

STATISTICS AND MAPPING OF ASTEROID CONCENTRATIONS IN THE PROPER ELEMENTS' SPACE

A. CARUSI

Laboratorio di Astrofisica Spaziale, CNR, Frascati, Italy

E. MASSARO

Istituto di Fisica, Università di Palermo

and

Unità di ricerca GIFCO-CNR, Palermo, Italy

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A statistical investigation of asteroid families has been carried out using a new clustering technique, developed by A.I. Gavrishin. Proper elements for 2764 asteroids (1810 numbered and 954 PLS asteroids) have been computed. Using these data the Gavrishin z^2 method gives only fifteen significant classes. Five of them are coincident with the Hirayama families, Themis, Eos, Coronis, Phocaea and Flora, that cannot be univocally subdivided. The new PLS families are clearly recognized. Furthermore many small sized families, reported by several authors, lose their statistical relevance.

Isodensity maps of the asteroid belt and the distribution of the apparent mean opposition magnitude are also presented: they give an interesting representation of the internal structure of the main belt.

Key Words: Asteroids – asteroid families – Kirkwood gaps – resonances

1. INTRODUCTION

The discovery of an internal structure of the asteroid belt goes back to the nineteenth century when D. Kirkwood (1867) observed several gaps in the asteroid distribution vs their mean daily motion, n^* . These gaps correspond to the values of n which are in perfect resonance with Jupiter's mean daily motion n_J . No satisfactory explanation has been proposed until now for such an important resonance phenomenon, that seems to occur in the rest of the Solar System only in the case of the Saturnian rings. The deepest Kirkwood gaps correspond to the values of the ratio n_J/n equal to $1/4$, $1/3$, $2/5$, and $1/2$, less deep gaps, however occur also at $2/7$, $3/7$ and $4/9$ (figure 1). On the other hand at the resonances $1/1$ and $2/3$ the isolated Trojans and Hilda groups can be found, and the single 279 Thule lies at the commensurability $3/4$.

Further analyses of the minor planet population, performed taking into account other orbital parameters, show many interesting results. Hirayama (1918, 1919), who first used statistical techniques, discovered some concentrations of asteroids in the (a, e, i) osculating element space: he called "families" these groupings with a precise genetic meaning. In his successive studies, Hirayama (1923, 1928, 1933) followed by other authors (Brouwer 1951; Arnold 1969; van Houten *et al.* 1970; Lindblad and Southworth 1971; Williams 1975) used proper elements e' and i' instead of the osculating elements e and i : again noticeable groupings were recognized, in quite good agreement with those found previously. The results of Hirayama's studies have been accepted and extended by successive authors: new objects were added to the discovered families and new groups were identified. Hence the number of families and their elements reported in literature has increased continuously.

The methods adopted in searching asteroid families differ from author to author. At the very beginning, Hirayama and Brouwer used visual inspections rather than a statistical analysis, with the exception of the Hirayama's work of 1919. Consequently their family recognitions were not supported by quantitative estimates.

The developing of big computers capable to process a large amount of data provided new methods in searching asteroid families. The first computer-based research was the one performed by J.R. Arnold in 1969.

In the present paper we will describe the results of a statistical investigation of asteroid families and the method used in this research. Asteroids' density maps in the proper elements space are given, which allow an

*Kopal (1973) has pointed out that the existence of these gaps was first observed by K. Hornstein.

easy recognition of the asteroid families, and give an interesting representation of the whole belt. We also report several graphs referring to the distribution of the apparent mean opposition magnitude $B(a, 0)$ within each family.

2. THE z^2 METHOD

To search asteroid families we have used a new clustering method, recently developed by A.I. Gavrishin (1968, 1974), and adapted for computer elaborations by Coradini *et al.* (1977).

Preliminary results of the application of that method to the identification of asteroid families have been reported previously (Carusi and Massaro 1977); in this paper the final analysis is presented, which is different from the other one for the presence of some minor groups.

The z^2 method was developed essentially to classify geochemical samples: it has been used with appreciable success in the classification of lunar rocks and glasses (Coradini *et al.* 1976). A detailed description of such technique can be found in the reference cited and here we will report only a brief summary of it. The outstanding characteristic of the new statistical procedure is the possibility of an automatic classification in terms of homogeneous taxonomic units. By means of successive transformations the original multivariate distribution is reduced to an univariate quasi-gaussian distribution with mean value and variance equal to 0 and 1 respectively. If N is the total number of elements in the sample, $n_1 < N$ elements which will form the first class are selected by a test of membership to a suitably defined normal distribution. Afterwards the method is applied to the remaining $N - n_1$ elements to identify the second homogeneous class. By successive iterations all the following classes are obtained.

The membership of an element to a class is determined by means of an hypothesis test where the critical value is represented by a parameter q . The choice of the q value depends on the security level (s.l.) needed: the larger the q , the more general is the classification. For example $q = 3.0$ corresponds to a s.l. of 99.8%.

The z^2 method gives excellent results when a large number of variables and elements is used. In our case the small number of variables used (a, e', i') is balanced by the very large number of the elements.

The search for families has been done dividing the entire range of a into five overlapping intervals: (1.90 – 2.50), (2.25 – 2.75), (2.50 – 3.00), (2.75 – 3.25), (3.00 – 4.20) AU. This way it has been possible to check whether or not a “family” recognized in one interval is present also in the preceding or in the subsequent one.

The method divides all the elements in classes, defined by a given mean value and standard deviation for each variable. It also compares each class with the others and then gives an indication of the distances between the classes. No objective “*a priori*” criterion exists to select the most significant groupings, but the method itself puts some internal constraints. Firstly it does not work very well if the number of residual elements, during the classification, becomes less than approximately 30: the last classes have very little significance. Secondly, the method indicates the relative weight of each variable and provides a goodness index for the classification. The use of this index can help to choose the q value.

In our analysis a value $q = 1.4$, corresponding to a s.l. 91.8% has been used. We have also chosen to use decimal logarithms for e' and i' variables, because of their small variation ranges. This does not affect the quality of the results because the method does not require any linear relationship between the variables.

For each class listed in every printout, we have also calculated, for every variable, the variance of the sample means σ_m . We have then analyzed the distribution of the σ_m values in each output and we have accepted the classes belonging to the first half of the distribution. The accepted classes have been subdivided into two types: type 1, if the class has been identified in two intervals, without noticeable differences, and type 2, if it has been found mixed with other less important classes in one of the overlapping intervals. Special computations were made for the extreme intervals to ensure the overlapping: other special runs were processed, with $q = 1.35, 1.50, 2.00$ and with different ranges both for a and e' . Moreover, to analyze the behaviour of a single class, runs were made with only numbered asteroids.

3. RESULTS

The orbital elements of 1861 numbered asteroids are reported in *Minor Planets Ephemeris* (1976). Since the orbital elements of many of them have been recomputed with improved precision, we have preferred to determine *ex novo* the proper eccentricity e' and inclination i' for all the minor planets with semi-major axis in the range 1.9–4.2 AU, using the tables by Brouwer and van Woerkom (1950). A four-point Bessel interpolation has been used for the values of a lying in the centre of the range; a four point Newton interpolation for those values close to the minimum and the maximum of the range. To avoid the effects due to the singularities of induced oscillation in proximity of 1.94, 2.03 and 2.64, the asteroids very close to these values have been rejected.

Proper elements for a total number of 3606 minor planets (1810 from MPE and 1796 from PLS) have been obtained, in agreement with the van Houten *et al.* (1790) proper elements down to the fourth decimal digit. The analyzed sample consisted of 2764 minor planets, 1810 of which numbered and 954 PLS asteroids (class 1).

On the basis of the selection criteria reported in the previous section, 15 major groupings have been selected: they are reported in table 1. The “families” – a term we now use without any genetic meaning – are shown for decreasing values of a . The geometric means value $\langle\langle e' \rangle\rangle$ and $\langle\langle i' \rangle\rangle$ are also reported. We may note, however, that the difference with the arithmetic mean is on the fourth decimal digit. Columns 4, 6, and 8 contain the variables' variation ranges, and column 9 the classification type (1 or 2). The last column shows the families recognized by other authors which seem to be strongly related to ours. Two families only have no correspondence: Hilda, a well-known group with a high value of semi-major axis, and Astraea.

Inside some families the z^2 method finds some structures. In fact it works iteratively, and the most compact groups are first identified: lower density structures result sometimes as type 2 classes. That characteristic is shown in the column 9 of table 1. The internal differentiation of the families is reported in table 2, while in table 3 the family members are listed.

We point out that Coronis is split in two type 1 classes, one of which (Coronis) is fully contained in the other. In the case of Themis the situation is similar, but the Themis subfamily is a type 2 class. The subfamilies belonging to Eos and Flora, which is the most articulate group, are instead not completely contained each in the other. It is remarkable that such compact subfamilies are to be regarded as dependent on the method of analysis, instead of being actually significant structures: we prefer to consider those families without subdivisions, as reported in table 1. In section 6 we will discuss in greater detail all the families found.

The family H4 (Maria) has not been identified in our analysis: Maria is a peculiar family whose members' number always increases more slowly than in the other families. Van Houten *et al.* list also, among the PLS asteroids, 63 new members for Themis, 23 for Eos, 32 for Coronis, but add only 3 new members to the H4. This fact confirms the low growth rate of this family which Hirayama himself regarded as doubtful (Hirayama 1933).

Regarding the PLS sample, the families vH1, vH2, vH6, vH32, vH33, and vH34 are well recognized while vH30 is hardly detectable: the other families cannot be separated from the background. vH3 is one of these families, whereas it is partially recognized in the total sample.

Inside the numbered sample, H1, H2 and H3 have been quite well identified, together with Hilda and Hygiea. In the central region of the asteroid belt the situation is more complex, and none of the families reported by the previous authors can be precisely identified: they are often broken and mixed up.

The total sample examined utilizing a $q = 1.5$ shows in this zone some concentrations that may be related to A86, A87, A83, A73; they do not satisfy, however, our selection criteria. Therefore, the non-existence of well-defined classes in this region of the asteroid belt seems to be confirmed.

In the inner region, only the Flora group is recognized, and its subdivisions depend on the s.l. used. None of the Brouwer (1951) and Arnold (1969) little families can be precisely confirmed.

4. DENSITY MAPS

The structure of the asteroid belt can be visually investigated drawing the density maps in the (a, e', i') space. A complete set of the density maps, including the description of the computational technique, is given in Carusi

and Massaro (1978). Here we will report only those maps which are particularly relevant to the families' description.

The gap structure of the asteroid belt appears very clear in the numbered asteroid (a, i') map (figure 3) and in the total (a, i') map (figure 2); it is just visible, but not so clear in the PLS (a, i') map (figure 4). We recall that, being the maps obtained by linear interpolation, the zone of influence of each asteroid is quite large, consequently the gaps' limits result slightly confused.

An important feature of these maps consists of five well-defined maxima, that correspond to Themis, Eos, Coronis, Nysa, and Flora. They are all out of the region between the gaps $1/3$ and $2/3$, where the isodensity distribution is quite uniform. Note also how clearly a division of the asteroid belt into three rings appears in these maps: an inner ring, between 2.06 and 2.50 AU ($1/4 - 1/3$ resonances); a central ring, between 2.50 and 2.82 AU ($1/3 - 2/5$); and an external one, between 2.82 and 3.28 AU ($2/5 - 1/2$). Planet-crossing asteroids, like Apollo-Amor objects, and external groups like Trojans and Hilda, are then excluded from the main belt. The presence of families and minor gaps suggests the existence of a fine structure in the rings. Further subdivisions of the inner and the outer ring can be traced in correspondence to $2/7$ resonance (Flora and Nysa subrings) and to $3/7$ and $4/9$ (Coronis, Eos, and Themis subrings). Such subdivision is not as relevant as the previous one; however, interesting properties can be observed among the subrings. The ring and subring structures are shown in table 4; the most important resonances and their relationship with the families are listed in table 5.

A striking dishomogeneity between the outer ring and the others can be observed in the total (a, i') map (figure 2). In the outer ring the asteroids are concentrated in the three maxima corresponding to Themis, Eos, Coronis; in the central and in the inner rings the asteroid distribution is instead quite smooth. This fact is not so evident in the total (a, e') map (figure 5), where also the inner ring is characterized by two major concentrations (*i.e.* Flora and Nysa), comparable with those of the outer ring.

A strong correlation can be noted among the families derived by the z^2 method and the density maxima. It is clear also that all the groups are more separated on i' than on e' .

5. MAGNITUDE HISTOGRAMS

The relationships between the mean opposition magnitude $B(a, 0)$ of the asteroids and their position in the belt has been also studied. It is particularly interesting to find whether the distribution of $B(a, 0)$ inside each family is similar or not to the general distribution of the corresponding ring.

The $B(a, 0)$ histogram for all the minor planets of the main belt (2675 objects) is shown in figure 6. It is characterized by the presence of two very well-defined peaks: the first is located at $15.5 - 16.0$ and the second at $19.0 - 19.5$. Clearly, the former corresponds to the numbered and the latter to the PLS asteroids. In figure 6 is also reported the $B(a, 0)$ distribution for the asteroids of the families listed in table 3: the two histograms are quite similar, unless a lack of faint objects. This fact can be easily explained considering the differences in the a -distribution between the two populations (figure 1), so that the richest and most distant families have only a little number of elements with large values of $B(a, 0)$.

In the following figures the distributions in the three rings of the main belt are shown. The frequency curves relative to the inner subrings of Flora and Nysa (figures 7 and 8) have two completely different patterns. The same holds for the two families of which the $B(a, 0)$ distributions are very similar to the corresponding ring. The distribution of the central ring asteroids (figure 9) is very similar to that of the main belt, and very little can be deduced from the magnitude graphs of the pseudo-families in this ring. In the outer ring (figure 10) all the distributions show the same features, and follow the distribution of the main belt. The families Themis, Eos, Coronis have a similar behaviour. The only relevant difference is, therefore, between the Flora and Nysa subrings, due to the lack of faint objects in the former. This fact can be partially explained taking into account that the selection effects in PLS tend to increase the number of low inclination asteroids and that the Nysa subring shows a lower mean inclination than Flora. However, it is difficult that such a large discrepancy might be due to this effect only: if so, the distribution in the central ring and in Flora should be more similar.

6. DESCRIPTION OF THE FAMILIES

We will now examine in greater detail all the families, as reported in table 1.

– **HILDA** – It is the family with the largest semi-major axis. Hilda group is well known, and is located exactly at the 2/3 resonance. Actually, in our Hilda family, only ten of the 28 known objects present in that region are contained, and likely must be considered as a chance grouping in that population. Note that some errors in the computation of e' and i' may be due to the Brouwer's method, whose application is doubtful at resonances.

– **HYGIEA** – It is a type 2 family, largely overlapped by Themis on a and e' . It is formed essentially by PLS asteroids (67%), and in figure 2 can be recognized by the flat maximum over Themis. Hygiea is also present in the PLS map of figure 4. It is strictly related to vH34 Medea. However because of PLS selection effects, more conspicuous in the outer regions, the statistical relevance of this family is scarce.

– **THEMIS** – This is the major family in the outer ring, and collects about 19% of all the asteroids in its subring. Among the far and rich families, Themis has the lowest proper inclination and the largest proper eccentricity (figures 2 and 5). The z^2 method divides this family into two subfamilies: Erato, type 1, and Themis, type 2, both listed in table 2. Such subdivision is maintained in the particular elaborations and in the analysis of the numbered sample; hence, it is not due to the presence of PLS asteroids. However, owing to the reasons already explained in section 3 we consider Themis as a single class. It is remarkable that the $B(a, 0)$ distribution of the family reflects quite well the total subring distribution (figure 10).

– **EOS** – It is the most compact family, identified in the (a, i') total map (figure 3) by the absolute maximum. Eos is the only high inclination group among the rich families and it is characterized by lack of PLS asteroids with the exception of PLS 2504 and PLS 4645 (now numbered as 1812). This circumstance is explained by the great extension of the vH2 family on a , crossed by 4/9 resonance. It must be pointed out that only the two mentioned PLS minor planets fall into the Eos' a – range, whereas it contains all the 60 numbered asteroids. In table 2 the Eos-Dorothea subdivision provided by the z^2 method is reported: such subdivision is also preserved in the printouts of the numbered minor planets. Finally, the Eos $B(a, 0)$ distribution is the sharpest in the outer ring.

– **CORONIS** – Its general features are very similar to those of the above mentioned families. Coronis' also is split in two subfamilies, both type 1 class, and can be found also in the numbered asteroid printouts. This is the only rich family with low eccentricity and inclination.

– **CENTRAL RING** – In this region we have identified five type 2 families. Gefion is the smallest one, with nine asteroids only. It lies very close to the 2/5 resonance. Other authors report the presence of small families in that range, and in fact we cannot consider as proved the existence of Gefion. Io and Tyche are joined on a , and partially overlap on e' and i' : the z^2 method finds a strong correlation between them. Their PLS asteroids are essentially elements of vH33, which is so large to cover almost the entire central ring; hence, these two families are to be considered as a single one even if their correlation is not so strong as in the case of Themis with Erato or Eos with Dorothea. An inspection of the density map of figure 2 reveals a flat maximum comparable in extension to that of above mentioned groups. We want to point out that our method has unified LSI0 and vH33, differently from that reported by Lindblad and Southworth (1971). Similar arguments can be given for Astraea and Russia: we will note only that Astraea's asteroids have never been reported in families by other authors; in fact this family is formed to a great extent by PLS asteroids. In figure 3 it is visible a relative maximum corresponding to Russia, while no feature is related to Astraea. The possibility that these groups are to be considered as a single one is supported by the strong correlation indicated by the z^2 method.

– **PHOCAEA** – This family has the highest inclination, and for this reason it has no PLS asteroids. In figures 2 and 3 it appears completely separated on i' axis from the rest of the subring, therefore it has been identified despite its low density like the Hilda group. No evidence exists for a clustering on e' (figure 5).

– **VICTORIA** – It is a type 2 family, largely formed by B27, A77, and vH27 elements. In figure 2 it corresponds to the relative maximum between Nysa and Phocaea.

– **NYSA** – It is one of the principal families, and is mainly formed by vH32 asteroids. Nysa corresponds to the only noticeable maximum in the PLS map (figure 4). Its numbered minor planets belong to B24 and A74.

Because of its large content of PLS asteroids, the $B(a, 0)$ distribution is very different from the others, as we have previously discussed.

– BAPTISTINA-FLORA – Among the asteroid families the Flora group is peculiar, because it is very numerous and structured. Since the earliest researches, it has always been the richest family containing more than double the average number of the asteroids forming the other Hirayama families. Brouwer, for the first time, recognized some structures in this group; he divided it into four separated families, numbered from 6 to 9. Such division was represented in the (e', i') plane by three parallel straight lines arbitrarily drawn. According to Brouwer that can be explained taking into account the distribution of $\pi' + \vartheta'$: in fact, for the elements of B8, $\pi' + \vartheta'$ is always in the interval $(0^\circ.0 - 0^\circ.6)$, while for the families B7 and B9 $\pi' + \vartheta'$ is distributed on the entire circumference. Brouwer claims that without the B8 identification, it would be impossible to separate B7 from B9. Actually, this fact has small relevance, because the $\pi' + \vartheta'$ distribution of B8 greatly changes adding some objects from B7 – B9 very close to the B8 limits, and consequently the above mentioned division cannot be traced again.

Arnold found the internal structure of the Flora group only in a “qualitative” way, without any specification; Lindblad and Southworth admitted their failure in recognizing this division.

The z^2 method gives Flora as separated in three groups: however, the division Flora-Berolina-Iduberga has to be considered as a spurious result, due essentially to the followed procedure, as explained in section 3. We conclude that there is only one very large family.

Quite different is the case of Baptistina, because it is located just beside the 2/7 resonance. Only one of the 34 objects of this family falls below that resonance, and correspondingly only one member of Flora is on the Baptistina side. In table 3 it can be noted that the numbered part of Baptistina is formed by elements of B8–B9, A8–A9 families. We argue that the division induced by the 2/7 resonance may be the only significant one in the Flora group.

7. CONCLUSIONS

On the basis of our analysis we conclude that two most important arguments are firmly established by the present distribution of bodies in the asteroid belt: (i) the internal subdivision of the main belt in three well defined rings; (ii) the existence of a small number of local concentrations, or “families”.

Other divisions of the asteroid belt, based on the presence of the Kirkwood gaps, have been already found by other authors. Chebotarev (1967) subdivided the whole belt in seven rings, four of which in the main belt, but no other structural difference was pointed out by him.

The existence of the three main rings is confirmed by the relevant discrepancies, exhibited in the density maps, as we discussed in section 4. We argue that the selection effects on the sample cannot completely justify those differences: in fact adding the PLS asteroids the structural profile of the numbered sample does not change, with the only remarkable exception of the Nysa maximum (see figures 2, 3, 4). Moreover, the subring division is not too evident, and depends only on the presence of minor gaps and families.

Checking the existence of the families, the present analysis identifies only a small number of them, and eight of the fifteen families in table 1 can be regarded as doubtful. This fact contrasts the general trend of the previous researches, but it seems to be confirmed by the following evidence: only the large, rich families, firstly defined by Hirayama, have always been recognized by all the authors, whereas the minor families were sometimes identified, or partially confirmed, or undetected at all, as in the case of B27, to which eight members are assigned in Brouwer’s classification. This family is included by Arnold in A77, with 16 members, 7 of which are also B27 members: only one of the nine left was not yet discovered in 1951. Lindblad and Southworth (1971) identify instead a 5-members family, while in the present work 6 B27 and 10 A77 members are included in Victoria (see table 3). Completely different is the case of B16, again with 8 members: only 3 of them are included in Arnold’s A66 and 2 in A67, whereas 7 are recognized by Lindblad and Southworth; in the present research

B16 disappears completely. We want to point out that most of the little families are contained in the central ring, whose flat and homogeneous structure was previously discussed.

Our conclusion is that the asteroids' groupings, significant for the interpretation of the structure of the asteroid belt, are only those coincident with the 4-5 density maxima in figure 2. The question if they were grouped by dynamical processes, or they are to be considered as the remnants of primary larger bodies, or finally if they appear only because of the data incompleteness (as shown in the case of Nysa), is still open.

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Table 1 Families of asteroids and their relevant parameters obtained by means of the z^2 method.

Name	elem.	<a>	Range a	<e'>	Range e'	<<1'>>	Range 1'	Type	Remarks
Hilda	10	3.9580	3.9332-3.9810	.1718	.1411-.2134	.1287	.1045-.1646	2	2/3 Resonance
Hygiea	21	3.1544	3.1317-.3.1782	.1322	.0982-.2045	.0946	.0850-.1047	2	viB4
Themis	114	3.1429	3.0756-.3.2128	.1577	.1236-.2167	.0260	.0164-.0414	1-2	H1, B1, A1, viB1
Eos	62	3.0156	3.0060-3.0239	.0735	.0549-.0933	.1776	.1682-.1858	1-2	H2, B2, A2
Coronis	61	2.8751	2.8316-2.9182	.0490	.0365-.0646	.0368	.0334-.0402	1	H3, B3, A3, viB3
Gefion	9	2.7901	2.7673-2.8134	.1358	.1296-.1478	.1641	.1586-.1698	2	viB6
Io	27	2.6755	2.6515-2.7097	.1675	.1500-.1875	.2161	.1977-.2394	2	A69, B19, L645, viB3
Tyche	41	2.6019	2.5482-2.6500	.1644	.1269-.2451	.2371	.2200-.2636	2	viB3, (B20, A70)
Astraea	36	2.5737	2.5262-2.6264	.2172	.1765-.2641	.0758	.0552-.1044	2	
Russia	15	2.5538	2.5372-2.5739	.1971	.1894-.2071	.1215	.0957-.1485	2	A73, L643
Nysa	77	2.4009	2.3379-2.4676	.1722	.1440-.2012	.0459	.0335-.0640	1	B24, A74, viB2
Victoria	26	2.3765	2.3295-2.4344	.1999	.1738-.2290	.1855	.1474-.2306	2	B27, A77, viB27
Phocaea	29	2.3654	2.3058-2.4284	.2396	.1821-.3158	.4434	.3879-.5135	1	H5, B5, A5
Baptistina	34	2.2848	2.2487-2.3157	.1560	.1349-.1881	.1041	.0833-.1263	2	B9, A9
Flora	162	2.2220	2.1798-2.2535	.1417	.1112-.1737	.0812	.0483-.1320	1-2	H6, B(6789), A(6789) viB6

Table 2 Splitting in subfamilies of Themis, Eos, Coronis, and Flora as indicated by the z^2 method.

Family	Subfam.	n. elem.	<a>	<e'>	<<1'>>	Type
Themis	Themis	69	3.1449	.1613	.0260	2
	Erato	45	3.1399	.1520	.0235	1
Eos	Dorothea	14	3.0160	.0651	.1713	2
	Eos	48	3.0155	.0759	.1795	1
Coronis	Dreada	35	2.8840	.0507	.0368	1
	Coronis	26	2.8632	.0466	.0367	1
Flora	Iduberga	9	2.2422	.1551	.1247	2
	Beorlina	47	2.2356	.1398	.0810	1
	Flora	106	2.2143	.1415	.0776	2

Table 3 Listing of family members and comparison with other classifications. (H: Hirayama - B: Brouwer - A: Arnold - vH: van Houten *et al.* - LS: Lindblad and Southworth).

[illegible]

Table 5 Relation between the asteroid families and the resonances with Jupiter.

Resonance (n λ_{ast} / n _{ast})	n ("'/d)	a (AU)	families
1/1	299.0510	5.201	Trojans
3/4	398.7340	4.293	Thule
2/3	448.5760	3.969	Hilda
3/5	498.4180	3.700	
1/2	598.1010	3.276	Hygiea Themis
4/9	672.8640	3.029	Eos
3/7	697.7850	2.956	Coronis
2/5	747.6260	2.823	Gefion Io Tyche Astraea Russia
1/3	897.1520	2.500	Nysa Victoria Phocaea Baptistina
2/7	1046.6770	2.256	Flora
1/4	1196.2020	2.064	

Table 4 Ring and subring structure of the asteroid belt and its correspondence with the families.

Ring	Subring	Families	N. of el.	Fam. el.	Percentage
Inner	Flora	Flora	290	162	55.86
	Nysa	Baptistina Phocaea Victoria Nysa	854 564	328 166	38.41 29.43
Central	none	Gefion Io Tyche Rumia Astraea	851	128	15.04
Outer	Coronis Eos	Coronis Eos	195-970 185-590	61 62	31.28 33.51
	Themis	Themis Hygiea	590	258 135	26.60 22.88
Main asteroid belt			2675	714	26.69

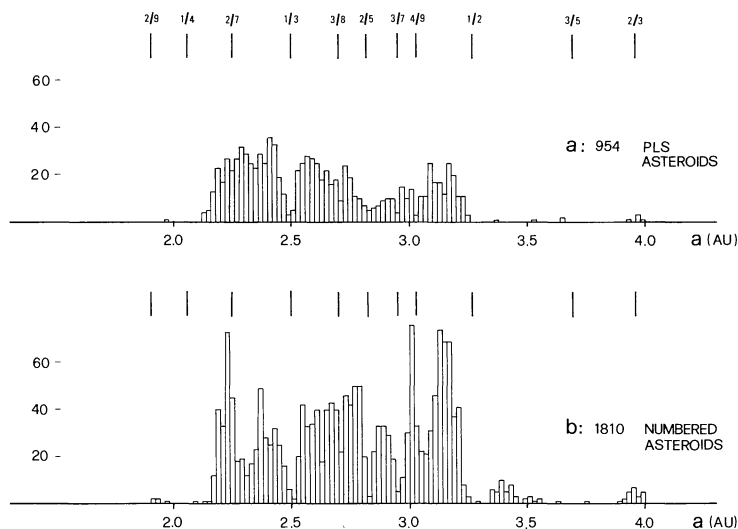


Figure 1 Distributions of the semi-major axes of the two asteroid populations: a) PLS and b) numbered asteroids.

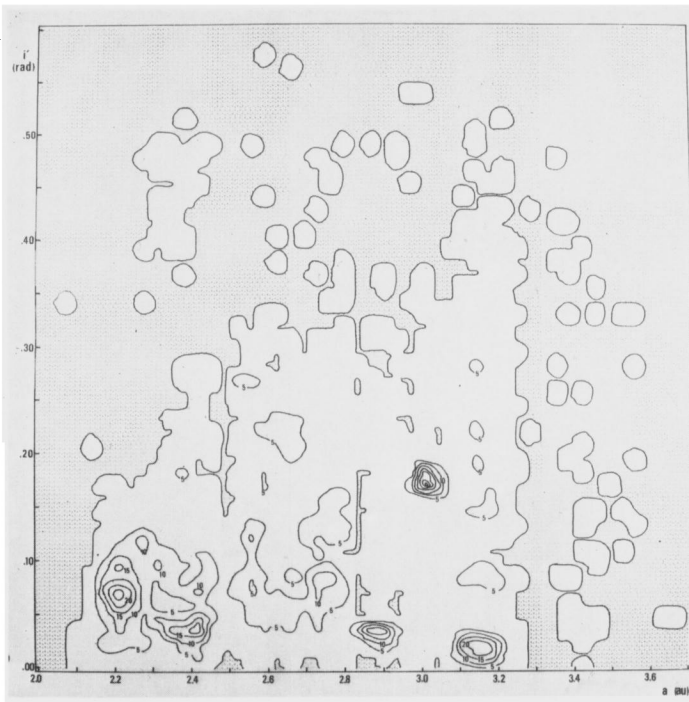


Figure 2 Isodensity curves in the (a, i') plane for the cumulate population of PLS and numbered asteroids (2707 objects).

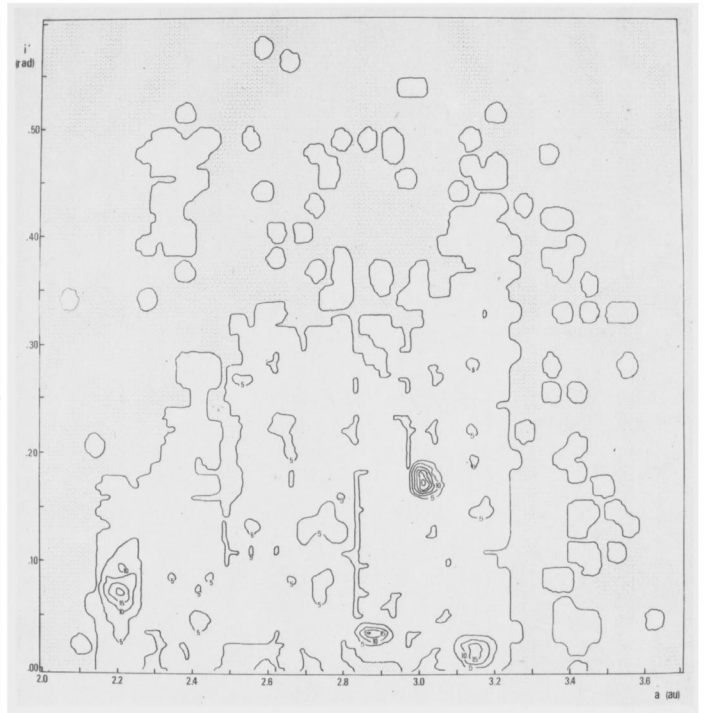


Figure 3 Isodensity curves in the (a, i') plane for the numbered asteroid population (1762 objects).

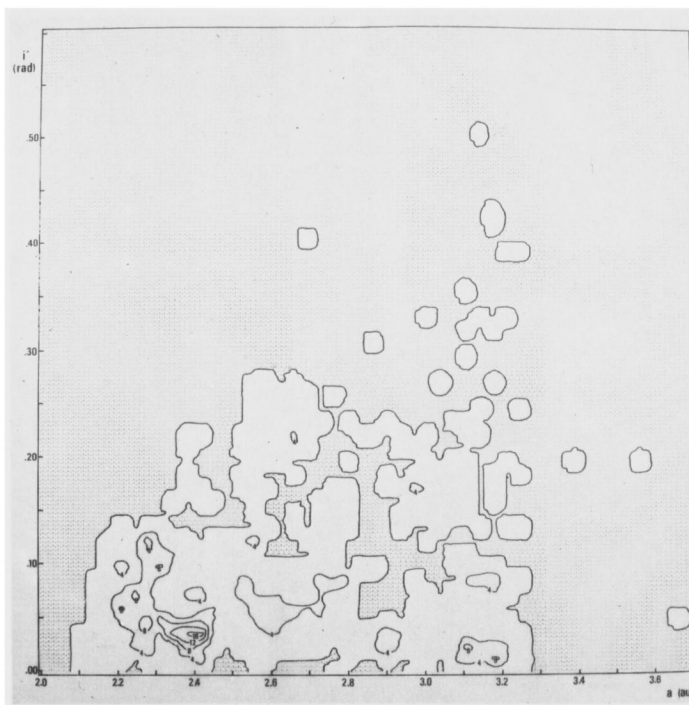


Figure 4 Isodensity curves in the (a, i') plane for the PLS asteroid population (945 objects).

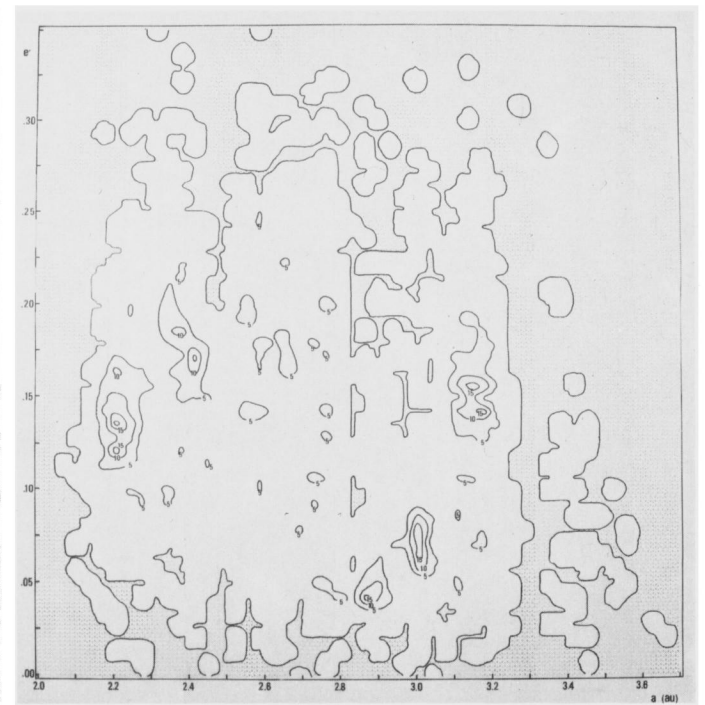


Figure 5 Isodensity curves in the (a, e') plane for the cumulate population of PLS and numbered asteroids (2707 objects).

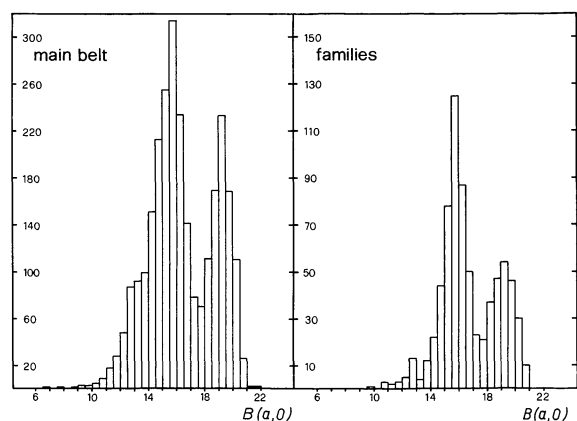


Figure 6 $B(a, O)$ distribution for the whole asteroid population of the main belt and the asteroids of the families listed in table 1 (Hilda has been excluded).

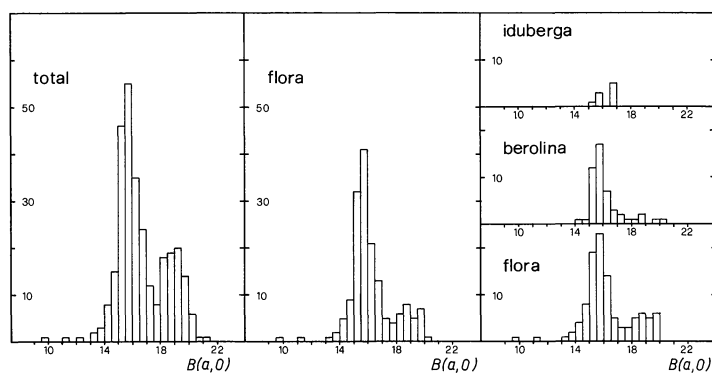


Figure 7 $B(a, O)$ distribution for the asteroid population of the Flora subring (inner ring) and of the families therein contained.

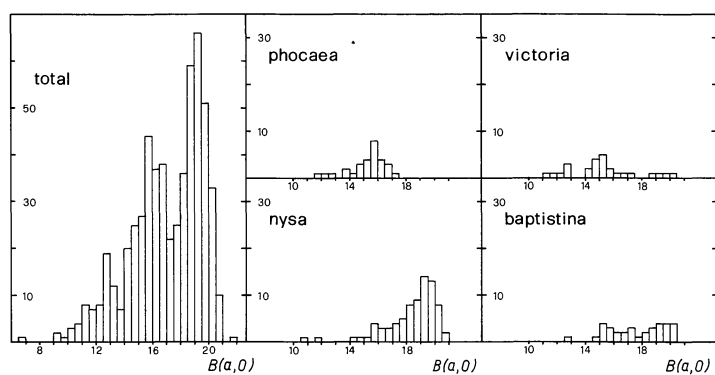


Figure 8 $B(a, O)$ distribution for the asteroid populations of the Nysa subring (inner ring) and of the families therein contained.

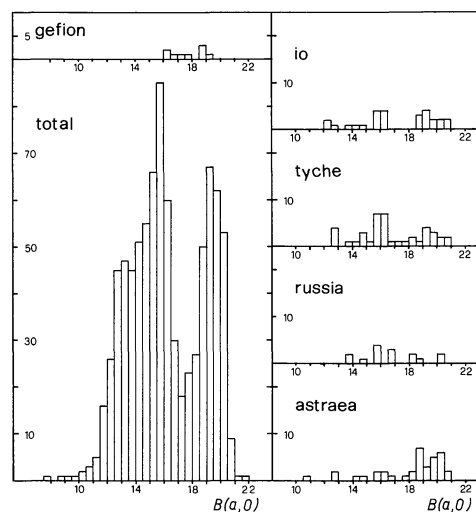


Figure 9 $B(a, O)$ distribution for the asteroid populations of the central ring and of the families therein contained.

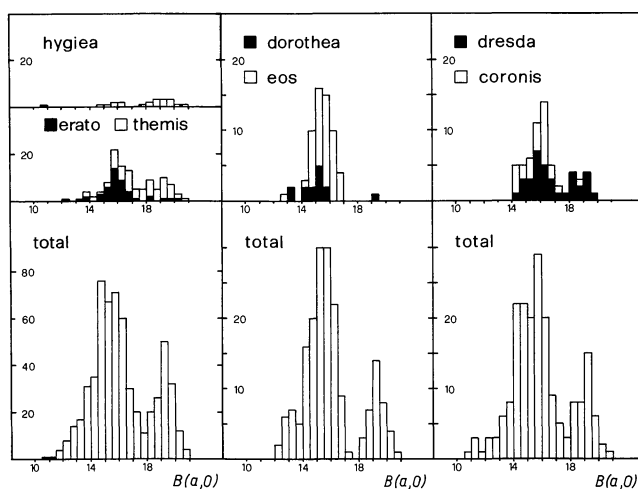


Figure 10 $B(a, O)$ distribution for the asteroid populations of the three subrings of Coronis, Eos, and Themis (outer ring) and of the respective families.