A RELATION IN FAMILIES OF PERIODIC SOLUTIONS

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Abstract. We show the existence of a general relation between the parameters of periodic solutions in dynamical systems with ignorable coordinates. In particular, for time-independent systems with an axis of symmetry, the relation takes the form $\partial T/\partial A = -\partial\Phi/\partial E$, where T is the period, A is the angular momentum, Φ is the angle through which the system has rotated after one period, and E is the energy.

1. The General Relation

We wish to call attention to a curious relation which is obeyed by families of periodic solutions in dynamical systems. This relation does not seem to have been noticed so far, although it is rather simple.

We begin by recalling some classical notions (Whittaker, 1937, Section 38; Goldstein, 1956, Section 7.2). Consider a dynamical system with n degrees of freedom, defined by a Lagrangian

$$L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \tag{1}$$

and suppose that there are k *ignorable coordinates*. We may choose them to be $q_1, \ldots, q_k$, and the Lagrangian reduces to

$$L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t). \tag{2}$$

There are then k integrals of the motion

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i, \qquad (i = 1, \ldots, k) \tag{3}$$

and the system can be reduced to $n-k$ degrees of freedom by the process of *ignoration of coordinates*. A new function, called the *Routhian*, is introduced

$$R = \sum_{i=1}^{k} \beta_i \dot{q}_i - L. \tag{4}$$

The k Equations (3) are solved for $\dot{q}_1, \ldots, \dot{q}_k$ and the resulting expressions are substituted in (4), so that R takes the form

$$R(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, t, \beta_1, \ldots, \beta_k). \tag{5}$$

As is easily shown, R is then the Lagrangian of a reduced system with $n-k$ coordinates $q_{k+1}, \ldots, q_n$. The quantities $\beta_1, \ldots, \beta_k$ appear as parameters in this reduced system.

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Once a solution of the reduced system has been found, the corresponding solutions of the original system are obtained by computing the ignorable coordinates from

$$q_i = \int \frac{\partial R}{\partial \beta_i} \, \mathrm{d}t, \qquad (i = 1, \ldots, k). \tag{6}$$

Note that k integration constants will appear as arbitrary additive constants in the q_i.

We shall assume now that the Lagrangian (2) is periodic with respect to time, and we take the period equal to 1 for convenience

$$L(q_{k+1}, \ldots, \dot{q}_n, t + 1) = L(q_{k+1}, \ldots, \dot{q}_n, t). \tag{7}$$

The Routhian (5) is then also periodic with period 1. From now on we restrict our attention to *periodic solutions of the reduced system*, i.e. solutions such that the reduced variables and their derivatives come back to their initial values after one period

$$q_i(1) = q_i(0), \qquad \dot{q}_i(1) = \dot{q}_i(0), \qquad (i = k + 1, \ldots, n). \tag{8}$$

For given values of $\beta_1, \ldots, \beta_k$, we may expect such periodic solutions to exist in general, because (8) represents a system of $2(n-k)$ conditions on the $2(n-k)$ initial values.

We consider now a particular periodic solution of the reduced system

$$q_i(t), \quad (i = k + 1, \ldots, n) \tag{9}$$

and the corresponding solutions of the original problem. According to (6), after one period each ignorable coordinate q_i has increased by a quantity T_i, given by

$$T_i = q_i(1) - q_i(0) = \int_0^1 \frac{\partial R}{\partial \beta_i} \, \mathrm{d}t, \quad (i = 1, \ldots, k). \tag{10}$$

The T_i have no reason to vanish in general; thus, the ignorable coordinates do not come back to their initial values, and the solutions of the original problem are not periodic in general.

The quantities T_i, which we shall call *generalized periods*, do not depend on the integration constants in (6). They do not depend either on the origin of time, i.e., there is more generally

$$T_i = q_i(t + 1) - q_i(t). \tag{11}$$

Thus, the generalized periods T_i can be considered as intrinsic parameters of the periodic solution. They often have a simple physical meaning, as we shall see below.

Another intrinsic parameter of the periodic orbit is the *Lagrangian action* (Synge, 1960, Section 65), computed over one period

$$S = \int_0^1 R \, \mathrm{d}t. \tag{12}$$

So far we have been considering a particular periodic solution of the reduced system, corresponding to given values of the parameters $\beta_1, \ldots, \beta_k$. Suppose now that we let these parameters vary. The periodic solution will also vary, and will generate a *k-parameter family of periodic solutions*, defined by

$$q_i(t, \beta_1, \ldots, \beta_k), \quad (i = k+1, \ldots, n). \tag{13}$$

Each generalized period becomes a function of the parameters

$$T_i(\beta_1, \ldots, \beta_k), \quad (i = 1, \ldots, k). \tag{14}$$

The Lagrangian action S, defined by (12), also becomes a function of the β_i. We compute the partial derivatives of this function, using (5)

$$\frac{\partial S}{\partial \beta_i} = \int_0^1 \left(\sum_{j=k+1}^{n} \frac{\partial R}{\partial q_j} \frac{\partial q_j}{\partial \beta_i} + \sum_{j=k+1}^{n} \frac{\partial R}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \beta_i} + \frac{\partial R}{\partial \beta_i} \right) \mathrm{d}t. \tag{15}$$

Substituting Lagrange's equations

$$\frac{\partial R}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial R}{\partial \dot{q}_j} \right), \tag{16}$$

we obtain

$$\frac{\partial S}{\partial \beta_i} = \int_0^1 \left[\sum_{j=k+1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial R}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \beta_i} \right) + \frac{\partial R}{\partial \beta_i} \right] \mathrm{d}t. \tag{17}$$

The first term vanishes because the q_j, R, and their derivatives all take again the same values after one period. Taking (10) into account, we are left with

$$\frac{\partial S}{\partial \beta_i} = T_i, \quad (i = 1, \ldots, k). \tag{18}$$

The Lagrangian action S does not have an immediate physical meaning in general. Therefore, provided that the number of ignorable coordinates is two or more, it is convenient to eliminate S from the relations (18) and to write them in the form

$$\frac{\partial T_i}{\partial \beta_j} = \frac{\partial T_j}{\partial \beta_i}, \quad (i, j = 1, \ldots, k). \tag{19}$$

These equations relate in a simple way the generalized periods T_i and the integrals β_i inside a family of periodic orbits.

2. The Case of a Time-Independent Lagrangian

We consider now the very frequent case where the Lagrangian does not contain the time explicitly. We assume also, as in the preceding section, that the system has k ignorable coordinates, so that the Lagrangian has the form

$$L(q_{k+1},\ldots, q_n, \dot{q}_1,\ldots, \dot{q}_n). \tag{20}$$

As is well known (Whittaker, 1937, Section 42), the time can then be considered as an additional ignorable coordinate. To show this, we introduce a new independent variable τ and new dependent variables $Q_1,\ldots, Q_n$ defined by

$$q_i = Q_i \quad \text{for} \quad i \neq k+1; \qquad q_{k+1} = \tau; \qquad t = Q_{k+1}, \tag{21}$$

from which we obtain

$$\dot{q}_i = Q'_i/Q'_{k+1} \quad \text{for} \quad i \neq k+1; \qquad \dot{q}_{k+1} = 1/Q'_{k+1}, \tag{22}$$

where the prime represents derivation with respect to τ. In effect, we have simply exchanged the roles of q_{k+1} and t. The new Lagrangian L^* is given by

$$L^*\,\mathrm{d}\tau = L\,\mathrm{d}t, \tag{23}$$

or, substituting (21) and (22),

$$\begin{aligned} &L^*(Q_1,\ldots, Q_n, Q'_1,\ldots, Q'_n, \tau) = L(\tau, Q_{k+2},\ldots, Q_n, \\ &Q'_1/Q'_{k+1},\ldots, Q'_k/Q'_{k+1}, 1/Q'_{k+1}, Q'_{k+2}/Q'_{k+1},\ldots, Q'_n/Q'_{k+1})Q'_{k+1}. \end{aligned} \tag{24}$$

This new Lagrangian contains τ explicitly; on the other hand, it does not depend on $Q_1,\ldots, Q_{k+1}$. Thus we have a system with $k+1$ ignorable coordinates, to which the treatment of the previous section can be applied. We obtain relations

$$\frac{\partial T_i^*}{\partial \beta_j^*} = \frac{\partial T_j^*}{\partial \beta_i^*}, \quad (i, j = 1,\ldots, k+1) \tag{25}$$

where the β_i^* and the T_i^* are the integrals and the generalized periods for the new Lagrangian L^*. Using (24) and (3), we have

$$\begin{aligned} \beta_i^* &= \frac{\partial L^*}{\partial Q'_i} = \frac{\partial L}{\partial \dot{q}_i} = \beta_i, \quad (i = 1,\ldots, k), \\ \beta_{k+1}^* &= \frac{\partial L^*}{\partial Q'_{k+1}} = L - \sum_{i=1}^{k} \frac{\partial L}{\partial \dot{q}_i}\frac{Q'_i}{Q'_{k+1}} - \frac{\partial L}{\partial \dot{q}_{k+1}}\frac{1}{Q'_{k+1}} - \\ &\quad - \sum_{i=k+2}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{Q'_i}{Q'_{k+1}} = L - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i = -E, \end{aligned} \tag{26}$$

where E is the total energy of the system (Whittaker, 1937, Section 41). Thus, the first k integrals are the same as in the original system, while the last integral, associated to the ignorable coordinate Q_{k+1} which corresponds to the original time, is minus the total energy.

One minor difference with Section 1 is that the new independent variable τ does not grow monotonously, but oscillates for a periodic solution, since it corresponds to one of the original coordinates. Thus, a periodic solution will now be characterized by the property that the variables $Q_{k+2},\ldots, Q_n$ of the reduced system and their derivatives come back to their initial values when τ itself comes back to its initial value. Integrals from 0 to 1 in the equations of the previous section should be replaced here by closed path integrals, taken along one revolution of the periodic solution. The generalized periods are

$$T_i^* = \oint \mathrm{d}Q_i = \oint \mathrm{d}q_i = T_i, \quad (i = 1, \dots, k),$$
$$T_{k+1}^* = \oint \mathrm{d}Q_{k+1} = \oint \mathrm{d}t = T. \tag{27}$$

Thus, the first k generalized periods are the same as in the original system, and the last generalized period T_{k+1}^* is simply the ordinary period T of the orbit, i.e., the time taken by one revolution.

Using (26) and (27), we can now go back to the original system, described by the Lagrangian (20), and replace the relations (25) by

$$\frac{\partial T_i}{\partial \beta_j} = \frac{\partial T_j}{\partial \beta_i}, \quad (i, j = 1, \dots, k + 1), \tag{28}$$

where $T_1, \dots, T_k$ and $\beta_1, \dots, \beta_k$ are the generalized periods and the integrals associated with the k ignorable coordinates $q_1, \dots, q_k$, while T_{k+1} and β_{k+1} are defined by

$$T_{k+1} = T, \qquad \beta_{k+1} = -E. \tag{29}$$

3. Applications

In the remainder of this paper, we shall consider the case of a time-independent system with an axis of symmetry (Whittaker, 1937, section 39): we assume that q_1 is an ignorable coordinate, and moreover that an increase of q_1 by a quantity α corresponds to a rotation of the physical system through an angle α around the z axis. Then the associated integral β_1 is the angular momentum with respect to the z axis, which we call A. For periodic solutions, the generalized period T_1 is the angle through which the configuration has rotated after one period; we call that angle Φ.

Equations (28) and (29) give then

$$\frac{\partial T}{\partial A} = -\frac{\partial \Phi}{\partial E}, \tag{30}$$

a remarkable relation between period, rotation angle, angular momentum, and energy inside families of periodic orbits. We shall now consider some specific applications of this relation.

(a) We consider first the classical problem of the plane motion of a particle in a central field. This problem has two degrees of freedom. There are two ignorable coordinates, the position angle and the time, associated to two integrals, the angular momentum and the energy. Thus, the reduced system has zero degrees of freedom: the problem is completely integrable. Every solution is periodic in the reduced system;

this corresponds to the fact that every solution is a rosette curve (Landau and Lifchitz, 1960), repeating itself after a period T and a rotation Φ around the origin. Thus, the family of periodic orbits corresponds here to the totality of solutions. Each solution is characterized by its energy E and angular momentum A; for an arbitrary force law, the period and the rotation angle are therefore functions $T(E, A)$, $\Phi(E, A)$. The relation (30) applies to these functions.

The relation shows in particular that if all the orbits are closed, i.e. $\Phi \equiv 0$, then the period is a function of energy only. This is the case for the inverse-square law, and also for a force proportional to distance.

The 'isochron' model of spherical star clusters (Hénon, 1959) was defined by the property that the period is a function of energy only. It encompasses the two cases above, and also a more general case where the orbits are not closed. It was found that the rotation angle Φ is a function of A only. This observed property is now explained by the relation (30).

(b) We consider now the motion of a particle in a time-independent, three-dimensional, axisymmetric field. This problem has three degrees of freedom, and can be reduced to one by using the integrals of angular momentum and energy; it cannot be integrated further in general. For given values of A and E, isolated periodic orbits are found; when A and E are varied, two-parameter families are generated (see for instance Mayer and Martinet, 1973). These families should satisfy the relation (30).

(c) Another application of interest is the N-body problem. As is well known, the system can be reduced from $3N$ to $3N-6$ degrees of freedom (Whittaker, 1937, sections 157 to 159). We consider here for simplicity the system already reduced to $3N-4$ degrees of freedom by the following restrictions: the centre of mass is at rest at the origin, and the angular momentum is carried by the z-axis. Periodic solutions will then form two-parameter families, with A and E as parameters, and the relation (30) once more applies.

A further simplification of the relation is possible here, because of the similarity property of solutions of the N-body problem (Landau and Lifchitz, 1960, Section 10). Given any particular periodic solution, an infinity of other periodic solutions is obtained by multiplying distances by λ^2, times by λ^3, and velocities by λ^{-1}; λ is an arbitrary parameter. This one-parameter family of periodic solutions is a subset of the full two-parameter family to which the original periodic solution belongs. The energy E is multiplied by λ^{-2} and the angular momentum A is multiplied by λ. The angle Φ, being dimensionless, does not change. Therefore we have

$$\Phi(\lambda^{-2}E, \lambda A) = \Phi(E, A). \tag{31}$$

Deriving with respect to λ and substituting $\lambda = 1$, we obtain

$$A \frac{\partial \Phi}{\partial A} - 2E \frac{\partial \Phi}{\partial E} = 0 \tag{32}$$

and combining with (30)

$$\frac{\partial T}{\partial A} = -\frac{A}{2E}\frac{\partial \Phi}{\partial A}. \tag{33}$$

Because of the similarity property, when one studies periodic solutions (or indeed arbitrary solutions) of the N-body problem, it is sufficient to consider the solutions which have a given normalized value of the energy E. Periodic solutions form then simple one-parameter families, with A as parameter, and (33) can be written

$$\frac{\mathrm{d}T}{\mathrm{d}A} = -\frac{A}{2E}\frac{\mathrm{d}\Phi}{\mathrm{d}A}. \tag{34}$$

This relation has been verified numerically in the case of the plane three-body problem (Hénon, 1975); in fact, it was the observation that an extremum of T along the family coincided with an extremum of Φ which led to the relation (34), and then to the general relation.

We remark that (34) can be written in the even simpler form

$$\frac{\mathrm{d}T}{\mathrm{d}\Phi} = -\frac{A}{2E}. \tag{35}$$

For a generalized N-body problem in which the force is proportional to the power m of the separation, this relation becomes

$$\frac{\mathrm{d}T}{\mathrm{d}\Phi} = \frac{m+3}{2m+2}\frac{A}{E}. \tag{36}$$

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