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ASTRONOMY AND ASTROPHYSICS

The Analytic Theory of Fluid Disks Orbiting the Kerr Black Hole

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Summary. Abramowicz et al. (1977) have recently shown that a sharp cusp exists on the inner edge of the accretion disk (with constant angular momentum), orbiting the Kerr black hole. The cusp resembles very much a similar cusp located on the Roche lobe in the close binary case (Lagrange L_1 point) and therefore its existence is very important from the physical point of view. In this paper we will show that the existence of the cusp is a typical phenomenon for any angular momentum distribution. We will also discuss the physical importance of the cusps. It is proved that the inner edge of any stable disk cannot be closer to the black hole than the marginally bound circular orbit, $r=r_{mb}$.

Key words: relativity — accretion disks — cusps

1. Introduction

The theory of non-geodesic, perfect fluid disks has been recently studied by Fishbone and Moncrief (1976), Fishbone (1977) and Abramowicz et al. (1977). Fishbone and Moncrief (FM) described the hydrodynamical structure of isentropic disks,

$$s=\text{const} \tag{1}$$

while Abramowicz et al. (AJS) did it in a more general case of barytropic disks,

$$p=p(\varepsilon). \tag{2}$$

Here s, p and ε denote the specific entropy, pressure and total energy density. Both FM and AJS assumed *stationary, axially symmetric, perfect fluid disks with a purely azimuthal flow*. These assumptions and (2) will be employed in this paper. There are five differential equations (four Einstein's field equations and one relativistic Euler's equation) which determine the hydrodynamical structure of a disk model. The authors integrated only the relativistic Euler equation and therefore in both FM and AJS solutions some unspecified functions appear. However, if one assumes that disk's self-gravitation is negligible and the background gravitational field is known, then only one unspecified function remains. It describes the law of disk's rotation. This function in the zero—viscosity case cannot be determined and has to be assumed a priori.

The $c=1=G$ convention and $+---$ signature is used in this paper. This means that all the physical quantities are measured in terms of length. The length unit is taken to be the mass of the black hole: $MG/c^2=1$. We also adopt the *standard, spherical coordinate system* in which the metric tensor does depend neither on the time coordinate, t, nor on the azimuthal coordinate, φ[1]. The convention adopted here is that the matter rotates in the positive direction of φ and that the angular momentum of the black hole can be either positive (direct disks, direct orbits) or negative (retrograde disks, retrograde orbits). In the standard coordinate system the line element of the spacetime can be written as

$$ds^2=g_{tt}dt^2+2g_{t\varphi}dtd\varphi+g_{\varphi\varphi}d\varphi^2+g_{rr}dr^2+g_{\theta\theta}d\theta^2. \tag{3}$$

On Figure 1 some most important features of the disk are presented and some important lines and points named. We will use subscripts "in" and "out" in order to mark the values of physical (or geometrical) quantities in the inner and outer edges of the disk. For example r_{in} denotes the "inner radius" of the disk on the equatorial plane. For a given disk and any physical quantity Q the quantities Q_{in} and Q_{out} do not depend on t, φ, r, θ coordinates. We will use the definition

$$\rho^2=g_{t\varphi}^2-g_{tt}g_{\varphi\varphi}.$$

In the Newtonian limit ρ is the distance from the axis of rotation.

2. Relativistic Concept of Angular Momentum Density

a) Newtonian Definition

The angular momentum per unit inertial mass (we will call it also "angular momentum density" or simply

[1] The reader interested in an invariant formulation of our method can consult Appendix 1 or e.g. the papers of Bardeen (1973) or Abramowicz (1974)

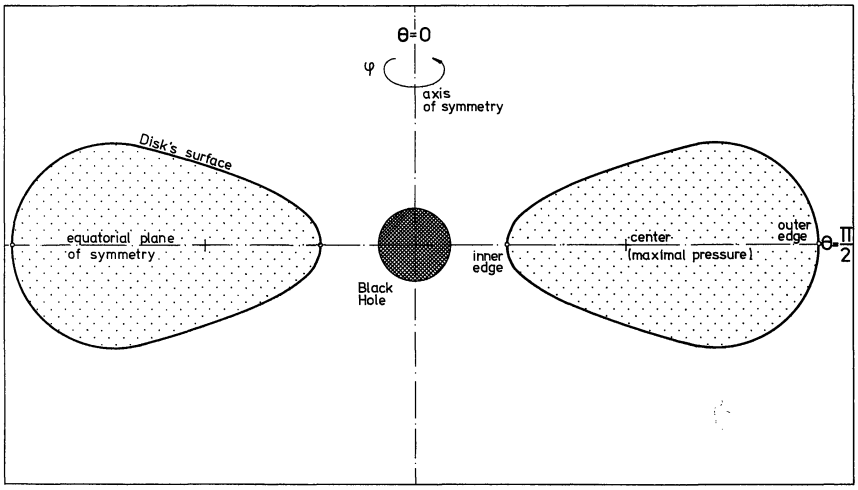

Fig. 1. The meridional section of the disk and the black hole

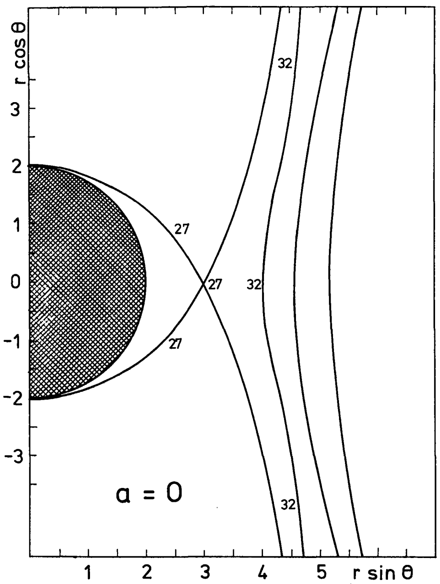

Fig. 1a. The von Zeipel cylinders for the test fluid orbiting the Schwarzschild black hole. The numbers give the values of l/Ω

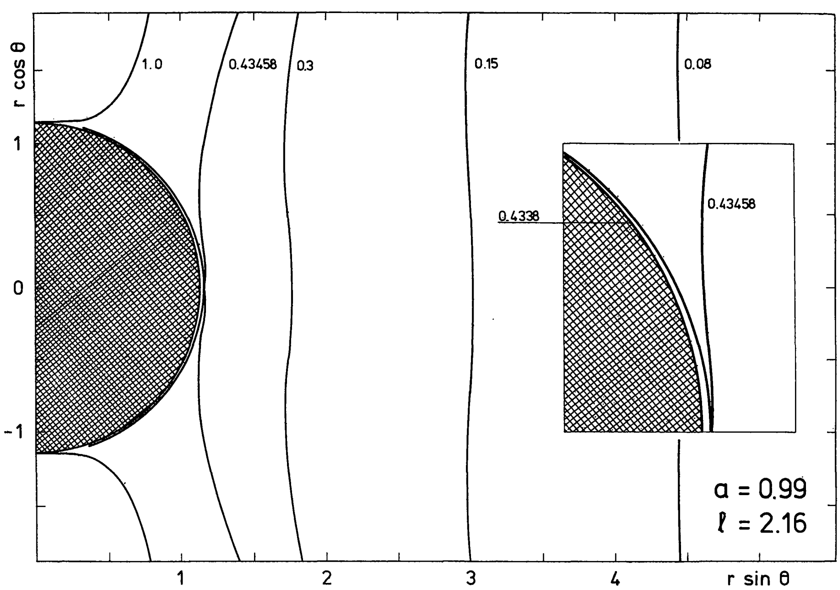

Fig. 1b. The von Zeipel cylinders for the test fluid with constant angular momentum ($l=2.16$) orbiting the Kerr black hole ($a=0.99$). The numbers give the values of Ω

"angular momentum") is defined in the Newtonian theory of gravitation by the formula:

$$l_N = \Omega R^2, \quad (4)$$

where Ω is the angular velocity and R is the distance from the axis of rotation. In the case of barytropic fluid von Zeipel's theorem guarantees that the surfaces $\Omega=\text{const}$ coincide with the surface $l_N=\text{const}$. This means that $l_N=l_N(R)$ and $\Omega=\Omega(R)$. Therefore, we can characterize a law of the rotation of the body by a specification of the form of the function

$$l_N = l_N(\Omega). \quad (5)$$

We will call this function the von Zeipel's formula. The criterion for the convective stability simple states that l_N has to increase outward,

$$\frac{d}{dR} l_N \geqq 0. \quad (6)$$

For an arbitrary motion in an axisymmetric, stationary gravitational field l_N is a constant of motion (does not change on a flow line):

$$\left(\frac{\partial}{\partial t} + v^i \nabla_i\right) l_N = 0. \quad (7)$$

Here v^i is the velocity and ∇_i is the operator of the covariant derivative. In addition to these, l_N is simply connected with the scalar of rotation, ω, of the congruence of the flow lines:

$$-\omega^2 = \tfrac{1}{4} R^{-2} (\nabla_i l_N) \cdot (\nabla^i l_N). \quad (8)$$

The definition of ω is: $-\omega^2 = (1/2)\omega^{ik}\omega_{ik}$; ω^{ik} is the antisymmetric part of $\nabla^i u^k$.

b) Relativistic Generalisation

A relativistic generalisation of the concept of the angular momentum density has to mimic as close as possible the above properties of l_N. The most frequently used relativistic definitions of the angular momentum density are the following:

$$l = -u_\varphi / u_t, \quad (9)$$

$$l_0 = -u_\varphi, \quad (10)$$

$$l_* = -u_\varphi u^t. \quad (11)$$

The relativistic angular velocity is defined by

$$\Omega = u^\varphi / u^t. \quad (12)$$

Here $u^i = (u^t, u^\varphi, 0, 0)$ is the four velocity of the fluid expressed in the standard coordinate system (3). Note, that $l_* = l/(1-\Omega l)$.

The reader can consult Table 1 in order to convince himself that the choice made in this paper, $l = -u_\varphi/u_t$, seems to be the most natural and reasonable.

Table 1. Properties of the angular momentum per unit mass

1. Definition	Newtonian: $l_N = \Omega R^2$	Relativistic: $l = -u_\varphi/u_t$	Relativistic: $l_0 = -u_\varphi$	Relativistic: $l_* = -u_\varphi u^t$
2. Newtonian limit	ΩR^2	ΩR^2	ΩR^2	ΩR^2
3. von Zeipel's theorem	Fulfilled	Fulfilled	Not fulfilled	Fulfilled
4. Is it a constant of motion?	Yes	Yes	Only in the vacuum	No
5. Criterion of stability	Simple: l_N has to increase outward	Simple: l has to increase outward	Simple only in the vacuum, l_0 has to increase outward	No simple criterion
6. Scalar of rotation	$\omega^2 \sim (\nabla l_N)^2$	$\omega^2 \sim (\nabla l)^2$	No simple relation	No simple relation
7. Keplerian distribution	Monotonic	Minimum at r_{ms}	Minimum at r_{ms} singular at r_{ph}	Minimum, singular at r_{ph}

r_{ms} = radius of marginally stable circular orbit
r_{mb} = radius of marginally bound circular orbit
r_{ph} = radius of circular photon orbit

From the definitions (12) and (9) it follows that

$$l = -\frac{g_{t\varphi} + \Omega g_{\varphi\varphi}}{g_{tt} + \Omega g_{t\varphi}}, \qquad \Omega = -\frac{g_{t\varphi} + l g_{tt}}{g_{\varphi\varphi} + l g_{t\varphi}}. \quad (13)$$

c) Von Zeipel's Cylinders

Let us define the angular momentum per baryon, j, the energy per baryon, $\mathscr{E}$, and the injection energy, Φ, by the following formulae (Bardeen, 1973):

$$j = -\frac{p+\varepsilon}{n} u_\varphi, \quad (14)$$

$$\mathscr{E} = \frac{p+\varepsilon}{n} u_t, \quad (15)$$

$$\Phi = \frac{p+\varepsilon}{n u^t}, \quad (16)$$

where n is the number baryon density. In the barytropic case the surfaces $l=\text{const}$ and the surfaces $\Omega=\text{const}$ coincide so one can introduce the function

$$F = (1-\Omega l) \exp \int_{l_{\text{in}}}^{l} (1-\Omega l)^{-1} \Omega dl, \quad (17)$$

where l_{in} denotes the value of l on the inner edge of the disk. In the isentropic case all the surfaces $\Omega=\text{const}$,

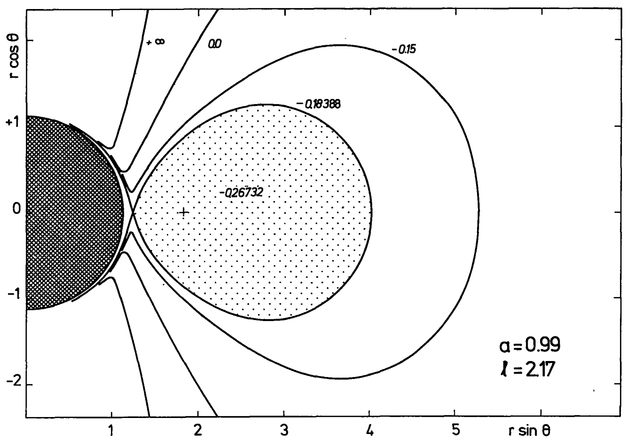

Fig. 2a. The meridional section of the disk ($l=2.17$) orbiting the Kerr black hole ($a=0.99$). The numbers give the values of the potential W. The fluid fills up the equipotential surface with the cusp. (This is only an example: the fluid can fill any closed equipotential surface—see Figure 2b)

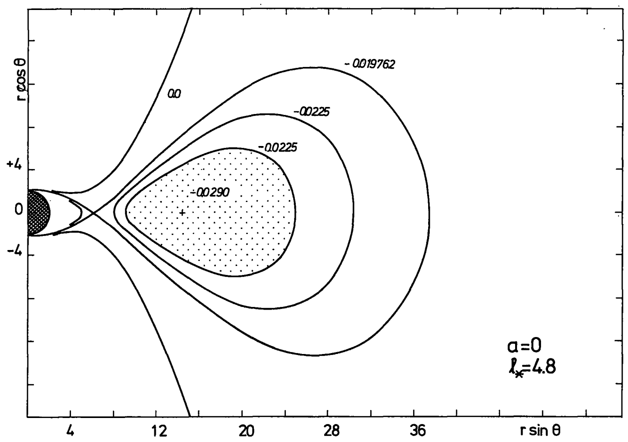

Fig. 2b. The same as in Figure 2a for the case $l_*=4.8$, $a=0$. There is the cusp on the closed equipotential surface $W=-0.019762$

$l=\text{const}, j=\text{const}, \mathscr{E}=\text{const}$ and $\Phi=\text{const}$ coincide and:

$$\Phi=\mathscr{E}_{\text{in}}F, \tag{18}$$

$$j=\mathscr{E}_{\text{in}}l(1-\Omega l)^{-1}F, \tag{19}$$

$$\mathscr{E}=\mathscr{E}_{\text{in}}(1-\Omega l)^{-1}F, \tag{20}$$

where $\mathscr{E}_{\text{in}}=\text{const}$ denotes the value of $\mathscr{E}$ on the inner edge of the disk (Abramowicz, 1974).

We will call the surfaces of constant l and Ω the von Zeipel's cylinders. In the case when the background gravitational field is known and a special form of the von Zeipel's formula is assumed one can easily find the explicit form of the functions

$$l=l(r,\theta), \quad \Omega=\Omega(r,\theta) \tag{21}$$

by solving Equation (13). In the case of a static gravitational field (e.g. Schwarzschild metric):

$$l/\Omega=-g_{\varphi\varphi}/g_{tt}. \tag{22}$$

This means that the equation of the family of von Zeipel's cylinders does not depend on the assumed rotation law, $l=l(\Omega)$. In the case of a stationary field with $g_{t\varphi}\neq 0$ (e.g. Kerr metric) the equation of the family of von Zeipel's cylinders will depend on the rotation law. Figure 1a and 1b gives two examples of the behaviour of the von Zeipel's cylinders.

The most remarkable property of the von Zeipel's cylinder is their cylinder-like topology inside the matter distribution (Abramowicz, 1974).

3. AJS Solution

The quantity of $A=u^t$ is called the redshift factor[2] and the quantity $U=u_t$ is the energy of the fluid element (per unit inertial mass). From the condition $u^iu_i=1$ it follows that:

$$A=(g_{tt}+2\Omega g_{t\varphi}+g_{\varphi\varphi}\Omega^2)^{-1/2}, \tag{23a}$$

$$U=-(g_{\varphi\varphi}+2lg_{t\varphi}+l^2g_{tt})^{-1/2}\rho, \tag{23b}$$

$$AU=(1-\Omega l)^{-1}. \tag{23c}$$

The AJS solution of the relativistic Euler equation,

$$\frac{\nabla_i p}{p+\varepsilon}=\nabla_i \ln A-\frac{l\nabla_i\Omega}{1-\Omega l} \tag{24a}$$

can be written in two equivalent forms

$$W-W_{\text{in}}=\ln(FA)_{\text{in}}-\ln(FA)=-\int_0^p\frac{dp}{p+\varepsilon}, \tag{24b}$$

$$W-W_{\text{in}}=\ln U-\ln U_{\text{in}}-\int_{l_{\text{in}}}^{l}\frac{\Omega dl}{1-\Omega l}=-\int_0^p\frac{dp}{p+\varepsilon}. \tag{24c}$$

The subscript "in" can be replaced here by "out". The quantity $W=W(p)$ is equal in the Newtonian limit to the total potential (gravitational plus centrifugal) expressed in the units of c^2. Therefore, at infinity $W=0$.

In the isentropic case considered by FM one has

$$\int_0^p\frac{dp}{p+\varepsilon}=\ln(h/h_{\text{in}}) \tag{25}$$

with $h=(p+\varepsilon)/n$ being the specific enthalpy. In any case the equipressure (equipotential) surfaces are given by the equation $W=\text{const}$.

For a known background gravitational field and a given von Zeipel's relation it is easy to determine the explicit form of the equation

$$W(r,\theta)=\text{const} \tag{26}$$

[2] If τ denotes the proper time of an observer with the four velocity $(u^t, u^\varphi, 0, 0)$, then $d\tau=dt/A$

from (24) and (21). Thus, the problem of constructing the disk model is completely solved.

Note, that from (24c) it follows that

$$\ln \frac{U_{\text{out}}}{U_{\text{in}}} = \int_{l_{\text{in}}}^{l_{\text{out}}} \frac{\Omega dl}{1-\Omega l}.$$

For a known background gravitational field and a given von Zeipel's relation this equation links r_{in} and r_{out}.

4. Two Explicit Solutions

Now, we are going to give explicit solutions for disks which obey two von Zeipel's relations

$$l = \text{const}, \tag{27}$$

$$l/(1-\Omega l) = l_* = \text{const}. \tag{28}$$

The first relation was used by AJS and recently by Fishbone (1977). The second relation was used by FM. We will assume that the background gravitational field is given by the Kerr solution:

$$g_{tt} = 1 - \frac{2r}{r^2 + a^2 \cos^2\theta}, \tag{29}$$

$$g_{t\varphi} = \frac{2ar \sin^2\theta}{r^2 + a^2\cos^2\theta}, \tag{30}$$

$$g_{\varphi\varphi} = -\left[r^2 + a^2 + \frac{2a^2 r \sin^2\theta}{r^2 + a^2\cos^2\theta}\right]\sin^2\theta, \tag{31}$$

with $|a| \leq 1$ being the total angular momentum of the Kerr black hole (Note, that in our notation, the total mass of the black hole is equal to one: $M=1$).

a) Solution with $l=$const

This solution is important for the following physical reasons:

First of all, according to Seguin (1975) configurations with $l=$const are marginally stable with respect to the axisymmetric perturbations (convective transport of the angular momentum). There are some reasons to believe (Weir, 1975) that if some specific mixing processes are present then the disk will tend to reach such a marginally stable state.

Secondly, if in a region of the disk the characteristic time scale of the dissipation of the angular momentum,

$$t_{\text{diss}} \approx l(u^i V_i l)^{-1} \approx l\left(v^r \frac{\partial l}{\partial r}\right)^{-1} \tag{32}$$

is much longer than the characteristic time scale of the radial drift,

$$t_{\text{drift}} \approx r/v^r \tag{33}$$

then the fluid moves inward with practically no dissipation of the angular momentum. In such a region we will have $l \approx$const. Note, that for the nearly Keplerian disk we have $\partial l/\partial r \approx 0$ for $r \approx r_{ms}$. (On the circle $r = r_{ms}$ the marginally stable circular orbit of a free test particle is located. It is believed that this corresponds to the inner edge for the nearly Keplerian accretion disks (see Novikov and Thorne, 1973). Therefore,

$$\frac{t_{\text{diss}}}{t_{\text{drift}}} \gg 1. \tag{34}$$

Thus, the nearly Keplerian disk will have $l=$const in the vicinity of its inner edge. Adopting formulae (17), (23) and (24) to the case $l=$const and remembering that at infinity we have $\Omega = l/r^2 = 0$, $A = (1 - l^2/r^2)^{-1/2} = 1$ we can write the final result:

$$l = \text{const}, \tag{35a}$$

$$\Omega = -\frac{g_{t\varphi} + l g_{tt}}{g_{\varphi\varphi} + l g_{t\varphi}}, \tag{35b}$$

$$A = (g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi})^{-1/2}, \tag{35c}$$

$$W = -\ln[A(1-\Omega l)] = \ln U. \tag{35d}$$

b) Solution for $l_*=$const

This solution has no strong physical motivation. We present it only in order to compare with FM solution of the same problem. Simple calculations similar to those for $l=$const case now gives

$$l_* = \text{const}, \tag{36a}$$

$$\Omega = -\frac{2l_* g_{t\varphi} + g_{\varphi\varphi} + (4l_*^2\rho^2 + g_{\varphi\varphi}^2)^{1/2}}{2l_* g_{\varphi\varphi}}, \tag{36b}$$

$$W = -\ln A + l_* \Omega. \tag{36c}$$

c) Nature of the Solutions $l=$const and $l_*=$const

Figure 3 gives two examples of the behaviour of the equipotential surfaces, one for $l=$const and one for $l_*=$const case. The matter can fill each of the closed surfaces $W=$const. Note, that in both cases one of the closed equipotential surfaces has a sharp cusp located on the inner edge of the disk. This, however, does not contradict the results of FM stated that no disks with $l_*=$const have any cusps.

In the FM method the cusps are explicitly excluded by the assumption that the pressure gradient cannot be zero on the disk boundary.

One can gain the best insight into the nature of both $l=$const and $l_*=$const solutions by the examination of the behaviour of the potential W on the equatorial plane of the Kerr black hole. This behaviour is qualitatively the same for disks with $l=$const and $l_*=$const and does not depend on the value of the Kerr parameter, a. It is shown on Figure 3. For $W<0$ we have closed equipotential surfaces and for $W>0$ the equipotential

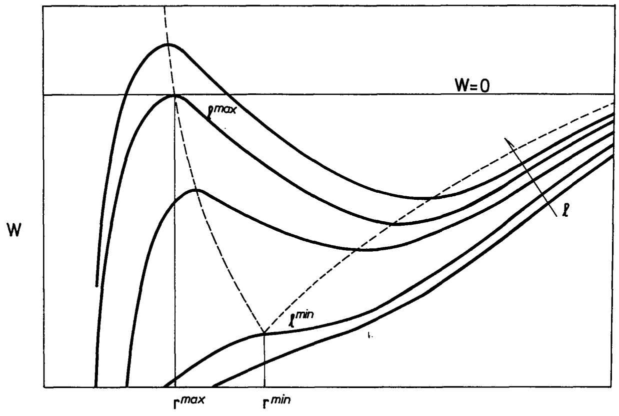

Fig. 3. The behaviour of the potential W on the equatorial plane, for l=const and l_*=const disks. The broken line shows the locii of the minima (center) and the maxima (cusp). The definitions of $l^{\max}$, $l^{\min}$, $r^{\max}$, $r^{\min}$ can be easily read out from this figure. Note the close analogy with the behaviour of the effective potential for free particles (Bardeen, 1973)

surfaces are open. The minima of W correspond to the centers (i.e. points in which the pressure is maximal) and the maxima to the cusps. The disk-like configuration is possible (center exists) only when

$$l \geqq l^{\min}(a) \tag{37}$$

in the case of l=const, or

$$l_* \geqq l_*^{\min}(a) \tag{38}$$

in the case of l_*=const.

For the critical values $l = l^{\min}$ (or $l_* = l_*^{\min}$) the center and cusp are in the same point $r^{\min}$ (or $r_*^{\min}$). It is evident, that $r^{\min}$ (or $r_*^{\min}$) is the minimal value of r for the center and the maximal one for the cusp. The minimal value of r for the cusp (and therefore also for the inner edge of the disk) is reached when the cusp is located on the marginally open equipotential surface, $W=0$. Such a situation is possible when $l = l^{\max}$ (or $l_* = l_*^{\max}$). Thus, we can write

$$r_{\rm in} \geqq r^{\max}(a) \tag{39}$$

in the case of l=const, or

$$r_{\rm in} \geqq r_*^{\max}(a) \tag{40}$$

in the case of l_*=const. Note, that in the case l=const we have

$$r^{\min} = r_{ms}, \tag{41a}$$

$$r^{\max} = r_{mb}. \tag{41b}$$

Figure 4 gives the behaviour of the critical values of l and r (l_* and r_*) for both direct ($al > 0$) and retrograde ($al < 0$) disks.

Note, that l=const disks are typically "more relativistic" than l_*=const disks.

d) *Convective Stability of the Models*

Seguin (1975) has shown that a configuration is stable with respect to the (local axisymmetric) convective perturbations if l increases outward. It is clear that all the models with l=const are then marginally stable. We have checked that all the models with l_*=const are also stable. (See Fig. 5 for an example of l_*=const distribution of the angular momentum, l.)

5. Physical Importance of the Cusps

We have shown that the cusp exists typically (for not too small and not too big angular momenta) for disks with angular distributions l=const and l_*=const. It is easy to note that for any stable distribution of the angular momentum the existence of the cusp will also be typical. On Figure 5 the dotted line represents a "general" (stable) angular momentum distribution. This line crosses the Keplerian angular momentum distribution in two points: the first corresponds to the cusp and the second to the center of the disk.

It is easy to show (see Appendix 2) that the cusp cannot be closer to the black hole then r_{mb} and that the same is also true for the inner edge of the disk:

$$r_{\rm in} \geqq r_{mb}. \tag{42}$$

This result in the special case of l=const disk was obtained by AJS and Fishbone (1977). The less restrictive (and in a sense trivial) condition $r_{\rm in} \geqq r_{ph}$ was obtained by FM.

The existence of the cusp is a very important phenomenon. It is clear that the accretion near the inner edge of the disk will be determined by the existence of the cusp which resembles very much a similar cusp located on the Roche lobe in the close binary system (Lagrange L_1 point). Having in mind this analogy Paczyński (1977) has proposed that the accretion into the black hole would be driven through the vicinity of the cusp due to a little overcoming of a critical surface, $W = W_{\rm cusp}$, by the surface of the disk, i.e. by a little violation of the hydrostatic equilibrium (Fig. 6). In this case no viscosity is needed in order to support the accretion in the vicinity of the cusp. This new scenario for the accretion onto the black hole removes the difficulties (pointed out by Stoeger, 1976) connected with the incorrect inner boundary condition adopted in the classical Novikov and Thorne (1973) accretion disk model. We will discuss this very important problem in details in a separate publication. Here we will give only a rough estimation of the maximal accretion rate following Paczyński's scenario for the accretion through the cusp.

Let v_r denote the drift velocity measured in the local rest frame of an observer whose motion is a pure rotation described by Equation (12). Let us also assume that in the vicinity of the inner edge of the disk we have l=const. This assumption follows from some arguments similar

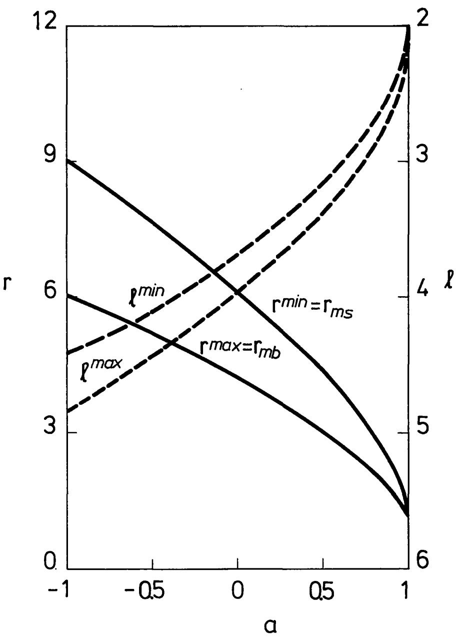

Fig. 4a. The critical values of l and r for l=const disks.

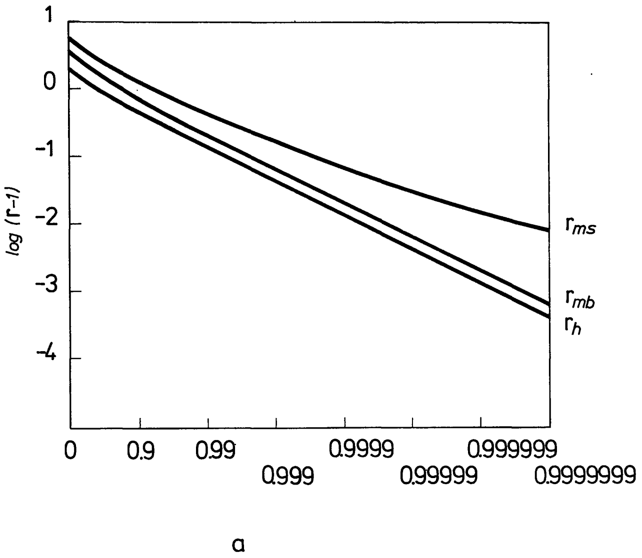

Fig. 4b. The critical values of r for l=const disks and $a \approx 1$ (see also Bardeen, 1973)

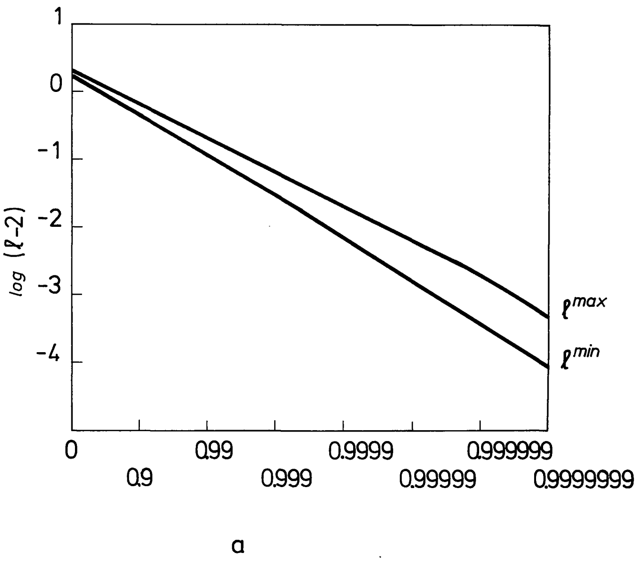

Fig. 4c. The critical values of l for l=const disks and $a \approx 1$

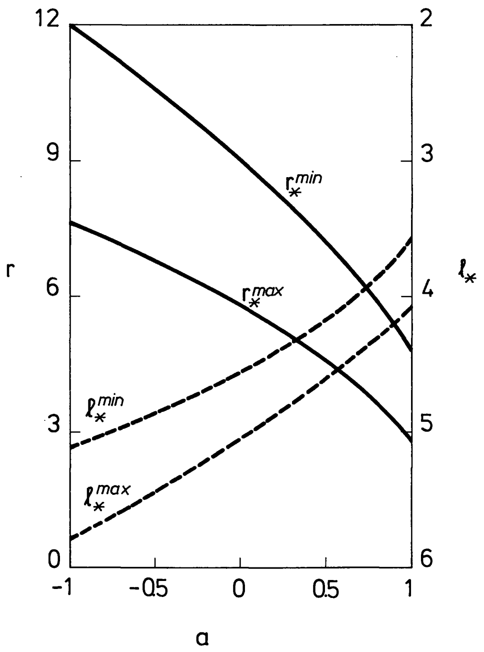

Fig. 4d. The critical values of l_* and r_* for l_*=const disks

to those used in derivation of formula (34). The relativistic Bernoulli equation (Novikov and Thorne, 1973) can be written in the following form

$$he^W(1-v_r^2)^{-1/2} = \text{const (on a flow line).} \quad (43)$$

Using this equation one can easily show that the flux of the rest mass density,

$$f = \frac{\varepsilon_0 v_r}{(1-v_r^2)^{1/2}}, \quad (44)$$

will be maximal when the radial drift velocity is equal to the velocity of sound, $v_s = (dp/d\varepsilon)^{1/2}$. Therefore, assuming that $v_s \ll 1$ and denoting $\beta = v_r/v_s$ one can write

$$f = \beta f^{\max} = \beta \varepsilon_0 v_s + O(v_s^3). \quad (45)$$

Now, let us assume that the fluid itself is not relativistic and obeys a polytropic equation of state

$$p = K\varepsilon_0^{1+1/N}, \quad \varepsilon_0 = \mu n \quad (46)$$

with K, N and μ being constant. In this case the rela-

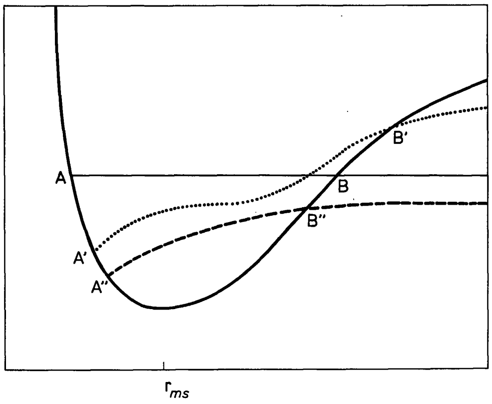

Fig. 5. Keplerian distribution of the angular momentum (A, A', A'', B'', B, B') compared with three examples of the angular momentum distributions inside disk: the l=const distribution (AB), l_*=const distribution ($A''B''$) and a "general" distribution ($A'B'$). The letter A indicates cusp and the letter B indicates center of the disk

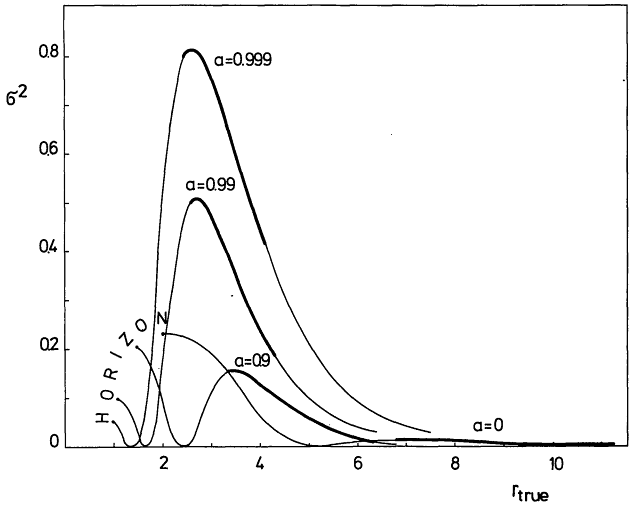

Fig. 7. The shear magnitude for the $l=\frac{1}{2}(l_{ms}+l_{mb})$ distribution on the equatorial plane of the disk. The heavy full lines correspond to the part of the disk between its center and cusp. The shear magnitude is plotted vs. $r_{true}=\int(-g_{rr})^{1/2}dr$. Note, that the maximum of the magnitude of shear almost exactly coincides with the cusp and that there exists a point between cusp and horizon where shear is equal to zero. This is important for the correct formulation of the inner boundary condition for the "integrated stress" (Novikov and Thorne, 1973; Stoeger, 1976)

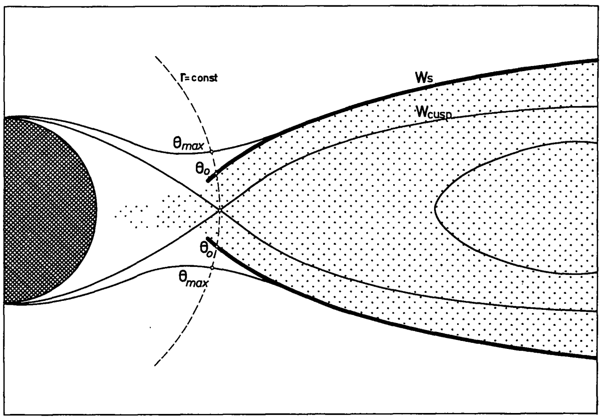

Fig. 6. Paczyński's scenario for the accretion. The surface S is defined by r=const, $|\theta-\pi/2|\leqq\theta_0$, $0\leqq\varphi\leqq 2\pi$. The surface S_{max} is defined by the similar condition with θ_{max} instead of θ_0. W_s denotes the equipotential surface which far away from the cusp coincide with the surface of the disk. The actual disk's surface is marked by the heavy full line

tivistic Bernoulli equation can be written as

$$K(1+N)\varepsilon_0^{1/N}+\tfrac{1}{2}v_r^2+W=W_S. \tag{47}$$

Here $K(1+N)^{1/N}$ and $\frac{1}{2}v^2$ are the first order terms in the expansions of $\ln h$ and $\ln(1-v_r^2)^{-1/2}$ with respect to $1/c^2$. The Bernoulli constant, W_S, is assumed to be the same for all the flow lines. This follows from the fact that far from the cusp the hydrostatic equilibrium is fulfilled, and that at the surface of the disk ($W=W_S$) both v_r and ε_0 are equal to zero. From the Bernoulli equation it follows that

$$f^{max}=\left(\frac{W_S-W}{N+1/2}\right)^{N+1/2}\left[\frac{N}{K(N+1)}\right]^N. \tag{48}$$

The accretion rate, $dM_0/d\tau$, measured by the "comoving" observer[3] is equal to the amount of the rest mass outflowing the disk in a unit proper time through the surface S (see Fig. 6):

$$\frac{dM}{d\tau_{comov}}=\int_S f dS. \tag{49}$$

The maximal possible rate of accretion is equal

$$\frac{dM}{d\tau_{comov}}=\int_{S_{max}} f^{max}dS=G(a,l)H(N)K^{-N}\Delta W^{1+N}, \tag{50}$$

where

$$\Delta W=|W_{cusp}-W_s|. \tag{51}$$

The factor $G(a,l)$ contains the information on the space time geometry and the behaviour of the potential W near the cusp:

$$G(a,l)=4\pi\left[\frac{g_{\varphi\varphi}g_{\theta\theta}}{W_2(1-v_\varphi^2)}\right]^{1/2}_{\substack{r=r_{cusp}\\ \theta=\pi/2}}, \tag{52}$$

[3] The "comoving" observer has $v_r=0$ and the four velocity given by $u^i=(u^t, u^\varphi, 0, 0)$

where

$$v_\varphi = lU/\rho A \tag{53}$$

is the azimuthal velocity measured in the locally non rotating frame (Bardeen, 1973) and

$$W_2 = \left(\frac{\partial W}{\partial \theta^2}\right)_{\theta=\pi/2} = -\frac{1}{2}\,\frac{Z_0+Z_1+2l(Y_0+Y_1)+l^2(X_0+X_1)}{Z_0+2lY_0+l^2X_0}, \tag{54}$$

with

$$X_0 = 1-2/r \qquad X_1 = 2a^2/r^3,$$
$$Y_0 = 2a/r \qquad Y_1 = -2a(1+a^2/r^2)/r, \tag{55}$$
$$Z_0 = -\left(r^2+a^2+\frac{2a^2}{r}\right), \quad Z_1 = r^2+a^2+(2a^2/r)\,(2+a^2/r^2).$$

Note, that

$$dS = \frac{2\pi(g_{\varphi\varphi}g_{\theta\theta})^{1/2}}{(1-v_\varphi^2)^{1/2}}\,d\theta. \tag{56}$$

The function $H(N)$ is defined by

$$H(N) = \frac{\sqrt{\pi}}{2}(1+1/N)^{-N}(N+1/2)^{-N-1/2}\,\frac{\Gamma(N+3/2)}{\Gamma(N+2)}. \tag{57}$$

In the usual physical units the formula for the accretion rate reads

$$\frac{dM_0}{d\tau_{\text{comov}}} = G_* H_* K^{-N} \Delta W^{N+1} \qquad (g/s), \tag{58}$$

where

$$G_* = 0.75\,(M/M_\odot)^2 G, \tag{59}$$

$$H_* = c^{2N+2} H \tag{60}$$

and G, H, ΔW are given in the units adopted in this paper (gravitational constant, velocity of light and the black hole mass equal to unity). The functions $G(a,l)$ and $H(N)$ are tabulated in Tables 2 and 3.

Formula (58) gives us the accretion rate in the frame of the comoving observer. The accretion rate in the frame of a stationary observer, $dM_0/d\tau_{\text{stat}}$, is equal

$$\frac{dM_0}{d\tau_{\text{stat}}} = \frac{dM_0}{d\tau_{\text{comov}}}\left(\frac{g_{tt}+2\Omega g_{t\varphi}+\Omega^2 g_{\varphi\varphi}}{g_{tt}}\right)^{1/2}. \tag{61}$$

In Paczyński's scenario for accretion it is assumed that $r_{\text{cusp}} \leqq r_{ms}$.

6. Discussion

The theory presented here does not include any dissipative processes and assumes that the disk's self-gravitation is negligible. The examples of $l=$const and $l_*=$const disks do not pretent to describe any realistic, astrophysical object[4]. They only give an idea of the advantages of our method: analytic form and simplicity. Now, we would like to make some estimations for the limitations of the applicability of the theory due to our simplified assumptions.

The work on a more sophisticated theory which will include all the processes omitted here is now in progress.

Table 2. The geometrical factor $G(a,l)$

a	$l^{\max}$	$\frac{1}{2}(l^{\max}+l^{\min})$	$l^{\min}$
−0.9999	0.957 E 03	0.134 E 04	0.340 E 04
−0.99	0.951 E 03	0.132 E 04	0.337 E 04
−0.9	0.892 E 03	0.124 E 04	0.314 E 04
−0.6	0.710 E 03	0.980 E 03	0.244 E 04
−0.3	0.547 E 03	0.748 E 03	0.182 E 04
0	0.402 E 03	0.544 E 03	0.128 E 04
0.3	0.275 E 03	0.366 E 03	0.824 E 03
0.6	0.166 E 03	0.215 E 03	0.450 E 03
0.9	0.692 E 02	0.849 E 02	0.149 E 03
0.99	0.353 E 02	0.410 E 02	0.579 E 02
0.9999	0.260 E 02	0.291 E 02	0.332 E 02

Note, that we use here the computer notation: 0.260 E 02 = $0.260\ 10^2$

Table 3. The function $H(N)$

N	0	1.5	3
$H(N)$	1.11	0.062	0.00226

a) Dissipative Processes

The dissipative processes such as the heat generation, transport of energy and angular momentum, radiation etc., have their source in the fact that the fluid (with a finite viscosity, η) rotates non-rigidly. The shear tensor, σ_{ik}, which characterises the non-rigidity of the rotation has the non-zero components:

$$-\sigma_{i\varphi} = \frac{1}{\Omega}\,\sigma_{it} = \frac{1}{2}A^3\rho^2\partial_i\Omega, \tag{62}$$

where $i=r,\theta$. Therefore, for a given angular momentum distribution (i.e. given von Zeipel's formula) and a given background gravitational field (e.g. Kerr spacetime with given a) σ_{iK} can be explicitly found in the form $\sigma_{ik} = \sigma_{ik}(r,\theta)$. The heat generation rate per unit volume, $\dot{\varepsilon}_{\text{heat}}$, is equal to

$$\dot{\varepsilon}_{\text{heat}} = 2\sigma^2\eta, \tag{63}$$

where $\sigma^2 = -\frac{1}{2}\sigma^{ik}\sigma_{ik}$ is the magnitude of the shear

[4] According to the discussion given in § 4a, however, the inner part of the nearly Keplerian disk will have $l=$const distribution of the angular momentum

tensor. From (62) it follows that:

$$\sigma^2 = -\left[\frac{\rho A}{2U(1-\Omega l)}\right]^2 [(\partial_r\Omega)^2 g_{rr}^{-1} + (\partial_\theta\Omega)^2 g_{\theta\theta}^{-1}]. \tag{64}$$

In Figure 7 we show the behaviour of this quantity on disk's equatorial plane for the case $l = \frac{1}{2}(l_{ms} + l_{mb}) = \text{const.}$ Note, that in the usual physical units, $(s)^{-2}$, the shear magnitude, σ_*^2 is equal to

$$\sigma_*^2 = \left(\frac{c^3}{GM_\odot}\right)^2 \left(\frac{M_\odot}{M}\right)^2 \sigma^2 = 4.1209\ 10^{10} \left(\frac{M_\odot}{M}\right)^2 \sigma^2. \tag{65}$$

This expression in the Newtonian theory of gravity reads:

$$\sigma_*^2 = \left(\frac{c^3}{GM_\odot}\right)^2 \left(\frac{M_\odot}{M}\right)^2 \frac{l^2}{\beta^4}, \tag{66}$$

where $\beta = r/(GM/c^2)$. It means, that in the Newtonian theory the magnitude of shear, when expressed in our units, can be written as

$$\sigma^2 = l^2/\beta^4.$$

Note, that the magnitude of shear for Keplerian orbits is equal in the Newtonian theory: $\sigma^2 = 9/16\ \beta^{-3}$. From Figure 7 and Equation (65) it follows that the dissipative effects will be, in general, more important in the case of galactic black holes, $M \approx 3M_\odot$, than in the case of supermassive black holes, $M \approx 3\ 10^7\ M_\odot$, and that they will be most important for the holes with the maximal angular momentum, $a \approx 1$.

If the heat generation rate is very small then the dissipative effects can be treated as a perturbation. The relativistic equations for the non-perfect fluid can be in this case linearised with respect to the *zero-order structure* discussed in details in this paper. Solution of these equations will give the *first-order structure* of the disk, i.e. the explicit disk model including the explicit model of the spectrum of radiation from disk. This zero-and-first-order-structure approach will probably be very fruitfull in the case of the nearly Keplerian disks.

The zero-and-first-order-structure approach is valid only if the heat generation rate is very small. This, however, depends strongly not only on the magnitude of shear, σ^2 but also on the viscosity coefficient, η which is not known with the sufficient accuracy. Therefore, we will not give here any numerical estimations.

b) Self-gravitation

The self-gravitation of the disk is important in two cases: *First,* if the total mass of the disk, m, is comparable with the mass of the black hole, M, and *second,* if it causes an instability.

In the first case our approach is valid, but one has to solve the Einstein field equations in order to determine the functions g_{ik}. This can be done by using a modification of Ostriker's SCF method. In the second case the discussion given by Bardeen (1973) will be probably a good starting point to solve the problem.

Let us examine the first possibility, i.e. let us check whether the self gravitation is negligible in the sense, that

$$\frac{m}{M} \ll 1.$$

In order to find the total mass of the disk, m, we will employ the well known Tolman's formula:

$$m = \iiint_{\text{disk}} (T_t^t - T_\varphi^\varphi - T_r^r - T_\theta^\theta)\sqrt{-g}\, d^3x, \tag{67}$$

where $d^3x = dr d\varphi d\theta$, $g = \det\{g_{ik}\}$ and

$$T_K^i = (p+\varepsilon)u^i u_k - \delta_k^i p \tag{68}$$

is the stress energy tensor (with $u^\theta u_\theta = u^r u_r = 0$). We will assume the $l = \text{const}$ case and the polytropic equation of state (46) with

$$p \ll \varepsilon. \tag{69}$$

After some simple algebraic manipulations Tolman's formula gives:

$$m = 2\pi\varepsilon_c \iint_{\text{disk}} \left(\frac{1+\Omega l}{1-\Omega l}\right)\left(\frac{W_{\text{in}} - W}{W_{\text{in}} - W_c}\right)^{1/N} (r^2 + a^2\cos^2\theta)\sin\theta\, d\theta dr, \tag{70}$$

where subscript c refers to the center of the disk.

In the usual physical units, grams, the formula for disk's mass, m_*, reads

$$m_* = \left(\frac{GM_\odot}{c^2}\right)^3 \left(\frac{M}{M_\odot}\right)^3 \varepsilon_c m. \tag{71}$$

This means, that in the usual units the m/M ratio can be expressed as:

$$\left(\frac{m}{M}\right)_* = 1.6192\ 10^{-18} \left(\frac{M}{M_\odot}\right)^2 \varepsilon_c m. \tag{72}$$

For a given background gravitational field each disk with constant angular momentum is characterised by two parameters: the value of the angular momentum, l, and the value of the potential on the boundary, W_{in}. We will use two others:

$$f_l = \frac{l - l_{ms}}{l_{mb} - l_{ms}}, \tag{73}$$

$$f_W = \frac{W_{\text{in}} - W_c}{W_{\text{cusp}} - W_c}. \tag{74}$$

The function $m = m(a, N, f_l, f_W)$ is shown in Figure 8. From this figure and Equation (72) one can conclude,

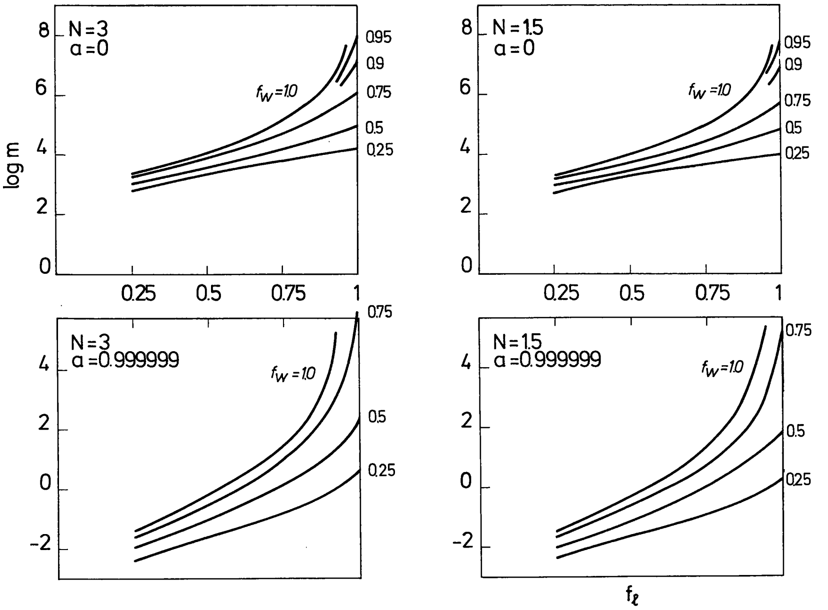

Fig. 8. The effects of the self-gravitation in the l=const disk models (see text for details)

that for the galactic black holes ($M \approx 3 M_\odot$) our approach is valid even in the case of superdense accretion disks ($\varepsilon_c \gtrsim 10^{17}$ g/cm^3). For the supermassive black holes ($M \approx 3\ 10^7\ M_\odot$) our approach is valid if $\varepsilon_c \lesssim 1$ g/cm^3.

7. Conclusions

The accretion disks with negligible self gravitation have sharp cusps on their inner edges. The cusp resembles the Lagrange L_1 point on the Roche lobe in the close binary case and its physical role is very similar to the L_1 point: Due to the existence of the cusp the accretion onto the black hole is driven through the vicinity of the cusp with no viscosity needed. The cusp (and also the inner edge) of any stable disk cannot be closer to the black hole than the radius of the marginally bound circular orbit.

Typically, the effects of the dissipative processes are more important for the galactic black holes ($M \approx 3 M_\odot$) than for the supermassive holes ($M \approx 3\ 10^7\ M_\odot$). Self gravitation is probably not important for galactic holes but it is important for the supermassive holes. Neither the dissipative effects, nor the self gravitation are included in our method.

Fishbone and Moncrief (1976) and Fishbone (1977) have found no cusps in their accretion disk models. There is, however, no conflict between their and our results because they explicitly excluded cusps from their solutions by an extra assumption that the pressure gradient cannot be zero on the disk boundary.

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Appendix 1

Here we give an invariant formulation of the most important definitions used in the body of the paper. We have assumed, that the spacetime is stationary and axially symmetric. This means that there are two Killing-vector-fields present:

η^i, which has open trajectories and is unit and time-like at infinity, and

ξ^i, which has closed trajectories and is everywhere spacelike.

In the standard coordinate system (3) we have

$$g_{tt} = \eta^i \eta_i, \quad g_{t\varphi} = \eta^i \xi_i, \quad g_{\varphi\varphi} = \xi^i \xi_i. \tag{A1}$$

The definitions of A and Ω can be read out from the assumed form of the four velocity:

$$u^i = A(\eta^i + \Omega \xi^i). \tag{A2}$$

The invariant definition of l is:

$$l = -u^i \xi_i / u^k \eta_k. \tag{A3}$$

Appendix 2

Here we give a formal proof that $r_{\mathrm{in}} \geqq r_{mb}$.

1*) According to our convention:

$$\Omega > 0, \quad u^t > 0. \tag{A4}$$

2*) Remark: For $r > r_{ph}$ we have

$$0 < \Omega_K l_K < 1, \tag{A5}$$

where K means "Keplerian".

3*) The center of the disk, $\theta = \pi/2$, $r = r_c$, is located on a Keplerian orbit (since $V_i p = 0$ there).

4*) *Lemma:* Everywhere in a disk (with a continuous angular momentum distribution) we have $1 - \Omega l > 0$.

Proof: $1 - \Omega l > 0$ for $\theta = \pi/2$ and $r = r_c$ (from 2* and 3*). If $\Omega l = 1$ in a point then $(u^\varphi/u^t)(u_\varphi/u_t) = -1$ in this point i.e. $u^t u_t + u^\varphi u_\varphi = 0$ in this point. This is, however, not possible.

5*) *Lemma:* $(\partial \ln u_t/\partial l)_{r,\theta} > 0$.

Proof: $(\partial \ln u_t/\partial l)_{r,\theta} = u_t u^\varphi = \Omega(1 - \Omega l) > 0$ (from 1* and 4*).

6*) *Theorem:* A stable disk with continuous angular momentum distribution has $r_{\rm in} \geqq r_{mb}$. Proof:

$$W_{\rm in} = \ln (u_t)_{\rm in} + \int_{l_{\rm in}}^{l_\infty} \frac{\Omega dl}{1 - \Omega l}. \tag{A6}$$

From 1* and 4* and assumed stability it follows that the integral in (A6) is positive. Therefore:

$$W_{\rm in} \geqq \ln (u_t)_{\rm in}. \tag{A7}$$

The condition: $l_{\rm in} \geqq l_K$ is necessary for equilibrium. It follows from Lemma 5* that $(u_t)_{\rm in} \geqq (u_t)_K(r_{\rm in})$ and therefore:

$$W_{\rm in} \geqq \ln [(u_t)_{\rm in}]_K. \tag{A8}$$

But $(u_t)_K > 1$ for $r < r_{mb}$, so we have

$$W_{\rm in} > 0 \quad \text{for} \quad r < r_{mb}. \tag{A9}$$

The condition for a surface $W = $const to be closed is $W \leqq 0$. Thus, we have proved that it is not possible to have any closed $W = $const surface in the region $r < r_{mb}$. Therefore, $r_{\rm in} \geqq r_{mb}$, Q.E.D.

References

Abramowicz, M. A.: 1974, *Acta Astron.* **24**, 45

Abramowicz, M. A., Jaroszyński, M., Sikora, M.: 1978, *Astron. Astrophys.* **63**, 221

Bardeen, J. M.: 1973, in *Black Holes,* eds. B. DeWitt and C. DeWitt, Gordon and Breach, New York

Fishbone, L. G.: 1977, *Astrophys. J.* **215**, 323

Fishbone, L. G., Moncrief, V.: 1976, *Astrophys. J.* **207**, 962

Novikov, I., Thorne, K. S.: 1973, in *Black Holes,* eds. B. DeWitt and C. DeWitt, Gordon and Breach, New York

Paczyński, B.: 1977 (private communication)

Paczyński, B., Sienkiewicz, R.: 1972, *Acta Astron.* **22**, 73

Seguin, F. H.: 1975, *Astrophys. J.* **197**, 745

Stoeger, W. R., S. J.: 1976, *Astron. Astrophys.* **53**, 267

Weir, A. D.: 1975, *Mem. Soc. Roy. Sci. Liège,* 6ᵉ serie, t. VIII, 37