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Relativistic, Accreting Disks

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Summary. An analytic theory of the hydrodynamical structure of accreting disks (without self-gravitation but with pressure) orbiting around an axially symmetric, stationary, compact body (e.g. black hole) is presented. The inner edge of the marginally stable accreting disk (i.e. disk with constant angular momentum density) has a sharp cusp located on the equatorial plane between r_{ms} and r_{mb}. The existence of the cusp is also typical for any angular momentum distribution. The physical importance of the cusp follows from the close analogy with the case of a close binary system (L_1 Lagrange point on the Roche lobe). The existence of the cusp is thus a crucial phenomenon in such problems as boundary condition for the viscous stresses, accretion rate etc.

Key words: accreting disks — relativity — hydrodynamics

1. Introduction

Accreting disks models are of interest in the theory of X-ray sources. The standard model of such a disk follows the well-known Novikov and Thorne (1973) scenario:

Close binary—flow of matter with large angular momentum from the normal star through the Lagrange L_1 point toward the black hole—formation of the orbiting disk around the black hole—accretion onto the black hole and generation of the heat due to viscous stresses acting against shearing motion of matter—radiation of the heat.

All the models of the accreting disks known in 1976 have been calculated with the assumption that disk's self-gravitation was negligible. In this paper we will describe a very simple, analytic method of computation of the hydrodynamical structure of the disk when its self-gravitation is not important and when the equation of state is barytropic. This method is a simple generalisation of classical Roche's approach to the theory of rotating bodies. It is based on a general theory of the equipotential surfaces inside any relativistic, differentially rotating, perfect fluid body, which has been studied e.g. by Boyer (1965) and Abramowicz (1974).

The configurations described in this paper are called the "accreting disks" but one can also call them the "toroidal stars", because due to the pressure-gradient forces they are not necessarily thin. Recently Fishbone and Moncrief (1976) have constructed some special models of configurations of this type and gave their general theory which is—as opposed to our method—rather complicated.

We use $(+---)$ signature and $c=1=G$ convention.

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2. The Equipotential Surfaces

Theory presented here is not new. It was originated by the significant work of Boyer (1965) and then studied by many authors. Originally it was a study of the conditions for matching material, rotating bodies to exterior gravitational fields. The main result of the theory ("Boyer's condition") states that the boundary of any perfect fluid, barytropic, stationary body has to be an equipotential surface (in the sense defined below). It should be noticed that some authors made their searches for a material source for the Kerr metric trivially wrong because they did not use Boyer's condition or they used it incorrectly. A coordinate-independent formulation of the theory can be found in Abramowicz (1974). Here we will describe it in a special ("standard") coordinate system (Bardeen, 1973).

Let us assume that the external gravitational field is stationary and axially symmetric. This means that in the standard coordinate system the metric is given by:

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \qquad (1)$$

where $\partial_t g_{ik} = 0 = \partial_\varphi g_{ik}$ i.e. the metric depends neither on the time coordinate, t, nor on the azimuthal angular coordinate, φ. Let us assume that the body is made of a perfect fluid which rotates in the φ direction. In the

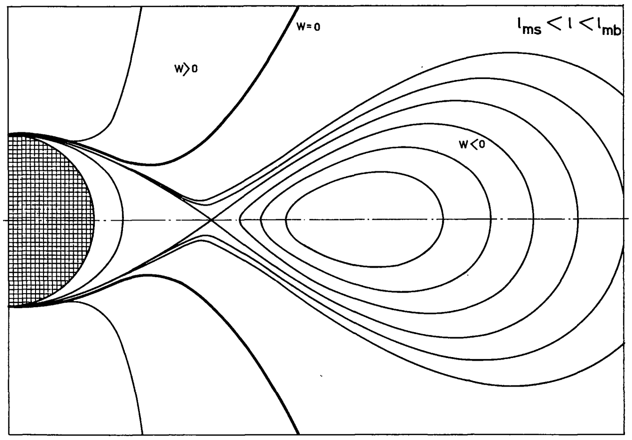

Fig. 1. Equipotential surfaces for the marginally stable disk with $l_0 = 3.77\ M$ orbiting the Schwarzschild black hole. The interior of the hole is dashed. This is the meridional section of the disk. The topology of the equipotential surfaces is the same also in the corresponding case of the Kerr metric

terms of the stress-energy tensor, T^i_k, and the four velocity, u^i, of the fluid it means that:

$$T^i_k = (p+\varepsilon)u^i u_k - \delta^i_k p, \tag{2}$$

$$u^i = (u^t, u^\varphi, 0, 0), \tag{3}$$

where ε and p are the total energy density and the pressure. Now, let us define the angular velocity of the rotating fluid, Ω, and its angular momentum per unit mass, l:

$$\Omega = u^\varphi/u^t, \qquad l = -u_\varphi/u_t. \tag{4}$$

From these definitions it follows that:

$$(u_t)^{-2} = -\frac{g_{\varphi\varphi} + 2l g_{t\varphi} + l^2 g_{tt}}{g^2_{t\varphi} - g_{tt} g_{\varphi\varphi}}, \tag{5}$$

$$\Omega = -\frac{g_{tt} l + g_{t\varphi}}{g_{t\varphi} l + g_{\varphi\varphi}}, \quad l = -\frac{g_{t\varphi} + \Omega g_{\varphi\varphi}}{g_{tt} + \Omega g_{t\varphi}}. \tag{6}$$

The equation of motion of the fluid, $h^k_i \nabla_j T^j_k = 0$, can be written as

$$\frac{\nabla_i p}{p+\varepsilon} = -\nabla_i \ln(u_t) + \frac{\Omega \nabla_i l}{1 - \Omega l}. \tag{7}$$

For a barytropic body, i.e. for the body with an equation of state

$$p = p(\varepsilon) \tag{8}$$

it follows from Equation (7) that:

1) There exists an invariant function, $\Omega = \Omega(l)$, which characterizes the body's rotation.

2) The surfaces of constant pressure ("equipotential surfaces") are given by Boyer's condition

$$\int_0^p \frac{dp}{\varepsilon + p} = W_{\rm in} - W = -\ln\frac{(u_t)}{(u_t)_{\rm in}} + F(l), \tag{9}$$

where the subscript "in" refers to the inner edge of the disk, and:

$$F(l) = \int_{l_{\rm in}}^{l} \frac{\Omega dl}{1 - \Omega l}. \tag{10}$$

Now, if one knows the external gravitational field of the compact companion, i.e. if one knows the functions

$$\begin{aligned} g_{tt} &= g_{tt}(r,\theta) \\ g_{t\varphi} &= g_{t\varphi}(r,\theta) \\ g_{\varphi\varphi} &= g_{\varphi\varphi}(r,\theta) \end{aligned} \tag{11}$$

one can easily find all the equipotential surfaces by the following procedure:

a) Specify the function $\Omega = \Omega(l)$ and then find the explicit form of the function $F(l)$ using Equation (10). One of the most important is the case of the marginally stable disk in which $l = l_0 = \text{const}$ (Seguin, 1975).

b) Solve Equation (6) in order to obtain $l = l(r,\theta)$ and $\Omega = \Omega(r,\theta)$. In the case of the marginally stable disk the solution is:

$$\Omega(r,\theta) = -\frac{l_0 g_{tt} + g_{t\varphi}}{l_0 g_{t\varphi} + g_{\varphi\varphi}}. \tag{12}$$

c) Knowing all this compute $W = W(r,\theta)$ using Equation (9) and then, knowing equation of state, compute $p = p(r,\theta)$. In the case of the marginally stable disk the result is

$$W(r,\theta) = \ln u_t. \tag{13}$$

3. Results

We have examined in details only the case of the marginally stable disks. On Figure 1, the equipotential surfaces for any possible disks with $l_0 = 3.77\ M$ orbiting the Schwarzschild black hole (with the total mass M) are presented. The matter can fill each of the closed equipotential surfaces [i.e. surfaces $W(r,\theta) = \text{const} < 0$]. One of these surfaces has a sharp cusp on the equatorial plane. It is easy to show that this cusp has to be located between the marginally bound circular orbit of a free test particle orbiting the black hole (r_{mb}) and the marginally stable circular orbit (r_{ms}). The proof is based on the fact that the location of the center of the disk and the cusp can be found in an extremly simple way. Note, that in these points the fluid moves freely as there are not the pressure-gradient forces ($\nabla_i W = 0 = \nabla_i p$). Therefore in these points we have

$$l_0 = l_K(r), \qquad \theta = \pi/2, \tag{14}$$

where l_K is the angular momentum of the Keplerian circular motion. For the Kerr black hole with the total mass M and the total angular momentum aM one can

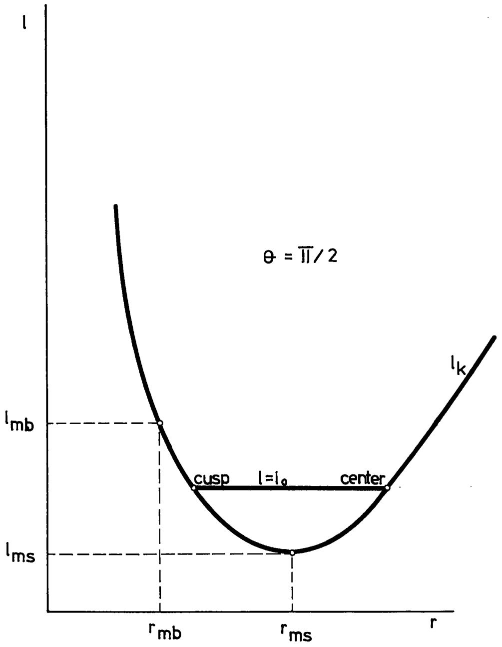

Fig. 2. Location of disk's center and its cusp

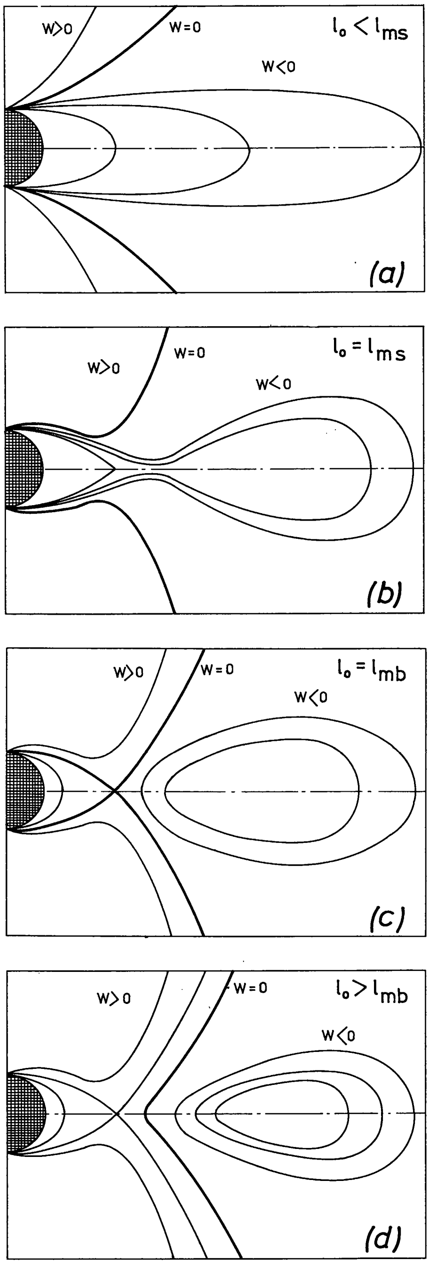

Fig. 3a—d. Topology of the equipotential surfaces (see text for details)

compute l_K using Equation (6) and the formula (Bardeen, 1973):

$$\Omega = \Omega_K = \pm M^{1/2}/(r^{3/2} \pm a M^{1/2}), \tag{15}$$

where the upper sings are for the direct ($\Omega_K a > 0$) and the lower for the retrograde ($\Omega_K a < 0$) orbits. The behavior of the function $l_K(r)$ is shown in Figure 2. The $r = r_{ms}$ point is the upper limit for the location of the cusp (and the lower limit for the location of the disk's center). On the other hand $W(r) \gtrsim 0$ for $r \lesssim r_{mb}$, i.e. the equipotential surfaces are open.

If the matter fills the equipotential surface with the cusp the accretion is driven (over the cusp) by the pressure-gradient forces rather than by viscosity (note, that all the circular orbits with $r \leqq r_{ms}$ are unstable). The existence of the cusp for any stable angular momentum distribution is typical simply for the reason that the distribution of l is stable if l increases outward (Seguin, 1975). A "general" angular momentum distribution can therefore cross the Keplerian one typically in two points, exactly like in the l=const case—see Figure 2. One of these points corresponds to the cusp. We do not agree with Fishbone and Moncrief (1976) who stated that any disks with the special angular momentum distribution $l/(1-\Omega l)$=const will have no cusp at the equator[1].

[1] We have explicitly constructed (Kozlowski et al., 1977; preprint, will appear in *Astron. Astrophys.* some models of the disks with the same angular momentum distribution as that employeed by Fishbone and Moncrief. We have found cusps in these models. We have also found some arguments that any accreting disk will have l=const in the vicinity of its inner edge, if the dissipative processes are present

Let us define $l_{ms} = |l_K(r_{ms})|$, $l_{mb} = |l_K(r_{mb})|$ and $l_0 = |l_0|$. The five topological possibilities of the different disks' models are the following:

I): $l_0 < l_{ms}$. No disks are possible (Fig. 3a).

II): $l_0 = l_{ms}$. The disk exists as an infinitesimally thin, unstable ring located on the circle $r = r_{ms}$ (Fig. 3b).

III): $l_{ms} < l_0 < l_{mb}$. There are possible many disks without cusps and one with the cusp (Fig. 1).

IV): $l_0 = l_{mb}$. The cusp is located on the marginally closed (W=0) equipotential surface (Fig. 3c). In this case the difference, ΔW, between the potential on the surface and in the center of the disk is maximal. It is equal to (1/2) ln 3=0.55 for the maximal Kerr black

Table 1. The maximal difference between the values of the potential W on the boundary and at the center of the disk

a/M	0.0	0.5	0.99	0.99999999	1.0
ΔW	0.0431	0.0635	0.246	0.536	0.549

ΔW is given in the units of c^2

hole. Some other values of ΔW are given in Table 1. Note, that for a stable neutron star $\Delta W_{max} \approx 0.3$.

V): $l_0 > l_{mb}$. The disks have no cusps (Fig. 3d).

4. Conclusions

The hydrodynamical structure of the accreting disk orbiting around a black hole can be computed from some simple algebraic formulae for any physically acceptable rotation law, $\Omega = \Omega(l)$. The very important new phenomenon found in this paper is the existence of the sharp cusp on the inner edge of the accreting disk. This is like the L_1 Lagrange point in the close binary system: the gas falls from the disk over the cusp toward the black hole with no dissipation of the angular momentum. The accretion is driven by the pressure-gradient forces and viscosity is not necessary. It is clear, therefore, that the boundary condition for the viscous stresses near the inner edge of the disk has to be re-examined (Paczyński, 1976).

Because the accretion onto the black hole is determined by the existence of the cusp, Thorne's (1974) conclusion that the Kerr black hole cannot have a/M greater than 0.998 is not necessarily true.

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