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The Diffusion of Stellar Orbits Derived from the Observed Age-Dependence of the Velocity Dispersion

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Summary. The observed increase of the dispersion of stellar space velocities with age can best be explained by local fluctuations of the gravitational field in a galaxy. This irregular field causes a diffusion of stellar orbits in phase space. The value of the diffusion coefficient can be derived empirically from the observed increase of stellar velocities with age, even without any knowledge of the actual physical source of the irregular field. A disk star changes its space velocity at random by more than 10 km s^{-1} per galactic revolution. The diffusion in position is about 1.5 kpc after 2 10^8 years. The diffusion of stellar orbits hampers the determination of stellar birth places. It enhances the dissolution of stellar groups and would have severe implications for stellar dynamics of galaxies in general.

Key words: stellar kinematics — stellar dynamics — diffusion of stellar orbits — stellar birth places — dissolution of stellar groups

1. Introduction

Orbits of stars in a galaxy are usually calculated by considering only the regular part of the gravitational field caused by the smoothed-out distribution of matter which varies smoothly with position and time. The irregular part of the gravitational field in galaxies is generally considered to be negligible because the relaxation time due to encounters between the stars is some orders of magnitude larger than the age of the Universe (e.g. Chandrasekhar, 1960). However, other perturbation processes beside two-body encounters of stars may lead to much larger local fluctuations of the gravitational field. Spitzer and Schwarzschild (1951, 1953) discussed gravitational encounters between stars and large interstellar cloud complexes and their influence on stellar velocities. Even today, however, the existence of such complexes of interstellar matter, involving required masses of about $10^6\ \mathscr{M}_{\odot}$, remains a delicate and rather unsettled observational problem. Also since other possible perturbation mechanisms, discussed in Section 3, have not been firmly established as significant, the probable existence of a considerable irregular gravitational field in a galaxy has been either explicitely denied or at least effectively ignored in most studies of stellar orbits in galaxies.

It is the purpose of the present paper, (1) to point out that the observed increase in the velocity dispersion of stars with age does strongly suggest the existence of a significant component of the galactic gravitational field with a rather stochastic behaviour, and (2) to investigate the implications of such an irregular field for stellar orbits. The perturbations of stellar orbits can be studied quantitatively on the basis of the observed velocities, even without a detailed knowledge of the basic source of the irregular gravitational field. The effect of the irregular field on stars can be described formally by a diffusion process in velocity space. The result of this process we propose to call "diffusion of stellar orbits".

2. Observed Velocity Dispersions

It is well known that the dispersion of the peculiar space velocities of stars increases with the age of the objects. Since this fact provides the observational basis of the following sections, we shall collect here recent results on this relation. The data are mainly based on the observed space velocities of nearby stars as catalogued by Gliese (1969). A summary of these results has been published by Wielen (1974a). More details will be found in forthcoming papers by Jahreiß and Wielen (in preparation).

The velocity dispersion is defined as the root mean square of the stellar space velocity $\boldsymbol{v}$, or its components, U, V, W, measured relative to the local mean motion. In practice, the dispersion of disk stars changes only slightly if referred to the local circular velocity instead of to the local mean motion of the stars under consideration. The U-axis points towards the galactic center, the V-axis in the direction of galactic rotation

Table 1. Velocity dispersions of stars as a function of age

Group of stars		At $z\sim 0$			Integrated over z				Age
		$\sigma_U^{(0)}$ km s^{-1}	$\sigma_V^{(0)}$	$\sigma_W^{(0)}$	σ_U km s^{-1}	σ_V	σ_W	σ_v	$\langle\tau\rangle$ 10^9 yrs
Classical Cepheids					8	7	5	12	0.05
Nearby	*6d*	14	8	4	14	8	3	16	0.21
stars	*6c*	17	7	4	20	7	4	21	0.47
on or near	*6b*	14	11	8	15	12	8	21	1.0
the main	*6a*	27	18	11	31	20	11	39	2.3
sequence	5	34	21	21	42	26	25	56	5.0
McCormick	HK +8/+3	18	10	8	20	10	6	23	0.3
K+M	HK +2	21	16	13	22	17	13	31	1.4
dwarfs	HK +1	29	17	15	30	16	15	37	3.0
	HK 0	38	23	20	40	21	21	50	5.2
	HK −1	40	27	26	40	34	34	63	7.2
	HK −2/−5	66	27	23	67	29	25	77	9.0
All McCormick stars		39	23	20	48	29	25	62	5.0

and the W-axis to the north galactic pole. The velocity dispersions of nearby stars have not been corrected for observational errors because the bias is rather small. An uncertainty of $\pm 20\%$ in the distances of nearby stars causes an overestimation of the velocity dispersion by less than 2%.

The nearby stars are observed in a volume element which lies close to the galactic plane, essentially at $z=0$. Since the velocity dispersion σ will in general depend on the distance z from the galactic plane, we shall use as a more representative quantity the velocity dispersion averaged over z. The velocity dispersion σ within a cylinder perpendicular to the galactic plane, can be obtained directly from the velocities observed at $z=0$ by

$$\sigma_U^2 = \sum_i |W_i| U_i^2 / \sum_i |W_i| , \quad (1)$$

$$\sigma_V^2 = \sum_i |W_i| V_i^2 / \sum_i |W_i| , \quad (2)$$

$$\sigma_W^2 = \tfrac{1}{2} \sum_i |W_i| W_i^2 / \sum_i |W_i| , \quad (3)$$

where U_i, V_i, W_i are the observed components of the space velocity of star i at $z=0$, measured relative to the local mean velocity. The method of weighting each nearby star by its W-component (Wielen, 1974a) is based on the assumptions that the group of stars is well-mixed in z, that the oscillation period of the z-motion is independent of the amplitude in z, and that the motion of a star parallel to the galactic plane is decoupled from its z-motion. The factor 1/2 in σ_W^2 stems from the fact that for a harmonic variation of W, $W=W_0 \cos \omega_z (t-t_0)$, the square of the average dispersion, $\sigma_W^2 = \langle W^2 \rangle$, is just one half of $\langle W_0^2 \rangle$. For the younger stars, the sampling interval Δz around $z=0$ (distances up to about 20 pc from the Sun) is not really small compared to the amplitude of the z-motion, $z_{max} = |W_0|/\omega_z$. However, even if Δz is 50% of z_{max}, our procedure for obtaining σ is correct within 10%, because W varies only slowly with time or z for $|z| \leq 0.5\, z_{max}$.

Our results on the relation between the velocity dispersion and age are summarized in Table 1. $\sigma^{(0)}$ is the velocity dispersion at $z=0$, σ is the velocity dispersion averaged over z, and $\sigma_v = (\sigma_U^2 + \sigma_V^2 + \sigma_W^2)^{1/2}$ is the dispersion of the total space velocity $\boldsymbol{v}$. The groups of nearby stars on or near the main sequence, called *6d* to 5, are those defined by Wielen (1974a). The mean age, $\langle\tau\rangle$, of such a group is assumed to be half the lifetime of a main sequence star at the mean position of the group on the H-R diagram. The McCormick K+M dwarfs in Gliese's Catalogue are dated (Wielen, 1974a) by means of their Ca II emission intensity HK (Wilson and Woolley, 1970). All ages are based on a constant rate of formation of disk stars over 10^{10} years. In addition to these nearby stars, we have added as representatives for young stars the classical cepheids within 1 kpc (Wielen, 1974b). The velocity dispersions of the cepheids correspond already to an average over z, because the cepheids are not selected according to z. The ages of the cepheids have been derived from their pulsation periods.

The difference between $\sigma^{(0)}$ and σ, shown in Table 1, indicates a correlation between the W-component and the U- and V-components of the stellar velocities, even for stars of the same age. This is in fact the general case, while the familiar Schwarzschild distribution with its vanishing correlation between U, V and W is degenerate in this respect.

The mean mass of the McCormick K+M dwarfs does not vary significantly over the different age groups defined by HK. The close agreement between the relations $\sigma(\tau)$ obtained from main sequence groups (with different mean masses) and from the McCormick stars rules out any significant dependence of the velocity dispersion on stellar mass. Even open clusters of various ages, with much higher total masses, follow closely the relation $\sigma(\tau)$ obtained from field stars (Wielen, unpublished).

Finally, Table 1 indicates that the ratio of the velocity dispersion in different directions, $\sigma_U:\sigma_V:\sigma_W$, does not change significantly with age. Hence the total velocity dispersion, σ_v, is fairly representative for the relative increase of all dispersions with age.

3. Possible Mechanisms

What causes the observed increase in stellar velocity dispersion with age τ? There are three main classes of explanations: (1) variation of the typical velocity at birth with the time of formation, (2) acceleration by global gravitational fields, (3) acceleration by local fluctuations of the gravitational field.

Although it is rather probable that the initial velocities of stars do vary with the time of formation over 10^{10} years, this cannot explain the observed relation $\sigma(\tau)$. In Figure 1, we show the velocity dispersion σ as a function of the time of formation, $t_f = 10^{10}$ yrs $-\tau$, on a linear scale. It is obvious that the velocity dispersion of stars varies slowly with time t_f for old stars, but decreases rapidly for stars born in the last billion years. This would mean an undue preference of our present epoch, violating the general cosmological principle that neither our position nor our epoch is of special significance. Although we do not know at present which property of the interstellar medium actually determines the initial velocity dispersion of stars at birth, we would expect a rapid variation during the early times but a gentle variation at present, contrary to the observed run of $\sigma(t_f)$. We must emphasize that we are concerned here with stars belonging to the galactic disk only. The much higher velocity dispersion of halo objects ($\sigma_v \sim 200$ km s^{-1}) is certainly due to the particular conditions at their time of formation during the early phase of galactic evolution.

If the initial velocities are not responsible for the observed increase of σ with τ, then the stars must have been accelerated after their formation by gravitational processes. The effect of such an acceleration mechanism depends on the period of time over which it affects the stars. Therefore, the velocity dispersion should mainly be a function of the actual age of a star instead of its time of formation. If we could obtain the data presented in Figure 1 as a function of various epochs of observation, we would see merely a horizontal shift of the fitting curve as the epoch of observation varies. Hence in the case of an acceleration mechanism, our present epoch has no exceptional significance.

The most promising *global* acceleration mechanism is the gravitational field of a density wave (Barbanis and Woltjer, 1967). A stationary density wave, however, will usually produce an increase in velocity dispersion only of young stars up to an age of a few times 10^8 yrs. For stars older than 10^9 yrs, the contribution of the density wave to the velocity dispersion remains constant. This has been established by numerically calculating the orbits of many test stars (Wielen, 1975; Schwerdtfeger and Wielen, in preparation). The asymptotic behaviour can be understood qualitatively from the following linear approximation: Consider the radial velocity component U of a star. In the presence of a density wave, the star behaves like a forced harmonic oscillator: $U(t) \sim U_{\text{per}} \cos 2(\omega - \Omega_p)(t - t_p) + U_{\text{epi}} \cos \varkappa (t - t_{\text{epi}})$. The first term is due to the density wave, the second one represents the free epicyclic oscillation. U_{per} is the velocity amplitude of the periodic orbit, Ω_p is the rotational frequency of the two-armed spiral pattern, ω is the basic galactic rotation frequency, t_p is a phase constant, $\varkappa$ is the epicyclic frequency, and U_{epi} and t_{epi} are two integrals of motion fixed by the starting conditions. The final velocity dispersion is obtained by averaging over t and the individual values of U_{epi}: $\sigma^2_{U,\text{final}} = \sigma^2_{U,\text{per}} + \sigma^2_{U,\text{epi}} = 0.5\ (U^2_{\text{per}} + \langle U^2_{\text{epi}} \rangle)$. For the usually adopted density wave in our Galaxy (e.g. Wielen, 1973), we obtain from the periodic orbit (Wielen, 1975) $\sigma_{U,\text{per}} = 8.4$ km s^{-1} at $R \sim 10$ kpc. The value of $\sigma_{U,\text{epi}}$ is built up by the contributions of the mean velocity of stars at birth and of the initial velocity dispersion at birth. We expect $\sigma_{U,\text{epi}} \leqq 15$ km s^{-1}. This leads to $\sigma_{U,\text{final}} \leqq 17$ km s^{-1}. Hence the acceleration of stars by a stationary density wave cannot be responsible for the observed velocity dispersion of old disk stars ($\sigma_U \geqq 60$ km s^{-1}). Similar objections can be raised against other global acceleration mechanisms, such as the gravitational field of a central bar or of other stationary deviations from axial symmetry.

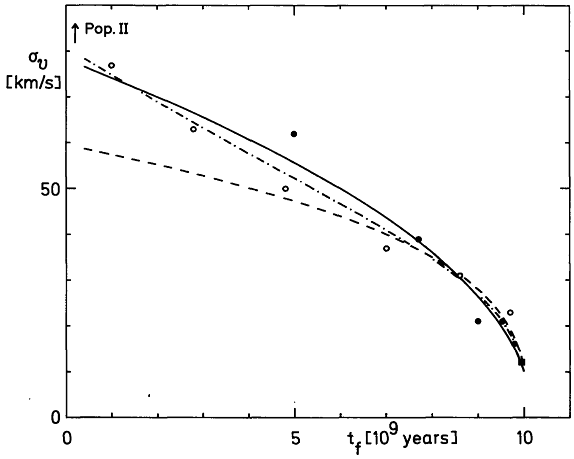

Fig. 1. Total velocity dispersion σ_v as a function of the time of formation t_f. Symbols: observed values. Curves: Theoretical fits based on different diffusion coefficients D. Full curve: constant D; dashed curve: velocity-dependent D; dash-dotted curve: velocity-time-dependent D.

The situation is more favourable if the density wave is not stationary. Barbanis and Woltjer (1967) proposed a sequence of transient density waves. If a galaxy switches over abruptly to another density wave of uncorrelated phase, $\sigma^2_{U,\mathrm{final}}$ increases by $(3/2+(\varkappa/(\omega-\Omega_p))^2/8)\,\sigma^2_{U,\mathrm{per}}=(1.56\,\sigma_{U,\mathrm{per}})^2$ per transition on the average. To explain the observed value of $\sigma_U \geqq 60$ km s^{-1} for the oldest disk stars, more than 20 consecutive density waves are required, each with a mean lifetime of less than 5 10^8 yrs. Slowly growing and decreasing density waves have no permanent effect on $\sigma_{U,\mathrm{final}}$, because the stars react "adiabatically". A more plausible deviation of a density wave from stationarity may be local and transient wriggles in a density wave of permanent grand design. This situation is already rather close to purely local accelerating mechanisms.

In a "*local*" acceleration mechanism, the corresponding gravitational field will vary "rapidly" as a function of position and (perhaps) time. The best studied process of this kind (beside the ineffective star-star encounters) is the Spitzer-Schwarzschild mechanism of stellar encounters with large concentrations of interstellar material. Spitzer and Schwarzschild (1953) had to postulate masses of these complexes of about $10^6\,\mathscr{M}_\odot$. Julian and Toomre (1966) have shown that such a concentration of matter will induce a wavelet in the stellar distribution. Such an amplification may lower the required mass of the initial perturbation considerably. While this picture is still the most convincing proposal for explaining the observed increase in velocity dispersion, it still lacks direct observational confirmation. Neither observations in our Galaxy nor studies of external galaxies have led to any firm conclusion about the "roughness" of the gravitational field produced by local inhomogeneities in the density distribution of interstellar matter or stars.

Other local acceleration processes beside the Spitzer-Schwarzschild mechanism are possible: There may be rapid fluctuations in the density of both the interstellar medium and young populations of stars, e.g. due to instabilities. Transient density wavelets (e.g. Julian, 1967), coexisting with a global density wave, will also be effective for gravitational scattering of stars. Local wriggles in a global density wave have already been mentioned. Generally speaking, optical pictures or H I maps of external galaxies usually give the impression that local irregularities are very common features in the structure of disk galaxies, although the corresponding irregularities of the gravitational field are difficult to estimate. Numerical experiments on the dynamical evolution of galaxies (e.g. Hohl, 1971; Miller et al., 1970) reveal also small-scale fluctuations of the gravitational field, but the still unsolved problem of the global stability of galactic disks (e.g. Toomre, 1974) casts doubts on the applicability of these results in real galaxies.

4. Diffusion in Velocity Space

The discussion of possible mechanisms has shown that *local* accelerating processes are most probably responsible for the observed increase in velocity dispersion with age. This identification is, however, mainly based on the inability or inefficiency of other processes rather than on a positive confirmation of a specific local mechanism. Hence it seems most appropriate to investigate the implications of the irregular gravitational field for stellar orbits in a general way, using only the essential properties of such a process instead of uncertain details. Most local accelerating processes, including the Spitzer-Schwarzschild mechanism, can be approximated by a sequence of independent and random perturbations of short duration. Each of these stochastically distributed impulses changes the velocity $\mathbf{v}$ of a star instantaneously by a (small) amount $\Delta\mathbf{v}_i$, but not the position at that time. Such a process is usually called a "diffusion in velocity space". We assume that the sum of the changes in velocity during a (short) period of time, Δt, vanishes on the average, $\langle\sum_i \Delta\mathbf{v}_i\rangle=0$, and that the average value of the sum of the squares of $\Delta\mathbf{v}_i$ increases proportionally to Δt,

$$\langle\sum_i (\Delta\mathbf{v}_i)^2\rangle = D_v \Delta t. \tag{4}$$

The quantity D_v is a diffusion coefficient. The basic idea of this paper is that a local accelerating process is mainly characterized by the corresponding diffusion coefficient D_v and that the value of D_v can be empirically determined from the observed increase in velocity dispersion with age, according to $D_v \sim d(\sigma_v^2)/d\tau$.

In general, the velocity of a star will change for two reasons: the regular variations along the orbit, and the irregular perturbations. In this section, we will neglect the orbital variations. Fortunately, this "force-free diffusion" combines conceptual simplicity with a rather accurate description of the time-dependence of the velocity dispersion. The full equations of motion will be treated in the next section. The diffusion coefficient, however, which we determine in the case of force-free diffusion, differs from the true diffusion coefficient D_v. Hence we shall call it the "apparent" diffusion coefficient C_v. In the absence of other forces beside the perturbations, the space velocity v of a star will be governed by the differential equation

$$d(v^2) = C_v dt. \tag{5}$$

Similar equations hold for the three components of the velocity, U, V, W. We distinguish the corresponding diffusion coefficients by indices, C_U, C_V, C_W. It holds that $C_v = C_U + C_V + C_W$.

In general, the diffusion will not only change the velocity dispersion but also the whole velocity distribution $f(U,V,W,t)$. Within our approximation, the change of f is described by a Fokker-Planck equation of the following form:

$$\frac{\partial f}{\partial t}=\frac{1}{2}\left[\frac{\partial}{\partial U}\left(C_U\frac{\partial f}{\partial U}\right)+\frac{\partial}{\partial V}\left(C_V\frac{\partial f}{\partial V}\right)+\frac{\partial}{\partial W}\left(C_W\frac{\partial f}{\partial W}\right)\right]. \tag{6}$$

Convenient discussions of the role of the Fokker-Planck equation in stellar dynamics can be found in the appendices of Chandrasekhar's book (1960) and in Hénon's review (1973). In accordance with our assumption of $\langle \Delta \boldsymbol{v}\rangle=0$, we neglect dynamical friction in Equation (6). This is phenomenologically justified since there is no indication that the velocity distribution or the velocity dispersion converges to a final state within 10^{10} yrs. Intuitively, that can be explained by saying that the mean kinetic energy of stars is far from being in equipartition with that of the perturbing fluctuations in density, because the effective mass of each perturbation is much higher than the mass of a star or even of an open cluster. Some of the general results presented below have already been obtained by Kuzmin (1961, 1973), who includes dynamical friction. Since the amount of dynamical friction is probably small for stars in galaxies, its inclusion has no practical implications. We aim here at a simplified but easily applicable procedure which gives actual numbers for the diffusion of stellar velocities and positions. Since the observational information on the change of the velocity distribution is much poorer than that on the velocity dispersion, we shall concentrate here mainly on the latter. The diffusion coefficient may depend on the velocity of the star and perhaps also on its position or explicitely on time. We shall study now some typical cases:

4.1. Constant Diffusion Coefficient

A constant value of the diffusion coefficient does not only represent the simplest case but fits the observations even better than more sophisticated versions. If C_v is constant, Equation (5) can be immediately integrated, yielding

$$v^2=v_0^2+C_v\tau. \tag{7}$$

Here v_0 is the initial velocity at the time of formation t_f, v is the statistically expected velocity at the present time of observation t_p, and $\tau=t_p-t_f$ is the present age of the star. Averaging over a group of stars with the same age, we derive

$$\sigma_v=(\sigma_{v,0}^2+C_v\tau)^{1/2}, \tag{8}$$

where σ_v and $\sigma_{v,0}$ are the velocity dispersions at present and at birth respectively. The relation (8) has been fitted to the observed data of Table 1. The rather nice fit for the total velocity dispersion σ_v, shown in Figures 1 and 2, yields

$$\sigma_{v,0}=10\ \mathrm{km\ s^{-1}},\quad C_v=6.0\ 10^{-7}\ (\mathrm{km\ s^{-1}})^2/\mathrm{yr}, \tag{9}$$

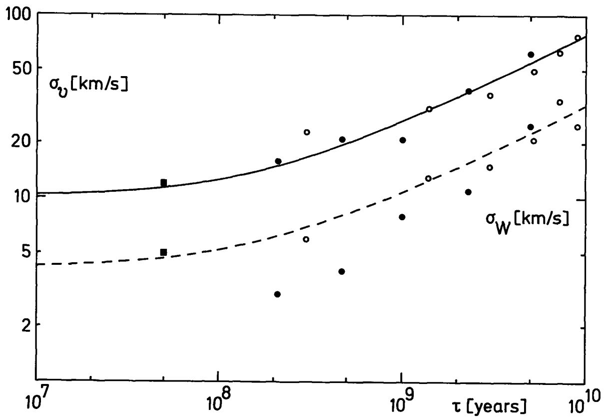

Fig. 2. Total velocity dispersion σ_v and velocity dispersion σ_W as a function of age τ. The curves are theoretical fits for an isotropic constant diffusion coefficient

with an estimated relative uncertainty of ±20% in both quantities. The diffusion coefficient C_v is mainly determined by the stars older than 10^9 yrs. The younger stars reflect mostly $\sigma_{v,0}$. Our value of $\sigma_{v,0}$ takes into account that $\sigma_{v,0}$ should be close to the observed dispersion in total space velocity of interstellar H I clouds (Takakubo, 1967). In reality, the initial velocity dispersion may depend slightly on the time of formation. There is, however, no serious error in using a constant value for $\sigma_{v,0}$, equal to the present one. Over the last billion years, the variation of $\sigma_{v,0}$ should be negligible, while for older stars, $\sigma_{v,0}$ is probably much smaller than the diffusion term $C_v\tau$.

In the lower part of Figure 2, we compare the observed data for the velocity dispersion in W with the relation $\sigma_W(\tau)=0.412\ \sigma_v(\tau)$. The reduction factor 0.412 is based on the assumption that the true diffusion coefficient is isotropic (see Section 5). The agreement is fair. In the case of constant diffusion coefficients, the Fokker-Planck Equation (6) is solved by an ellipsoidal Schwarzschild velocity distribution in which each velocity dispersion $\sigma_U(\tau)$, $\sigma_V(\tau)$ and $\sigma_W(\tau)$ varies according to Equation (8) after replacing the index v by U, V, or W. Hence the velocity distribution can be invariant except for the steady increase of the dispersions. Such a behaviour is in rather good agreement with the observed velocity distributions of the various age groups of the McCormick K+M dwarfs.

While the diffusion process acts primarily on the velocities of stars, it affects the stellar positions too. A perturbation in velocity, $\Delta \boldsymbol{v}_i$, at time t_i, causes a deviation in position, $\Delta \boldsymbol{r}_i(t)=\Delta \boldsymbol{v}_i(t-t_i)$ for $t>t_i$. Hence the diffusion in position is

$$\langle(\Delta r)^2\rangle=\left\langle\left(\sum_i \Delta \boldsymbol{v}_i(t-t_i)\right)^2\right\rangle=\int_0^\tau C_v(\tau-\tau')^2 d\tau'=\tfrac{1}{3}\,C_v\tau^3. \tag{10}$$

Contrary to the derived velocity dispersion $\sigma(\tau)$, however, the force-free result (10) for the diffusion of stellar positions is realistic for small ages only (see Section 5).

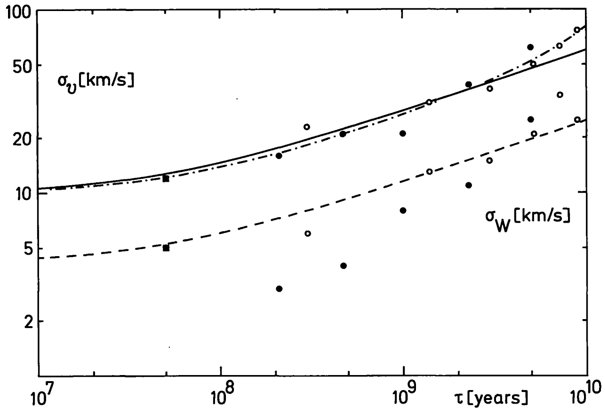

Fig. 3. As Figure 2, but here with theoretical fits for an isotropic velocity-dependent diffusion coefficient (full and dashed curves) and for a velocity-time-dependent coefficient (dash-dotted curve)

4.2. *Velocity-Dependent Diffusion Coefficient*

Gravitational two-body encounters between stars and other masses leads usually to a diffusion coefficient which decreases with increasing peculiar velocity v of a star (Chandrasekhar, 1960; Hénon, 1973; Spitzer and Schwarzschild, 1953). The following form of the diffusion coefficient reproduces the gross characteristics of a diffusion process, based on two-body encounters, if the velocities of the stars are larger than those of the scattering masses:

$$C_v = \gamma_v / v, \tag{11}$$

where the constant γ_v is determined by the diffusion mechanism only. From Equations (5) and (11), we obtain

$$v^3 = v_0^3 + \tfrac{3}{2}\gamma_v \tau. \tag{12}$$

In order to derive the increase of the velocity dispersion $\sigma_v(\tau)$ with age τ, we have to integrate v^2 over the distribution of initial velocities v_0. Since this distribution is not well-known and since the diffusion coefficient (11) is a poor approximation for small velocities, we assume that all the stars start with $v_0 = \sigma_{v,0}$, yielding

$$\sigma_v = (\sigma_{v,0}^3 + \tfrac{3}{2}\gamma_v \tau)^{1/3}. \tag{13}$$

The error introduced by this assumption is negligible for $\sigma_v \gg \sigma_{v,0}$ as well as for $\sigma_v \simeq \sigma_{v,0}$, and is usually small ($\lesssim 10\%$) for intermediate values. In Figure 3, we show a fit between the observed data (Tab. 1) and the relation (13) for

$$\sigma_{v,0} = 10\ \text{km s}^{-1}, \quad \gamma_v = 1.4\ 10^{-5}\ (\text{km s}^{-1})^3/\text{yr}. \tag{14}$$

The fit is adequate for ages up to about 3 10^9 yrs. For higher ages, the observed velocities show a faster increase with age than allowed by Equation (13). This may be an indication that the "constant" γ_v in Equation (11) is time-dependent. Let us assume that γ_v decreases with time t according to

$$\gamma_v(t) = \gamma_{v,p} \exp\left(-(t - t_p)/T_\gamma\right), \tag{15}$$

because the "roughness" of the gravitational field may decay with time. The constant $\gamma_{v,p}$ is the value of γ_v at the present time t_p, and T_γ is the decay time of γ_v. From Equations (15), (11) and (5), we derive (using $v_0 \sim \sigma_{v,0}$)

$$\sigma_v^3 = \sigma_{v,0}^3 + \tfrac{3}{2}\gamma_{v,p} T_\gamma (\exp(\tau/T_\gamma) - 1). \tag{16}$$

In Figure 3, the dash-dotted line represents the relation (16) for

$$\sigma_{v,0} = 10\ \text{km s}^{-1}, \gamma_{v,p} = 1.1\ 10^{-5}\ (\text{km s}^{-1})^3/\text{yr}, \quad T_\gamma = 5\ 10^9\ \text{yr}. \tag{17}$$

The fit of the observed data is now as good as in the case of a constant diffusion coefficient.

If the diffusion coefficient depends on the stellar velocity, then the diffusion alters not only the velocity dispersion but also the functional form of the velocity distribution. Spitzer and Schwarzschild (1951) have shown, however, that the change in the properly scaled velocity distribution with age is rather small.

It is therefore difficult at present to distinguish observationally between the cases of a constant diffusion coefficient and a properly chosen velocity- and time-dependent coefficient. The constant coefficient has the advantage of simplicity, while the velocity-dependent coefficient is more satisfying from a theoretical point of view. In the following, we shall present results for all the three cases [C_v = const, Eqs. (11) and (15)].

5. Epicyclic Theory of Diffusion

In contrast with Section 4 which dealed with force-free diffusion, we shall now take into account that the orbit of a star is mainly determined by the regular field of the Galaxy and is only perturbed by the diffusion in velocity space. We shall treat the problem within the epicyclic approximation (e.g. Lindblad, 1959) for an axisymmetric galactic field. Then the equations of stellar motion are given by

$$\ddot{\xi} - 2\omega\dot{\eta} = 4\omega A \xi, \tag{18}$$

$$\ddot{\eta} + 2\omega\dot{\xi} = 0, \tag{19}$$

$$\ddot{\zeta} + \omega_z^2 \zeta = 0, \tag{20}$$

where $\xi = R - R_0$, $\eta = R_0(\theta - \theta_0)$ and $\zeta = z$ are the deviations of the actual stellar orbit from the circular orbit of a reference star. The velocity components U, V, W, referred to the circular velocity at the actual position of the star, are given by

$$U = -\dot{\xi}, \tag{21}$$

$$V = \dot{\eta} + 2A\xi, \tag{22}$$

$$W = \dot{\zeta}. \tag{23}$$

The irregular gravitational field produces an extra force which should be added to the equations of motion (18)–(20). Within our description of the irregular field by a diffusion in velocity space, however, it is easier not to introduce the extra force terms explicitely but to mimic the effect by sudden changes in the velocity of a star. This can be treated formally as sequential initial value problems using the unaltered equations of motion inbetween two successive velocity perturbations.

We ask now for the difference between the orbits of a star with and without diffusion. Let $\Delta R = R$ (with diffusion)—R (without diffusion) be the difference between the distances from the axis of galactic rotation, $\Delta S = R\Delta\theta$ the corresponding difference in the direction of galactic rotation, $\Delta z = \Delta\zeta$ the difference in height above the galactic plane and ΔU, ΔV, ΔW the corresponding differences in velocity. Because of the linearity of Equations (18)–(23), the differential quantities ΔR, ΔS, Δz, ΔU, ΔV, ΔW fulfil the same equations of motion as the total quantities ξ, η, ζ, U, V, W:

$$(\Delta R)^{\cdot\cdot} - 2\omega(\Delta S)^{\cdot} = 4\omega A\Delta R, \tag{24}$$

$$(\Delta S)^{\cdot\cdot} + 2\omega(\Delta R)^{\cdot} = 0, \tag{25}$$

$$(\Delta z)^{\cdot\cdot} + \omega_z^2 \Delta z = 0, \tag{26}$$

$$\Delta U = -(\Delta R)^{\cdot}, \tag{27}$$

$$\Delta V = (\Delta S)^{\cdot} + 2A\Delta R, \tag{28}$$

$$\Delta W = (\Delta z)^{\cdot}. \tag{29}$$

Let us consider first a single velocity perturbation of amount δU_i, δV_i, δW_i at a time t_i. Up to $t = t_i$, there is no difference in the orbits: $\Delta R = 0, \ldots, \Delta W = 0$. For a time $t > t_i$, the orbits differ by

$$\begin{aligned}\Delta R &= -\varkappa^{-1} \sin \varkappa(t-t_i)\delta U_i \\ &\quad + (-2B)^{-1}(1-\cos \varkappa(t-t_i))\delta V_i,\end{aligned} \tag{30}$$

$$\begin{aligned}\Delta S &= (-2B)^{-1}(1-\cos \varkappa(t-t_i))\delta U_i \\ &\quad + (-2B)^{-1}[(\varkappa/-2B) \sin \varkappa(t-t_i) \\ &\quad - 2A(t-t_i)]\delta V_i,\end{aligned} \tag{31}$$

$$\Delta z = \omega_z^{-1} \sin \omega_z(t-t_i)\delta W_i, \tag{32}$$

$$\Delta U = \cos \varkappa(t-t_i)\delta U_i - (\varkappa/-2B) \sin \varkappa(t-t_i)\delta V_i, \tag{33}$$

$$\Delta V = (-2B/\varkappa) \sin \varkappa(t-t_i)\delta U_i + \cos \varkappa(t-t_i)\delta V_i, \tag{34}$$

$$\Delta W = \cos \omega_z(t-t_i)\delta W_i. \tag{35}$$

These solutions have been derived from the differential Equations (24)–(29) using the initial values $\Delta R = \Delta S = \Delta z = 0$, $\Delta U = \delta U_i$, $\Delta V = \delta V_i$, $\Delta W = \delta W_i$ at $t = t_i$. The complete solution for $\Delta R(t)$ etc. is given by the sum (over i) of the contributions of all the velocity perturbations between the time of formation t_f of a star and the time of observation, $t = t_p$. We now introduce our statistical assumptions on the velocity perturbations, namely that they are independent and randomly distributed. Then, on the average,

$$\sum_i \delta U_i = \sum_i \delta V_i = \sum_i \delta W_i = 0, \tag{36}$$

$$\sum_i \delta U_i \delta V_i = \sum_i \delta U_i \delta W_i = \sum_i \delta V_i \delta W_i = 0, \tag{37}$$

$$d\Big(\sum_i \delta U_i \delta U_i\Big) = D_U dt, \tag{38}$$

$$d\Big(\sum_i \delta V_i \delta V_i\Big) = D_V dt, \tag{39}$$

$$d\Big(\sum_i \delta W_i \delta W_i\Big) = D_W dt. \tag{40}$$

Equations (38)–(40) have been written in a differential form because the diffusion coefficient D may vary with time. Using the statistical assumptions (36)–(40), we derive the following expectation values for the orbital differences from Equations (30)–(35) as a function of the age $\tau = t_p - t_f$ of a star:

$$\langle \Delta R \rangle = \langle \Delta S \rangle = \langle \Delta z \rangle = \langle \Delta U \rangle = \langle \Delta V \rangle = \langle \Delta W \rangle = 0, \tag{41}$$

$$\begin{aligned}\langle (\Delta R)^2 \rangle &= \int_0^\tau (D_U \varkappa^{-2} \sin^2 \varkappa(\tau-\tau') \\ &\quad + D_V (2B)^{-2}(1-\cos \varkappa(\tau-\tau'))^2)d\tau',\end{aligned} \tag{42}$$

$$\begin{aligned}\langle (\Delta S)^2 \rangle &= \int_0^\tau (2B)^2 (D_U(1-\cos \varkappa(\tau-\tau'))^2 \\ &\quad + D_V((\varkappa/-2B) \sin \varkappa(\tau-\tau') \\ &\quad - 2A(\tau-\tau'))^2)d\tau',\end{aligned} \tag{43}$$

$$\langle (\Delta z)^2 \rangle = \int_0^\tau D_W \omega_z^{-2} \sin^2 \omega_z(\tau-\tau')d\tau', \tag{44}$$

$$\begin{aligned}\langle (\Delta U)^2 \rangle &= \int_0^\tau (D_U \cos^2 \varkappa(\tau-\tau') \\ &\quad + D_V(\varkappa/2B)^2 \sin^2 \varkappa(\tau-\tau'))d\tau',\end{aligned} \tag{45}$$

$$\begin{aligned}\langle (\Delta V)^2 \rangle &= \int_0^\tau (D_U(2B/\varkappa)^2 \sin^2 \varkappa(\tau-\tau') \\ &\quad + D_V \cos^2 \varkappa(\tau-\tau'))d\tau',\end{aligned} \tag{46}$$

$$\langle (\Delta W)^2 \rangle = \int_0^\tau D_W \cos^2 \omega_z(\tau-\tau')d\tau'. \tag{47}$$

The mixed second order moments are of minor interest in general, although the majority of them are not strictly zero. We postpone a discussion of the mixed moments $\langle \Delta U \Delta V \rangle$ and $\langle \Delta R \Delta S \rangle$ to Section 6. The evaluation of the integrals in Equations (42)–(47) is straightforward if the diffusion coefficients are either constant or depend explicitely on age, $D(\tau')$, or time, $D(t_f + \tau')$. In the case of velocity-dependent coefficients, the problem can, in principle, only be solved by iteration.

The velocity dispersion, measured now relative to the local circular velocity, of a group of stars of the same age is given by

$$\sigma_U^2 = \sigma_{U,0}^2 + \langle (\Delta U)^2 \rangle \tag{48}$$

and by similar equations for V, W and v. The quantity $\sigma_{U,0}$ represents the contribution of the random and systematic velocities at birth to the velocity dispersion at later times. $\sigma_{U,0}$ may vary with age as long as the stars are statistically not well-mixed. For simplicity, we shall identify $\sigma_{U,0}$ with the initial velocity dispersion at birth.

In Section 6, we shall study numerically the behaviour of $\langle(\Delta R)^2\rangle, \ldots, \langle(\Delta W)^2\rangle$ with age. It turns out that each quantity can be decomposed into two parts: (1) a steady increase with age, and (2) a periodic modulation. The latter oscillation is of no importance in practice. Hence we shall derive here the more important first part directly. For that purpose, we may average the Equations (42)–(47) over e.g. one epicyclic period. For example, in Equation (47) we may replace $\cos^2 \omega_z(\tau - \tau')$ by its average value 1/2, yielding:

$$\langle\langle(\Delta W)^2\rangle\rangle = \int_0^\tau (D_W/2)\, d\tau'. \tag{49}$$

This relation shows that the apparent diffusion coefficient $C_W = d\sigma_W^2/dt = d\langle\langle(\Delta W)^2\rangle\rangle/dt$ is smaller than the true coefficient D_W:

$$C_W = 0.5\, D_W. \tag{50}$$

The reason for the apparent disappearance of half of the energy provided by the diffusion is the following: Although the diffusion enhances primarily the kinetic energy at the time t_i when the velocity perturbation occurs, a part of this energy gain is subsequently transformed into potential energy by the orbital motion of the star in the regular field. I.e., the amplitude of the z-motion is increased on the average and that increase "absorbs" half of the energy provided by the diffusion process.

It is illustrative to derive the corresponding results for U and V not from Equations (45) and (46) but from the epicyclic energy integral

$$E = U^2 + (\varkappa/2B)^2 V^2. \tag{51}$$

The values of U^2 and V^2 averaged over one epicyclic period, are given by

$$\langle U^2\rangle = E/2, \tag{52}$$

$$\langle V^2\rangle = (2B/\varkappa)^2 E/2. \tag{53}$$

These equations show that the long-term increase in σ_U and σ_V is completely governed by the corresponding increase in E. The average ratio σ_U/σ_V shall be constant in time, equal to $(\kappa/-2B)$, both with and without diffusion. Hence the ratio of the apparent diffusion coefficients C_U and C_V is also equal to $(\varkappa/2B)^2$, on the average, and does not provide any information about the ratio D_U/D_V of the true coefficients. Introducing a "diffusion coefficient" D_E for the epicyclic energy E by $dE = D_E\, dt$, we derive from Equation (51) with our statistical assumptions (36)–(40)

$$D_E = D_U + (\varkappa/2B)^2 D_V \tag{54}$$

and hence, using Equations (52) and (53),

$$C_U = d\sigma_U^2/dt = (D_U + (\varkappa/2B)^2 D_V)/2, \tag{55}$$

$$C_V = d\sigma_V^2/dt = ((2B/\varkappa)^2 D_U + D_V)/2. \tag{56}$$

We have to insert a warning here: Although our statistical assumptions (36)–(40) are suitably chosen for describing the global effects of a typical diffusion mechanism on stellar orbits, they will in general not be strictly valid for a specific physical process. In such a situation, it can be misleading to calculate the diffusion coefficients and to insert these into our equations. Usually, one would have to go back to the basic Equations (30)–(35). As an extreme example, one could imagine a process in which each perturbation changes the velocity components U and V, but nevertheless conserves the epicyclic energy E. In this case, D_E would obviously vanish in spite of $D_U \neq 0$ and $D_V \neq 0$, thereby violating our Equation (54). The failure of our statistical equations were caused by the correlation between δU_i and δV_i introduced by the condition $\delta E_i = 0$. This shows that our procedure should not be considered as a general framework for all local accelerating mechanisms. The purpose of our procedure is to indicate the expected order of magnitude of the diffusion of stellar orbits, as long as we do not possess a better knowledge of the actual local accelerating mechanism.

Since we have neither from the observations nor from theoretical considerations any relevant information about the ratio between the diffusion coefficients D_U and D_V, we shall consider in the following the simplest case, namely isotropic diffusion:

$$D_U = D_V = D_W = D. \tag{57}$$

Using the conventional values for Oort's constants ($A = 15$ km s^{-1}/kpc, $B = -10$ km s^{-1}/kpc at $R \sim R_0 = 10$ kpc), we have $(\varkappa/2B)^2 = 2.5$, and hence Equations (55), (56) and (50) lead to

$$C_U = 1.75\, D,\ C_V = 0.70\, D,\ C_W = 0.50\, D, \qquad C_v = C_U + C_V + C_W = 2.95\, D. \tag{58}$$

The numerical value of the true diffusion coefficient D is obtained from $D = C_v/2.95$ by using the empirical values of C_v according to the Equations (9), (14) or (17). They are the *constant coefficient:*

$$D = 2.0\ 10^{-7}\ (\text{km s}^{-1})^2/\text{yr}, \tag{59}$$

velocity-dependent coefficient:

$$D = (4.7\ 10^{-6}\ (\text{km s}^{-1})^3/\text{yr})/v, \tag{60}$$

velocity-time-dependent coefficient:

$$D = (3.7\ 10^{-6}\ (\text{km s}^{-1})^3/\text{yr}) \exp(-(t - t_p)/5\ 10^9\,\text{yr})/v. \tag{61}$$

Table 2. Diffusion of stellar orbits for the constant diffusion coefficient [Eq. (59)]

Age	Expected rms deviation							
τ yrs	ΔU km s^{-1}	ΔV	ΔW	Δv	ΔR kpc	ΔS	Δz	Δp
$1\ 10^7$	1.5	1.4	1.2	2.4	0.008	0.008	0.008	0.014
$2\ 10^7$	2.2	1.9	1.3	3.2	0.025	0.023	0.017	0.038
$5\ 10^7$	4.2	2.7	2.3	5.5	0.108	0.091	0.025	0.143
$1\ 10^8$	5.9	3.8	3.2	7.7	0.300	0.392	0.035	0.495
$2\ 10^8$	8.4	5.4	4.5	10.9	0.410	1.356	0.050	1.418
$5\ 10^8$	13.3	8.5	7.1	17.3	0.67	4.47	0.078	4.52
$1\ 10^9$	18.8	12.0	10.1	24.5	0.92	12.7	0.111	12.8
$2\ 10^9$	26.7	16.9	14.3	34.7	1.30	35.8	0.156	35.8
$5\ 10^9$	42.2	26.7	22.5	54.8	2.09	141	0.247	141
$1\ 10^{10}$	59.7	37.7	31.9	77.5	2.94	399	0.349	399

Knowing the true diffusion coefficient D, we can estimate the neglected coefficient η of dynamical friction. According to Chandrasekhar (1960), it holds in the case of constant coefficients that $\eta = 3D/2\sigma^2_{\rm equi}$, where $\sigma_{\rm equi}$ is the total velocity dispersion in the hypothetical final Maxwellian equilibrium state. Since the observed relation $\sigma_v(\tau)$ does not seem to be significantly affected by dynamical friction, we estimate $\sigma_{\rm equi} > 100$ km s^{-1}. This leads to $\eta^{-1} > 3\ 10^{10}$ yrs. Hence dynamical friction should be of no importance for the diffusion of stars over the present age of our Galaxy. Furthermore, any allowance for dynamical friction would *increase* the derived differences between the actual and regular orbits: In order to explain the *same* observed increase of $\sigma(\tau)$, the diffusion coefficients would have to be increased to counteract dynamical friction which tends to decrease the stellar velocities.

Finally, we shall demonstrate that our assumption of an isotropic true diffusion coefficient is in accordance with the observed velocity dispersions of the most representative group of disk stars, namely the McCormick K+M dwarfs in Gliese's catalogue. The observed values of σ_U, σ_V, σ_W are 48, 29 and 25 km s^{-1}, respectively, according to Table 1. The values predicted from isotropic diffusion [Eq. (58)],

$$\sigma_U : \sigma_V : \sigma_W = \sqrt{C_U} : \sqrt{C_V} : \sqrt{C_W} = 0.77 : 0.49 : 0.41, \qquad (62)$$

are, for the same total velocity dispersion σ_v, 47, 30 and 25 km s^{-1} respectively, which is in perfect agreement with the observed data. Hence, the hypothesis of isotropic diffusion provides a simple explanation for the otherwise unexplained ratio of σ_W to σ_U and σ_V. This was noticed earlier by Kuzmin (1961).

6. Implications for Stellar Orbits

The regular orbit of a star can be rigorously calculated as soon as the regular gravitational field of a galaxy is known. A stellar orbit in the presence of a local irregular field can be described in a statistical sense only, because the local fluctuations are not known individually but only statistically, e.g. by means of the diffusion coefficient. The typical deviations between the actual and regular orbits can be best characterized by the root mean squared differences in stellar position and velocity, i.e. by $\Delta R_{\rm rms} = (\langle(\Delta R)^2\rangle)^{1/2}$ etc. These quantities describe the "uncertainty" of a stellar orbit in the presence of the irregular field.

In the Tables 2–4, we present numerical values for the effect of diffusion on stellar orbits. The quantities $\Delta p_{\rm rms}$ and $\Delta v_{\rm rms}$ give the total rms deviation in position and velocity, i.e. $\Delta p_{\rm rms} = \langle(\Delta \mathbf{r})^2\rangle^{1/2}$ and $\Delta v_{\rm rms} = \langle(\Delta \mathbf{v})^2\rangle^{1/2} = (\Delta U^2_{\rm rms} + \Delta V^2_{\rm rms} + \Delta W^2_{\rm rms})^{1/2}$. The results have been obtained by analytical or numerical integration of the epicyclic Equations (42)–(46), using the isotropic diffusion coefficients (59), (60) or (61), respectively. In the two cases where the diffusion coefficient depends on the velocity v, we have replaced v by σ_v according to Equations (13) or (16). This approximation is adequate for our present purpose. In the case of the time-dependent diffusion coefficient, τ is the *present* age of a star, i.e. $t_f = t_p - \tau$.

The data in the Tables 2–4 do not differ grossly. This is to be expected, because it is built into the three cases that they should fit the *same* observed relation $\sigma_v(\tau)$. In fact, many of the quantities in Tables 2–4, could have been obtained directly from the empirical relations, if these were not so "noisy". For example, $\Delta v_{\rm rms}(\tau)$ should be equal to $(\sigma^2_v(\tau) - \sigma^2_v(0))^{1/2}$. For ages up to a few billion years, the constant diffusion coefficient produces the smallest orbital diffusion. In that sense, the constant coefficient is the most conservative choice. In general, it is obvious that the diffusion of stellar orbits is rather drastic: A typical star has forgotten its initial peculiar velocity ($v_0 = 10$ km s^{-1}) within 2 10^8 yrs ($\Delta v_{\rm rms} = 11-14$ km s^{-1}). Also later, after each galactic revolution the total space velocity has changed again by a random amount of about 10 km s^{-1}. The spread in position is also remarkably large, especially in the "angular" coordinate $S = R\theta$.

Table 3. Diffusion of stellar orbits for the velocity-dependent diffusion coefficient [Eq. (60)]

Age	Expected rms deviation							
τ yrs	ΔU km s^{-1}	ΔV	ΔW	Δv	ΔR kpc	ΔS	Δz	Δp
$1\ 10^7$	2.2	2.1	1.9	3.6	0.013	0.013	0.012	0.022
$2\ 10^7$	3.3	2.9	1.9	4.8	0.037	0.035	0.025	0.057
$5\ 10^7$	6.1	3.8	3.2	7.9	0.159	0.135	0.035	0.211
$1\ 10^8$	8.1	5.2	4.4	10.6	0.428	0.571	0.048	0.715
$2\ 10^8$	10.8	7.0	5.8	14.1	0.519	1.858	0.064	1.931
$5\ 10^8$	15.4	9.9	8.3	20.1	0.80	5.68	0.092	5.73
$1\ 10^9$	20.0	12.8	10.7	26.1	0.97	15.1	0.119	15.1
$2\ 10^9$	25.9	16.3	13.8	33.6	1.25	39.0	0.151	39.0
$5\ 10^9$	35.6	22.5	19.0	46.2	1.78	136	0.209	136
$1\ 10^{10}$	45.2	28.6	24.1	58.7	2.23	347	0.264	347

Table 4. Diffusion of stellar orbits for the velocity-time-dependent diffusion coefficient [Eq. (61)]

Age	Expected rms deviation							
τ yrs	ΔU km s^{-1}	ΔV	ΔW	Δv	ΔR kpc	ΔS	Δz	Δp
$1\ 10^7$	2.0	1.9	1.7	3.2	0.011	0.011	0.010	0.019
$2\ 10^7$	2.9	2.6	1.7	4.2	0.033	0.031	0.022	0.051
$5\ 10^7$	5.5	3.4	2.9	7.1	0.142	0.121	0.032	0.189
$1\ 10^8$	7.3	4.7	3.9	9.6	0.386	0.514	0.043	0.645
$2\ 10^8$	9.9	6.4	5.3	12.9	0.476	1.697	0.059	1.763
$5\ 10^8$	14.4	9.3	7.7	18.7	0.74	5.31	0.085	5.36
$1\ 10^9$	19.0	12.2	10.2	24.8	0.92	14.6	0.113	14.6
$2\ 10^9$	25.6	16.1	13.6	33.2	1.23	40.0	0.149	40.0
$5\ 10^9$	39.5	25.0	21.1	51.3	1.98	165	0.232	165
$1\ 10^{10}$	61.8	39.1	33.0	80.2	3.05	554	0.362	554

After 10^{10} yrs, the number of revolutions of a star around the galactic center is uncertain by ± 6 revolutions, compared with the average number of 40 revolutions at $R=10$ kpc. The distance of old disk stars from the galactic plane is almost completely determined by the diffusion mechanism. The distances of these old stars from the galactic center differ typically by a few kpc from the original values at formation. Altogether, the diffusion produces a very significant mixing of stars in position and velocity space.

In Figure 4, we show the modulation of the increase of the deviations between the actual and regular orbits of stars. This modulation is caused by the orbital motion of the stars in the regular field. Since the amplitude of the modulation remains approximately constant in time, its relative importance decreases rapidly with time. The modulation is smaller in the velocities than in the positions. Especially in the total space velocity v, there is essentially no modulation. This justifies our procedure of obtaining the apparent diffusion coefficient by using the unmodulated relations $\sigma_v(\tau)$ according to Equations (8), (13) or (16). The observed values of the velocity dispersions do not show any sign of a modulation. This is to be expected, because the modulation is already small compared to the initial velocity dispersions.

In Figure 5, we show the elliptic contours of the rms deviations in the galactic plane. They are based on the quantities $\langle(\Delta R)^2\rangle$, $\langle(\Delta S)^2\rangle$ and $\langle \Delta R \Delta S\rangle$. The mixed moment $\langle \Delta R \Delta S\rangle$ is negative because of the differential galactic rotation. In spite of the assumed isotropy of diffusion, the contours are tilted. The major axis turns monotonically from $\ell=0°$ for $\tau=0$ to $\ell=90°$ for $\tau\rightarrow\infty$. There is a small *negative* vertex deviation if $2.5\, D_V > D_U$. For the constant isotropic diffusion coefficient, the moment $\langle \Delta U \Delta V\rangle$ varies as $-\sin^2 \varkappa\tau$ between 0 and $-(1.7 \text{ km s}^{-1})^2$. Hence it is always small compared with σ_U^2 and σ_V^2. The observed *positive* vertex deviation must be due to other effects. Both Figures 4 and 5 are based on a constant diffusion coefficient but are representative for the other cases too.

In order to verify the results derived from the epicyclic approximation, we have numerically integrated some "diffusion orbits". The regular field is that given by Contopoulos and Strömgren (1965) and Wielen (1973). A constant diffusion coefficient according to

Equation (59) is mimicked by changing the stellar velocity by small increments $\Delta \mathbf{v}$ after intervals of time Δt. The orientation of each $\Delta \mathbf{v}$ is chosen at random, while $\Delta t = 10^6$ yrs and $|\Delta \mathbf{v}| = (3D\Delta t)^{1/2} = 0.78$ km s^{-1} are kept constant. As an example, we show in Figure 6 the variation of $R(t)$ in a diffusion orbit which has been started with the circular velocity ($v_0 = 0$, $R_0 = 10$ kpc). The oscillations in R are excited and subsequently modified by the diffusion, while the regular field is revealed by the regular behaviour of each oscillation.

The statistical averages over many of such diffusion orbits are in good agreement with the results derived from the epicyclic approximation. Only one significant difference occurs: There is systematic outward drift of the diffusion orbits, i.e. $\langle \Delta R \rangle$ is positive and increases with age. At $\tau = 10^{10}$ yrs, $\langle \Delta R \rangle$ is of the order of $+1$ kpc. This drift is basically caused by the fact that for a finite deviation of an orbit from its circular reference orbit, the star makes a larger and longer excursion to positive values of ΔR than to negative values. Because the diffusion enhances the deviation of an actual orbit from a circular one, it thereby leads to the outward drift of stars. The occurrence of this drift in real galaxies, however, is questionable since we have used a fixed gravitational field for the orbital studies instead of treating the problem in a self-consistent manner.

Finally, we should emphasize that the expected rms deviations due to orbital diffusion probably include a drastic averaging in position and time. For example, the actual diffusion coefficient may be larger within a spiral arm than outside, or may decrease with increasing $|z|$. Furthermore, the diffusion is evident only in the velocity dispersion of stars older than 10^9 yrs, since the velocity dispersion of younger stars is governed by the initial velocities and by the orbital mixing in the gravitational field of the density wave. Hence we cannot decide empirically, whether the diffusion is caused by many small perturbations (say, $|\delta v_i| \sim 0.8$ km s^{-1} in intervals of $\Delta t \sim 10^6$ yrs) or by rare large perturbations (e.g., $|\delta v_i| \sim 17$ km s^{-1} with $\Delta t \sim 5\,10^8$ yrs). At present, however, there is no reason to assume that the diffusion is inoperative for younger stars. Of course, the typical time scale of the fluctuations of the irregular field, as seen by an orbiting star, will be known only after an identification of the source of this irregular field.

7. Implications for Places of Formation

Stellar orbits, calculated backwards in time, have been used to derive the places of formation of stars of known age (Strömgren, 1967; Yuan, 1969; Wielen, 1973; Grosbøl, 1976). The probable diffusion of stellar orbits affects these calculated birth places significantly. The diffusion introduces an unavoidable "cosmic error" into the presently observed positions and velocities which are used as initial values for the backward integration of the regular orbits. According to our results (Tables 2–4), this cosmic error increases with age and reaches $\Delta v_{\rm rms} \sim 10$ km s^{-1} and $\Delta p_{\rm rms} \sim 1.5$ kpc after only $2\,10^8$ yrs. It is therefore useless to measure the present space velocity of a star with a much higher accuracy than $\Delta v_{\rm rms}(\tau)$, for the purpose of deriving birth places. Hence only objects with rather small ages, up to at most 10^8 yrs, can be used to derive accurate individual places of formation. For older stars, it is important to use as many stars as possible in order to obtain at least some statistical information on the most probable regions of star formation. The typical uncertainty of a birth place can be derived from Figure 5. One has simply to reflect the ellipses through the ΔR-axis, because we are now going backwards in time.

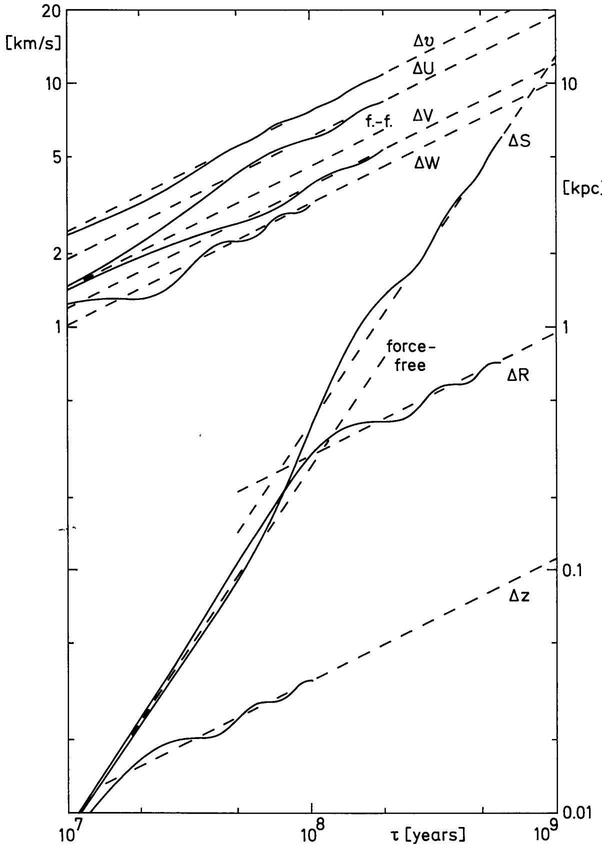

Fig. 4. The modulated increase of the expected rms deviations caused by orbital diffusion as a function of age τ. The asymptotic behaviour for small τ (f.-f. = force-free diffusion, using the true diffusion coefficient) and for large τ is indicated by the dashed lines

A more detailed treatment of the uncertainty of the calculated places and velocities of formation has been worked out by applying the Equations (42)–(47) to this problem. The reference orbit is now the regular orbit derived from the presently observed position and velocity of a star. Going backwards in time, the actual orbit deviates from this regular orbit in a similar way as for the forward direction of time. In fact, for a constant diffusion coefficient, the expected rms devia-

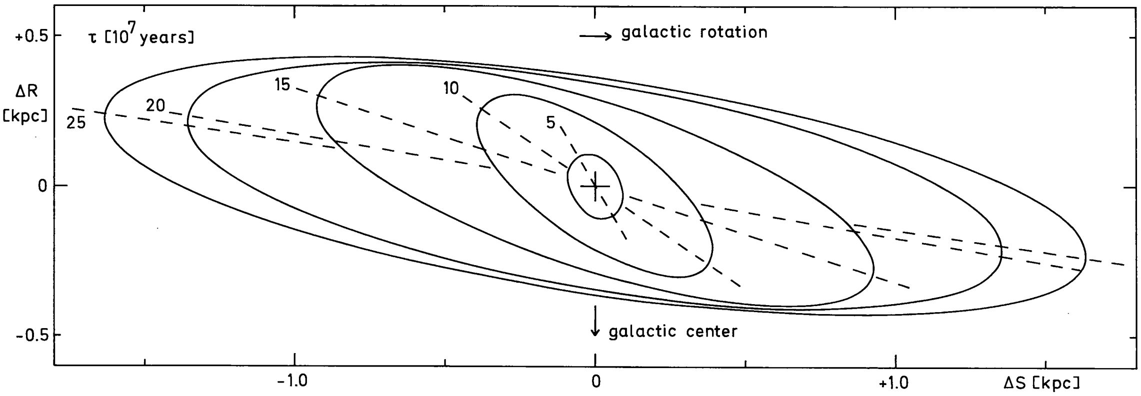

Fig. 5. Dispersion ellipses in the galactic plane for orbital diffusion as a function of age τ. For a normal distribution, the dispersion ellipse contains 39% of the total probability. The 50%-ellipses would be larger by a factor 1.18

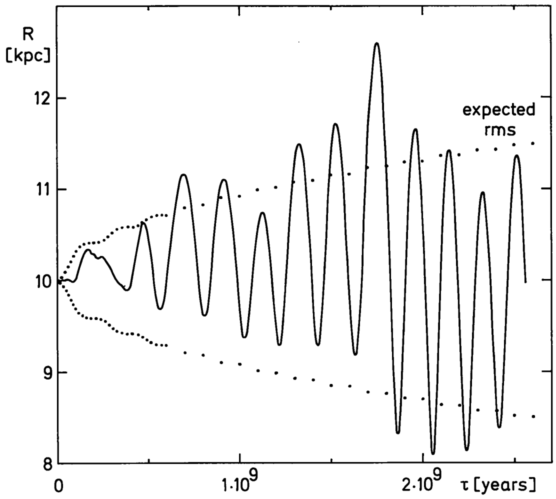

Fig. 6. Example of a diffusion orbit. The variation of the stellar distance R from the galactic center is shown as a function of age τ. The dots represent the expected rms deviation in R

tions (Table 2) are exactly the same for both arrows of time. For diffusion coefficients which decrease with increasing age, such as those described by Equations (60) or (61), the uncertainty of the places of formation is slightly smaller than the orbital diffusion, presented in Section 6 for the forward direction of time. The differences between the forward and backward directions, however, are so small that Tables 3 and 4 can be applied without any significant error for birth places too. The interpretation, however, is now slightly different. When going backwards in time, the increase of $\Delta v_{\rm rms}$, for example, does not imply a corresponding change of the actual velocity dispersion of stars but merely reflects the increasing loss of information on the actual velocity. This does not contradict the real decrease of the velocity dispersion as we approach the time of formation.

8. Dissolution of Stellar Groups

The diffusion of stellar orbits can enhance the dissolution of stellar groups by increasing the internal velocity dispersion of the group. We encounter here, however, the problem of the coherence of the irregular gravitational field over small distances. If two stars are close together in space, their perturbations at the same time will be strongly correlated. The distance over which simultaneous perturbations are significantly correlated, may be called the coherence length L of the irregular gravitational field. Its value depends on the physical mechanism which causes the irregular field. In our Galaxy, the coherence length may be of the order of a few hundred parsecs.

The dissolution of a stellar group will proceed in two phases. At first, the internal velocity dispersion $\sigma_{\rm int}$ will blow up the group until the diameter Q is larger than the coherence length L. Then the second phase starts during which the diffusion mechanism will support the dissolution of the group. The dispersing effect of diffusion during this second phase is properly described by $\Delta R_{\rm rms}$, $\Delta S_{\rm rms}$ and $\Delta z_{\rm rms}$ (Tables 2–4, Fig. 5), if the age τ is replaced by the period of time passed since the beginning of the second phase. The duration of the first phase is proportional to $(L-Q)/\sigma_{\rm int}$ which is about $5\ 10^7$ yrs for $\sigma_{\rm int}=10$ km s^{-1} and $L-Q=500$ pc. Already during the first phase, the velocity and position of the center-of-mass of the group is entirely affected by the diffusion mechanism, while the relative positions and velocities in the group are disturbed only by the smaller tidal effect in the irregular field for $Q(\tau)<L$. If the diffusion coefficient decreases with increasing stellar velocity, a group with a higher mean peculiar space velocity will survive longer than those with small mean velocities (Woolley and Candy, 1968).

The presence of a diffusion mechanism essentially prevents the existence of old moving groups of large spatial extent and small internal velocity dispersion.

This could be used as an observational test against the assumed diffusion of stellar orbits. At present, however, no such counter-example is established. Orbital diffusion may help to dissolve ageing spiral arms as well as clouds of escapers around star clusters.

9. Further Implications of Orbital Diffusion

If the diffusion of stellar orbits is established as a real phenomenon, this would have far-reaching consequences for stellar dynamics of galaxies in general. The irregular field does not allow any strict individual integrals of motion. Nevertheless, even in the presence of diffusion, a star would "be aware" of the existence of integrals of motion in the regular field. For example, a third integral in the regular field would still prevent the exchange of energy, gained by a star from diffusion, between the motions parallel and perpendicular to the galactic plane. Other properties of regular orbits would, however, become less prominent. For example, orbital resonance effects may largely be washed-out.

For a statistical description of stars in galaxies, the encounterless Liouville equation would not now be appropriate. The relaxation time T of young disk stars, defined by $\Delta v_{\rm rms}(T) \sim \sigma_{v,0}$, is about 2 10^8 yrs. The implications of diffusion for collective effects, such as spiral density waves, are difficult to assess.

The existence of a rather local acceleration mechanism would hamper numerical experiments on the dynamical evolution of galaxies. The self-consistent gravitational field used in these experiments essentially represents the regular field only. Even if the local fluctuations of the gravitational field could be self-consistently incorporated into the numerical model, the computing time would increase drastically, since a smaller grid size, a smaller time step and a larger number of stars are required for such a more detailed model. If the local fluctuations are mainly caused by the interstellar matter, a self-consistent treatment is still more difficult. Of course, the diffusion could be artificially imposed in a similar way as described for our numerical diffusion orbits (Section 6). Our procedure should then be generalized in order to take into account the non-zero coherence length of the supposed irregular field.

10. Concluding Remarks

We have shown that the observed increase of the stellar velocity dispersion with age seems to indicate the presence of a rather significant irregular gravitational field in our Galaxy, and hence, by analogy, presumedly in similar galaxies too. The diffusion of stellar orbits, caused by this irregular field, would be important: The typical relaxation time of young disk stars would be of the order of the period of revolution, about 2 10^8 yrs, in the solar neighbourhood. It is clear, however, that the diffusion of stellar orbits is at present merely a hypothesis, though a very plausible one in the author's opinion. The actual existence of the supposed diffusion mechanism for stars in galaxies cannot be considered as safely established as long as the physical source of this irregular field has not been identified with some certainty.

It has been our philosophy to use only that property which should be common to most local acceleration mechanisms, namely the resulting diffusion of stars. The value of the diffusion coefficient is not based on presently rather speculative details of possible acceleration mechnisms, but has been derived empirically from the *observed* increase of the stellar velocity dispersion with age. Hence our general results on the diffusion of stellar orbits will hopefully be approximately valid for whatever the physical source of the irregular field may turn out to be.

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