(v)\,\mathrm{d}v = \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left[-\frac{\mu v^2}{2kT}\right] 4\pi v^2\,\mathrm{d}v \tag{2}$$

where μ is the reduced mass of the particles a and X, k is the BOLTZMANN constant, and T the absolute temperature.

For a more exact analysis of gaskinetic conditions in stellar interiors one could consider generalizations of the BOLTZMANN equation (e.g. consideration of fluctuations, triple collisions at higher densities, several boundary conditions; cf. COHEN and THIRRING 1973) and the resulting corrected distribution functions. Till now, in the literature concerning stellar nuclear reaction rates one finds ad-hoc-assumptions about alteration of the MAXWELL-BOLTZMANN distribution function for stellar plasmas (see CLAYTON 1974, VASIL'EV, KOČAROV and LEVKOVSKIJ 1974).

By means of the distribution function (2) one gets from eq. (1)

$$\langle\sigma v\rangle = \left(\frac{\mu}{2\pi kT}\right)^{3/2} 4\pi \int_0^\infty \mathrm{d}v\, v^3 \sigma(v) \exp\left[-\frac{\mu v^2}{2kT}\right]. \tag{3}$$

So for a given type of nuclear reaction $\langle\sigma v\rangle$ depends on the temperature only and is independent of density. In the nonrelativistic case for the kinetic energy holds $E = \mu v^2/2$. With that, according to (1) and (3) the nuclear reaction rate may be expressed as

$$r = \left(1 - \frac{1}{2}\delta_{aX}\right) N_a N_X \left(\frac{1}{\pi\mu}\right)^{1/2} \left(\frac{2}{kT}\right)^{3/2} \int_0^\infty \mathrm{d}E\, E \exp\left[-\frac{E}{kT}\right] \tilde{\sigma}(E)\,. \tag{4}$$

To compute the thermonuclear reaction rate, additional expressions giving the details of $\tilde{\sigma}(E)$ for important reactions are required. Now we consider the cross section for low-energy nuclear reactions $X(a, b)\,Y$, far from any resonances. Such reactions involving a nucleon and a nucleus or else two nuclei are in general described in the BOHR picture. Starting from the BREIT-WIGNER formula one obtains for the cross section of low-energy nonresonant nuclear reactions the expression (cf. COX and GIULI 1968; σ for $\tilde{\sigma}$)

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-2\pi\eta(E)\right], \qquad \left(\frac{E}{B} \ll 1\right), \tag{5}$$

where E is the relative kinetic energy between the particles, B the nuclear barrier height, $S(E)$ the cross section factor, and $\exp[-2\pi\eta(E)]$ the GAMOW factor:

$$\eta(E) = \left(\frac{\mu}{2}\right)^{1/2} \frac{Z_a Z_X e^2}{\hbar} \frac{1}{E^{1/2}} . \tag{6}$$

Z_a, Z_X: atomic numbers of nuclei a and X
e: quantum of electric charge
$\hbar = h/2\pi$, h: PLANCK's quantum of action

Because of the very small cross sections for charged particle reactions at low energies only particles with $l = 0$ (zero relative orbital angular momentum) make by far the greatest contribution to $\sigma(E)$. Therefore, deriving the cross section formula (5) only partial waves with $l = 0$, i.e. head-on collisions, are considered. The structure of σ considering non-central collisions is treated by FRANK-KAMENECKIJ (1959).

The cross section factor $S(E)$ represents the intrinsic nuclear characteristics of the probability for the ocurrence of a nuclear reaction. $S(E)$ is a slowly varying function of energy over a limited energy range and is computed from the measured values of $\sigma(E)$ using (5). Usually one takes $S = \text{const.} = S_0$. A better approximation to $S(E)$ is a MACLAURIN series expansion up to the second order in E:

$$S(E) \approx S(0) + S'(0)\, E + \frac{1}{2!} S''(0)\, E^2 .$$

For certain reactions $S(E)$ is represented by a power function (see FOWLER, CAUGHLAN and ZIMMERMAN 1975).

Further modifications of the quantum mechanical cross section under conditions of stellar interiors are needed to take into consideration electron screening (SALPETER 1954, cf. COX and GIULI 1968) and non-sphericity of nuclear potentials (FRANK-KAMENECKIJ 1959).

With eqs. (5) and (6) according to (3) we get the low-energy nonresonant part of the thermonuclear reaction rate (for $S(E)$ only $S(0) \equiv S_0$ is taken):

$$r = \left(1 - \frac{1}{2}\delta_{aX}\right) N_a N_X \left(\frac{1}{\pi\mu}\right)^{1/2} \left(\frac{2}{kT}\right)^{3/2} S_0 \int_0^\infty \mathrm{d}E \exp\left[-\frac{E}{kT} - 2\pi\eta(E)\right] =$$

$$= \left(1 - \frac{1}{2}\delta_{aX}\right) N_a N_X \left(\frac{2}{\pi\mu kT}\right)^{1/2} S_0 \cdot 2L_0(x) , \tag{7}$$

where $L_0(x)$ is the parameter-dependent integral

$$L_0(x) = \int_0^\infty \mathrm{d}y \exp\left[-y - xy^{-1/2}\right] , \tag{8}$$

$x = \left(\frac{2\mu}{kT}\right)^{1/2} \frac{Z_a Z_X e^2 \pi}{\hbar}$ is positive real in physical case.

As already mentioned above, in the literature concerning nuclear physics and astrophysics the integral $L_0(x)$ is considered as not capable of being evaluated in a closed form. For large values of the parameter x one obtains the approximate expression (cf. e.g. COX and GIULI 1968)

$$L_0(x) \approx 2\left(\frac{\pi}{3}\right)^{1/2} \left(\frac{x}{2}\right)^{1/3} \exp\left[-3\left(\frac{x}{2}\right)^{2/3}\right]. \tag{9}$$

In the following we consider the more general integral

$$L_\alpha(x) = \int_0^\infty \mathrm{d}y\, y^\alpha \exp\left[-y - xy^{-1/2}\right] \tag{10}$$

containing $L_0(x)$ for $\alpha = 0$ and evaluate it in a closed form. With the aid of the expression obtained one can consider in the reaction rate formula (4) higher order terms in the MACLAURIN expansion of the cross section factor $S(E)$.

Corresponding to the approximation used for getting (9) we obtain for (10) in the large parameter approximation in case of real α (cf. appendix)

$$L_\alpha(x) \approx 2\left(\frac{\pi}{3}\right)^{1/2} \left(\frac{x}{2}\right)^{(2\alpha+1)/3} \exp\left[-3\left(\frac{x}{2}\right)^{2/3}\right].$$

3. The evaluation of the integral in a closed form

3.1. Evaluation via Consul's solution of the associated differential equation problem

In the sense of a method often useful in investigating parameter-dependent integrals (which can be traced back to N. H. ABEL) we think the integral in consideration as a function of the parameter, assume it sufficiently often differentiable in a certain range of the parameter and aspire to derive a differential equation for this function.

For that we substitute $y = x^2\lambda$ in the integral (10) considered for real α, x. With that arises

$$L_\alpha(x) = x^{2\alpha+2} \int_0^\infty d\lambda\, \lambda^\alpha \exp\left[-x^2\lambda - \lambda^{-1/2}\right].$$

Differentiation with respect to x yields

$$L'_\alpha(x) \equiv \frac{d}{dx} L_\alpha(x) = (2\alpha + 2)\, x^{2\alpha+1} \int_0^\infty d\lambda\, \lambda^\alpha \exp\left[-x^2\lambda - \lambda^{-1/2}\right] - 2x^{2\alpha+3} \int_0^\infty d\lambda\, \lambda^{\alpha+1} \exp\left[-x^2\lambda - \lambda^{-1/2}\right].$$

Resubstituting $\lambda = x^{-2}y$ one obtains

$$L'_\alpha(x) = x^{-1}\left\{(2\alpha + 2) \int_0^\infty dy\, y^\alpha \exp\left[-y - xy^{-1/2}\right] - 2 \int_0^\infty dy\, y^{\alpha+1} \exp\left[-y - xy^{-1/2}\right]\right\}.$$

Therefore holds

$$xL'_\alpha(x) = (2\alpha + 2)\, L_\alpha(x) - 2 \int_0^\infty dy\, y^{\alpha+1} \exp\left[-y - xy^{-1/2}\right].$$

Twofold differentiation of this equation with respect to x leads to

$$L'_\alpha(x) + xL''_\alpha(x) = (2\alpha + 2)\, L'_\alpha(x) + 2 \int_0^\infty dy\, y^{\alpha+(1/2)} \exp\left[-y - xy^{-1/2}\right]$$

$$L''_\alpha(x) + L''_\alpha(x) + xL'''_\alpha(x) = (2\alpha + 2)\, L''_\alpha(x) - 2 \int_0^\infty dy\, y^\alpha \exp\left[-y - xy^{-1/2}\right],$$

i.e., to a differential equation for $L_\alpha(x)$,

$$xL'''_\alpha(x) - 2\alpha L''_\alpha(x) + 2L_\alpha(x) = 0\,. \tag{11}$$

The differential equation (11)[1]) for the integral $L_\alpha(x)$ is given without derivation by BAGAI (1962; the integral (2.6.) considered there is produced from our integral after substituting $y = \lambda^2$).

Once given the values of $L_\alpha(x_0)$, $L'_\alpha(x_0)$, and $L''_\alpha(x_0)$, the solution of the linear homogeneous ordinary differential equation of third order (11) is uniquely determined. By choosing $x_0 = 0$, according to (10) we obtain the three initial values

$$L_\alpha(0) = \int_0^\infty dy\, y^\alpha \exp[-y] = \Gamma(\alpha + 1)\,,$$

further

$$L'_\alpha(0) = \left(\frac{dL_\alpha(x)}{dx}\right)_{x=0} = -\int_0^\infty dy\, y^{\alpha-(1/2)} \exp[-y] = -\Gamma\left(\alpha + \frac{1}{2}\right) \tag{12}$$

and

$$L''_\alpha(0) = \left(\frac{d^2L_\alpha(x)}{dx^2}\right)_{x=0} = \int_0^\infty dy y^{\alpha-1} \exp[-y] = \begin{cases}\infty & \text{for } \alpha = 0\\ \Gamma(\alpha) & \text{for } \alpha > 0\,,\end{cases}$$

because

$$\int_0^\infty dy\, y^{q-1} \exp[-py] = p^{-q}\Gamma(q), \qquad \operatorname{Re} p > 0, \quad \operatorname{Re} q > 0$$

holds (see GRADŠTEJN and RYŽIK 1971).

In the case of α being a non-negative integer, the special solution of the differential equation (11) with the initial conditions (12) is given as an infinite series by CONSUL (1964). In this paper, a factor 2 at $\Psi(2\nu + 1)$ is not printed off. The result with this factor given in MATHAI (1971) is correct, as it can be verified by differentiating the expression and inserting it into the differential equation:

$$L_\alpha(x) = \Gamma(\alpha + 1)\left\{\sum_{\nu=0}^{\alpha} \frac{(\alpha - \nu)!}{\alpha!(2\nu)!} x^{2\nu} + \sum_{\nu=\alpha+1}^{\infty} \frac{(-1)^{\nu-\alpha-1}}{(\nu - \alpha - 1)!\,(2\nu)!} x^{2\nu}\left[2\Psi(2\nu + 1) + \Psi(\nu - \alpha) - 2\ln x\right]\right\} -$$
$$- \Gamma\left(\alpha + \frac{1}{2}\right)\left\{x + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(2\nu + 1)!\, \prod_{\tau=1}^{\nu}\left(\tau - \alpha - \frac{1}{2}\right)} x^{2\nu+1}\right\}, \tag{13}$$

where $\Psi(z)$ is the logarithmic derivative of EULER's Gamma function, $\Psi(z) = (d/dz) \ln \Gamma(z)$. It holds (see GRADŠTENJ and RYŽIK 1971)

$$\Psi(1) = -C, \qquad \Psi(\varkappa) = -C + 1 + \frac{1}{2} + \ldots + \frac{1}{\varkappa - 1}, \qquad \varkappa \text{ integer} > 1;$$

C is the EULERian constant.

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[1]) In the case of integer and half-integer $\alpha \geq -1/2$, the differential equation (11) for the integral $L_\alpha(x)$, recurrence relations for the coefficients of the series representation of the solution, and the asymptotic representation of $L_\alpha(x)$ are given in M. ABRAMOWITZ and I. A. STEGUN (eds.), Handbook of Mathematical Functions, Washington 1964, pp. 1001–1002.

As it can be seen from (13), the solution is additively composed of a part nonanalytical at $x = 0$ (with $\ln x$) and a part analytical at $x = 0$. The power series constituents of the solution (13) are obviously convergent for all real x. The series representation (13) of the integral (10) is suitable for its numerical computation. Furthermore, this representation is advantageous for the approximate evaluation of the integral in the case of small x. According to (13) for $\alpha = 0$ holds

$$L_0(x) = 1 - \pi^{1/2} x - x^2 \ln x + \tfrac{3}{2} (1 - C)\, x^2 + O(x^3) \,. \tag{14}$$

3.2. Evaluation via representation as Laplace transform of a Meijer's G-function

The integral (10) may be understood as LAPLACE transform of the function $\varphi(y; \alpha, x) = y^\alpha \exp[-xy^{-1/2}]$, α, x complex:

$$L_\alpha(x) = \int_0^\infty \mathrm{d}y \exp[-y]\, \varphi(y; \alpha, x) \equiv \mathfrak{L}\{\varphi(y; \alpha, x);\, 1\} \,. \tag{15}$$

To calculate the transform we desire to represent $\varphi(y; \alpha, x)$ as an G-function $G^{m,n}_{p,q}\left(\lambda y \,\middle|\, {a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right)$, introduced into analysis by C. S. MEIJER in 1936 as a generalization of the hypergeometric function; see for definition ERDÉLYI et al. (1953), GRADŠTEJN and RYŽIK (1971), MATHAI and SAXENA (1973). The LAPLACE transform of a G-function yields a G-function again. Therefore, the result can be given explicitly in a closed form (cf. ERDÉLYI et al. 1953; in MATHAI and SAXENA 1973 one finds a far extending generalization of the following formula):

$$\int_0^\infty \mathrm{d}y \exp[-y]\, G^{m,n}_{p,q}\left(\lambda y \,\middle|\, {a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right) = G^{m,n+1}_{p+1,q}\left(\lambda \,\middle|\, {0, a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right),$$

$$|\arg \lambda| < \left(m + n - \frac{p}{2} - \frac{q}{2}\right)\pi, \qquad \operatorname{Re} b_h + 1 > 0, \qquad h = 1, \ldots, m \,. \tag{16}$$

In the first instance, we express the exponential function $\exp z$ by KUMMER's function (confluent hypergeometric function) $\Phi(a, b, z)$ (cf. GRADŠTEJN and RYŽIK 1971):

$$\exp z = \Phi(b, b, z) \,.$$

After that we take into consideration the connection between KUMMER's function and MEIJER's G-function (see MATHAI and SAXENA 1973) and obtain

$$\exp z = \Phi(b, b, z) = G^{m=1,\, n=1}_{p=2,\, q=1}\left(-\frac{1}{z} \,\middle|\, {a_1 = 1,\, a_2 = b \atop b_1 = b}\right).$$

Taking advantage of an elementary property of the G-function (see MATHAI and SAXENA 1973) from this follows

$$G^{1,1}_{2,1}\left(-\frac{1}{z} \,\middle|\, {1, b \atop b}\right) = G^{0,1}_{1,0}\left(-\frac{1}{z} \,\middle|\, 1\right).$$

Consequently holds

$$\exp[-xy^{-1/2}] = G^{0,1}_{1,0}(x^{-1} y^{1/2} \mid a_1 = 1) \,.$$

Consideration of a further property of the G-function (see ERDÉLYI et al. 1953; in MATHAI and SAXENA 1973 is given a highly effective generalization due to SAXENA) yields

$$\exp[-xy^{-1/2}] = \pi^{-1/2} G^{0,2}_{2,0}(4x^{-2} y \mid a_1 = 1/2,\, a_2 = 1) \,.$$

With the relation (see ERDÉLYI et al. 1953, MATHAI and SAXENA 1973)

$$z^\xi\, G^{m,n}_{p,q}\left(z \,\middle|\, {a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right) = G^{m,n}_{p,q}\left(z \,\middle|\, {a_1 + \xi, \ldots, a_p + \xi \atop b_1 + \xi, \ldots, b_q + \xi}\right) \tag{17}$$

one finally obtains the desired relation between φ and G:

$$\varphi(y; \alpha, x) = y^\alpha \exp[-xy^{-1/2}] = \pi^{-1/2} \left(\frac{x}{2}\right)^{2\alpha} G^{0,2}_{2,0}\left(\left(\frac{x}{2}\right)^{-2} y \,\middle|\, \frac{1}{2} + \alpha,\ 1 + \alpha\right). \tag{18}$$

According to (16) for the LAPLACE transform of this function holds

$$\int_0^\infty \mathrm{d}y \exp[-y]\, G^{0,2}_{2,0}\left(\left(\frac{x}{2}\right)^{-2} y \,\middle|\, a_1 = \frac{1}{2} + \alpha,\ a_2 = 1 + \alpha\right) = G^{0,3}_{3,0}\left(\left(\frac{x}{2}\right)^{-2} \,\middle|\, a_1 = 0,\, a_2 = 1/2 + \alpha,\, a_3 = 1 + \alpha\right).$$

Consideration of the important property (see ERDÉLYI et al. 1953, MATHAI and SAXENA 1973)

$$G^{m,n}_{p,q}\left(z^{-1} \,\middle|\, {a_1, \ldots, a_p \atop b_1, \ldots, b_q}\right) = G^{n,m}_{q,p}\left(z \,\middle|\, {1 - b_1, \ldots, 1 - b_q \atop 1 - a_1, \ldots, 1 - a_p}\right) \tag{19}$$

and of property (17) finally yields the representation of our integral (10) by a MEIJER's G-function:

$$L_\alpha(x) = \int_0^\infty \mathrm{d}y\, y^\alpha \exp[-y - xy^{-1/2}] = \pi^{-1/2} G^{3,0}_{0,3}\left(\left(\frac{x}{2}\right)^2 \,\middle|\, b_1 = 1 + \alpha,\ b_2 = \frac{1}{2},\ b_3 = 0\right), \quad |\arg x^2| < \frac{3}{2}\pi \,. \tag{20}$$

The relation following from this formula in the special case $\alpha = 0$,

$$L_0(x) = \pi^{-1/2}\, G_{0,3}^{3,0}\left(\left(\frac{x}{2}\right)^2 \middle|\, b_1 = 1,\ b_2 = \frac{1}{2},\ b_3 = 0\right) \tag{21}$$

was derived in another, mediate way by MATHAI (1971) (see also MATHAI and SAXENA 1973) who has represented G-functions of the type $G_{0,q}^{q,0}(z \mid b_1, \ldots, b_q)$ in a computable form via evaluation of their MELLIN-BARNES integral definition and who has identified the series expansion so obtained for $G_{0,3}^{3,0}(z \mid 1, \frac{1}{2}, 0)$ with CONSUL's series representation (13) (case $\alpha = 0$) of the integral $L_0(x)$.

According to a general result concerning the asymptotic behaviour of G-functions (see MATHAI and SAXENA 1973), via eq. (20) the order of magnitude of the integral $L_\alpha(x)$ for $x \to \infty$ is given by

$$L_\alpha(x) \sim 2\left(\frac{\pi}{3}\right)^{1/2}\left(\frac{x}{2}\right)^{(2\alpha+1)/3} \exp\left[-3\left(\frac{x}{2}\right)^{2/3}\right], \qquad |\arg x^2| < 4\pi\,. \tag{22}$$

For real α, x this is just the result (A.4) obtained in the appendix by approximate evaluation of the integral (10) or large values of the parameter x.

In the case of $x \to 0$ the function $G_{0,3}^{3,0}((x/2)^2 \mid 1 + \alpha, \frac{1}{2}, 0)$ shows the behaviour $|(x/2)^2|^\gamma$, $\gamma = \mathrm{Min}\,(1 + \mathrm{Re}\,\alpha, \frac{1}{2}, 0)$, i.e.

$$L_\alpha(x) = O\left(\left|\left(\frac{x}{2}\right)^2\right|^\gamma\right), \qquad \gamma = \mathrm{Min}\,(1 + \mathrm{Re}\,\alpha, 0), \quad \text{for} \quad x \to 0\,. \tag{23}$$

For real $\alpha > -1$ holds $\gamma = 0$ and we have

$$L_{\alpha>-1}(x) = O\,(\mathrm{const.}) \quad \text{for} \quad x \to 0\,.$$

This result follows from the definition (10) immediately. It holds explicitly according to (12): $L_{\alpha>-1}(0) = \Gamma(\alpha+1)$.

The MEIJER's function $G_{0,3}^{3,0}(z \mid 1 + \alpha, \frac{1}{2}, 0)$ satisfies the linear homogeneous differential equation of the third order (cf. ERDÉLYI et al. 1953, MATHAI and SAXENA 1973)

$$\begin{aligned}
0 &= \left[-z - \prod_{j=1}^{3}\left(z\frac{\mathrm{d}}{\mathrm{d}z} - b_j\right)\right] G_{0,3}^{3,0}\left(z \mid b_1 = 1 + \alpha,\ b_2 = \frac{1}{2},\ b_3 = 0\right) = \\
&= \left[-z - \left(z\frac{\mathrm{d}}{\mathrm{d}z} - (1 + \alpha)\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{2}\right) z\frac{\mathrm{d}}{\mathrm{d}z}\right] G_{0,3}^{3,0}\left(z \,\middle|\, 1 + \alpha,\ \frac{1}{2},\ 0\right) = \\
&= \left[-z^3\frac{\mathrm{d}^3}{\mathrm{d}z^3} - \left(\frac{3}{2} - \alpha\right) z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{\alpha}{2} z\frac{\mathrm{d}}{\mathrm{d}z} - z\right] G_{0,3}^{3,0}\left(z \,\middle|\, 1 + \alpha,\ \frac{1}{2},\ 0\right).
\end{aligned} \tag{24}$$

Substituting $z = (x/2)^2$ the eq. (24) goes over into the equation

$$\left[x\frac{\mathrm{d}^3}{\mathrm{d}x^3} - 2\alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\right] G_{0,3}^{3,0}\left(\left(\frac{x}{2}\right)^2 \,\middle|\, 1 + \alpha,\ \frac{1}{2},\ 0\right) = 0\,. \tag{25}$$

With that, in the case of real α, x the function $\pi^{-1/2}G_{0,3}^{3,0}((x/2)^2 \mid 1 + \alpha, \frac{1}{2}, 0)$ is immediately seen a solution of the differential equation (11) derived in (3.1.). So in general case the function $\pi^{-1/2}G_{0,3}^{3,0}((x/2)^2 \mid 1 + \alpha, \frac{1}{2}, 0)$ directly appears as continuation of the CONSUL series (13) to complex values of α, x.

Appendix: Large parameter approximation of the integral

The integrand of

$$L_\alpha(x) = \int_0^\infty \mathrm{d}y\, y^\alpha \exp\left[-y - xy^{-1/2}\right]; \qquad \alpha \text{ real},\ x > 0\,, \tag{A.1}$$

vanishes at the upper and at the lower limits of the integral. Its dominating exponential part

$$\exp\left[-f(y)\right] \equiv \exp\left[-(y + xy^{-1/2})\right]$$

assumes one strongly marked maximum at $y = y_0$. From

$$0 = f'(y) = 1 - \tfrac{1}{2}\, xy^{-3/2}$$

one obtains

$$y_0 = \left(\frac{x}{2}\right)^{2/3}.$$

It holds $f''(y_0) = \frac{3}{2}\,(x/2)^{-2/3} > 0$, i.e. there is a minimum for $f(y)$ at $y = y_0$.

The value of the integral (A.1) can be obtained in often sufficient approximation by considering the integrand in an appropriately small vicinity of the extremal point y_0 only. In the first instance such an approximate calculation (after a method due to LAPLACE) is carried out for the general simple integral defined on the non-negative real semiaxis

$$I = \int_0^\infty dy\, g(y) \exp\left[-uf(y)\right], \tag{A.2}$$

where u denotes a large parameter. The function $f(y)$ is assumed to be a convex function of y in the interval of integration. The dominating exponential part $\exp\left[-uf(y)\right]$ has one maximum at $y = y_0$: $f(y_0) > 0$, $f'(y_0) = 0$, $f''(y_0) > 0$, and we assume $y_0 \gg 0$. Further, $g(y)$ is assumed to be continuous in the range of integration and differentiable at y_0. By assumption, for $f(y)$ in a vicinity of y_0 holds

$$f(y) = f(y_0) + \tfrac{1}{2} f''(y_0)\,(y - y_0)^2 + \dots$$

With that, approximately one obtains for I

$$\begin{aligned}
I &\approx \int_0^\infty dy\, g(y) \exp\left[-u\{f(y_0) + \tfrac{1}{2} f''(y_0)\,(y-y_0)^2\}\right] = \\
&= \exp\left[-uf(y_0)\right] \int_{-y_0}^\infty d\lambda\, g(y_0 + \lambda) \exp\left[-\tfrac{1}{2}\, uf''(y_0)\,\lambda^2\right] \approx \\
&\approx \exp\left[-uf(y_0)\right] \int_{-\infty}^\infty d\lambda\, g(y_0 + \lambda) \exp\left[-\tfrac{1}{2}\, uf''(y_0)\,\lambda^2\right] = \\
&= \exp\left[-uf(y_0)\right] \left\{ g(y_0)\, 2 \int_0^\infty d\lambda \exp\left[-\tfrac{1}{2}\, uf''(y_0)\,\lambda^2\right] + O(u^{-3/2}) \right\} \approx \\
&\approx 2 \exp\left[-uf(y_0)\right] g(y_0) \frac{\pi^{1/2}}{2\left(\dfrac{uf''(y_0)}{2}\right)^{1/2}} = (2\pi)^{1/2} \frac{g(y_0)}{(uf''(y)_0)^{1/2}} \exp\left[-uf(y_0)\right]
\end{aligned} \tag{A.3}$$

at which according to GRADŠTEJN and RYŽIK (1971)

$$\int_0^\infty d\lambda\, \lambda^n \exp\left[-s\lambda^2\right] = \frac{\Gamma\left(\dfrac{n+1}{2}\right)}{2s^{(n+1)/2}}; \qquad \mathrm{Re}\, s > 0, \quad \mathrm{Re}\, n > -1$$

is considered for $s > 0$, $n = 0, 2, 4, \dots$.

$\Gamma(z)$ is EULER's Gamma function.

(Considering the integral (A. 2) in the complex y-plane one obtains via the method of steepest descent an approximate expression (cf. JEFFREYS and SWIRLES 1966) which gives (A.3) after formally passing over to real y-axis.)

According to (A.3), in our case for large values of the parameter x follows with

$$f(y) = y + xy^{-1/2}, \qquad g(y) = y^\alpha$$

and

$$y_0 = \left(\frac{x}{2}\right)^{2/3}$$

and

$$f(y_0) = 3\left(\frac{x}{2}\right)^{2/3}, \qquad f''(y_0) = \frac{3}{2}\left(\frac{x}{2}\right)^{-2/3}, \qquad g(y_0) = \left(\frac{x}{2}\right)^{2\alpha/3}$$

for $L_\alpha(x)$ the approximate expression

$$L_\alpha(x) \approx 2\left(\frac{\pi}{3}\right)^{1/2} \left(\frac{x}{2}\right)^{(2\alpha+1)/3} \exp\left[-3\left(\frac{x}{2}\right)^{2/3}\right]. \tag{A.4}$$

By (A.4) holds for $\alpha = 0$ (cf. also COX and GIULI 1968)

$$L_0(x) \approx 2\left(\frac{\pi}{3}\right)^{1/2} \left(\frac{x}{2}\right)^{1/3} \exp\left[-3\left(\frac{x}{2}\right)^{2/3}\right], \qquad x \gg 0 .$$

References

BAGAI, O. P.: 1962, Sankhya **A 24**, 55.
CLAYTON, D. D.: 1974, Nature **249**, 131.
COHEN, E. G. D., and THIRRING, W. (eds.): 1973, The BOLTZMANN Equation. Theory and Applications, Springer-Verlag, Wien-New York.
CONSUL, P. C.: 1964–1965, Math. Nachr. **28**, 169.
COX, J. P., and GIULI, R. T.: 1968, Principles of Stellar Structure ,Vol. I.: Physical Principles, Gordon and Breach, New York-London-Paris.
ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F., TRICOMI, F.: 1953, Higher Transcendental Functions, Vol. I.: The BATEMAN Manuscript Project, Mc Graw-Hill, New York-Toronto-London, in russian: 1973, Nauka, Moscow.
FOWLER, W. A., CAUGHLAN, G. R., ZIMMERMAN, B. A.: 1975, Ann. Rev. Astron. Ap. **13**, 69.
FRANK-KAMENECKIJ, D. A.: 1959, Fizičeskie processy vnutri zvëzd, Fizmatgiz, Moscow.
GRADŠTEJN, I. S., and RYŽIK, I. M.: 1971, Tablicy integralov, summ, rjadov i proizvedenij (revised under participation of GERONIMUS, Ju. V., and CEJTLIN, M. Ju.), Nauka, Moscow.
JEFFREYS, H., and SWIRLES, B.: 1966, Methods of Mathematical Physics, University Press, Cambridge, in russian: 1970, Izd. Mir, Moscow.
MATHAI, A. M.: 1971, Math. Nachr. **48**, 129.
MATHAI, A. M., and SAXENA, R. W.: 1973, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Lecture Notes in Mathematics, Vol. 348, Springer-Verlag, Berlin-Heidelberg-New York.
MITTLER, H. E.: 1977, Astrophys. J. **212**, 513.
VASIL'EV, S. S., KOČAROV, G. E., LEVKOVSKIJ, A. A.: 1974, Izvest. Akad. Nauk SSSR, Ser. Fiz. **38**, 1827.

Adresses of the authors:

H. J. HAUBOLD
Zentralinstitut für Astrophysik der AdW der DDR
Astrophysikalisches Observatorium Potsdam
DDR-15 Potsdam
Telegrafenberg
German Democratic Republic

R. W. JOHN
Zentralinstitut für Astrophysik der AdW der DDR
Sternwarte Babelsberg
DDR-1502 Potsdam-Babelsberg
Rosa-Luxemburg-Straße 17a
German Democratic Republic